Self-avoiding Lévy flights in one dimension

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We study the node-avoiding (NALF) and path-avoiding (PALF) extensions of the Lévy flights in one dimension both numerically and analytically. The asymptotic behavior of the PALF is determined exactly while that of NALF has been available from the mapping to a spin model. Monte Carlo results for both types of self-avoiding Lévy flights are used to study the convergence toward the asymptotic behavior. We find very large corrections to asymptotic scaling in NALF for a wide range of the Lévy index μ and also, surprisingly, that the moments of the end-to-end distance of the NALF are greater than those of the PALF when they both exist. Based on these observations we conclude that the morphology of the NALF is far more complex than that of the PALF or the random Lévy flights, and that the NALF and PALF are certainly in different universality classes in one dimension.

I. INTRODUCTION

Some time ago Mandelbrot[1] discussed certain random walks which do not have a fixed step size but rather are associated with a variable step size l distributed according to a probability distribution of the type

$$P(l) \propto l^{-1-\mu}, \tag{1.1}$$

with $\mu > 0$. These walks are called *Lévy flights* since $P(l)$ is a distribution of Lévy type,[2] and for $0<\mu<2$ they were found to have trajectories strikingly different from those of ordinary random walks. In particular, the mean step size of such walks is infinite and the trace of the sites visited by the walk forms a set of fractal dimension μ.[3]

Recently, some variants of the original Lévy flight have been proposed and discussed in relation to a wide range of physical phenomena such as chaos and turbulence,[4] adsorption of polymer chains on a surface,[5] and pattern recognition.[6] In this paper, we study the theoretical aspects of some self-avoiding extensions[7] of the Lévy flight, which are possibly relevant to the latter two physical problems listed. The two types of self-avoiding Lévy flights proposed earlier[7] are classified according to the severity of the constraints imposed on the walks: a node-avoiding Lévy flight (NALF) is a Lévy flight which does not visit the same site (or *node*) more than once, while a path-avoiding Lévy flight (PALF) does not have any overlapping or intersecting step (or *path*) in addition to the node-avoiding constraint (see Fig. 1). These Lévy flights are defined algorithmically as usual with the constraints used to remove particular walks from the ensemble of the *random* Lévy flights.

Since the mean step size is infinite for both types of Lévy flights, the xth moments of the end-to-end distance, $\langle R_N^x \rangle$, of the N-step Lévy flights are also infinite for most values of x. Thus, we will choose to consider the "zeroth moment" $\langle \ln R_N \rangle$ in most of what follows. If we assume for large N that

$$\langle R_N^x \rangle^{1/x} = AN^{\nu}(1 - BN^{-\Delta_1} + \cdots), \tag{1.2}$$

for all $0<x<x_0$, then it is clear that the $x \to 0$ limit gives

$$\langle \ln R_N \rangle = \nu \ln N + \ln A - B\, N^{-\Delta_1} + \cdots. \tag{1.3}$$

We will be particularly interested in the leading Flory exponent ν and the first correction exponent Δ_1 in this

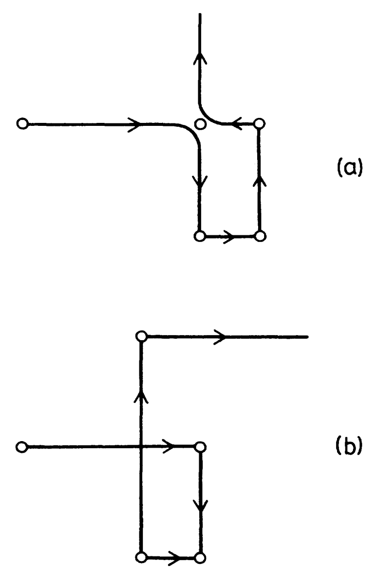

FIG. 1. An example of (a) a *random* Lévy flight which would be excluded by the node-avoiding constraint; and (b) one which would survive the node-avoiding constraint but would be excluded by the path-avoiding constraint.

equation. If, on the other hand, the leading correction is logarithmic, i.e., if

$$\langle R_N^x \rangle^{1/x} = AN^{\nu}(\ln N)^{\Theta} + \cdots , \quad (1.4)$$

then the zeroth moment will be of the form

$$\langle \ln R_N \rangle = \nu \ln[N(\ln N)^{\Theta/\nu}] + \ln A + \cdots , \quad (1.5)$$

and we will be interested in ν and Θ.

In this paper we study the simplest case of one dimension, treating the asymptotic properties of the PALF problem analytically and both the NALF and the PALF problems by Monte Carlo simulations for various N and μ on a rather extensive scale. These problems are interesting already in one spatial dimension for a number of reasons. For the NALF, we expect interesting crossovers among the classical Lévy flight, self-avoiding Lévy flight, and short-range self-avoiding walk regimes already in $d=1$ according to a mapping to a model of magnetic phase transitions obtained in Ref. 7. Also, some of the results of this reference have been disputed recently using Monte Carlo simulations.[8] In addition, simple real-space renormalization calculations[9] appear to suggest the two problems NALF and PALF to be in the same universality class (at least in two dimensions) and, as it turns out, the asymptotic properties of the PALF problem can be solved exactly in one dimension.

In Sec. II, we present the exact results for PALF in one dimension, and in Sec. III we summarize the expected behavior of the NALF on the basis of the mapping to a spin model[7] and discuss the relationship between the NALF and PALF that can be surmised from these and other arguments. One very surprising suggestion that arises from these discussions will be the possibility that the moments of the end-to-end distance of the NALF are larger than those of the PALF when they both exist and are finite. The Monte Carlo results are presented in Sec. IV and they evidently confirm this suggestion. The numerical results also show signs of very strong corrections to scaling and, when they are taken into account, the data can be regarded as consistent with the results implied by the spin mapping. A brief summary is given in Sec. V. The details of the calculations of Secs. II and IV are given in Appendixes A and B, respectively.

II. PATH-AVOIDING LÉVY FLIGHT IN ONE DIMENSION

In one dimension, path-avoiding Lévy flights (PALF) are relatively simple because each segment of the walk is extended in the same direction with no overlaps. This contrasts with the node-avoiding Lévy flights (NALF) which, even in one dimension, have much backtracking and many overlaps in general. The PALF in one dimension may also be quite different from those in higher dimensions because in one dimension their end-to-end distance is the simple sum of the lengths of each step. This makes it possible to evaluate the mean "zeroth moment" of the end-to-end distance, $\langle \ln R_N \rangle$, explicitly even in the long-range regime and to obtain the Flory exponent ν and the leading correction exponents.

Let us consider, without loss of generality, a continuous distribution of the step-size l where the probability for the step size l is given by $P(l)=\mu/l^{\mu+1}$ with a lower cutoff at $l=1$. Further denote the step size sequence of an N-step PALF in one dimension by $\{l_i\}$ with $i=1,2,3,\ldots,N$. Then we have

$$\begin{aligned}\langle \ln R_N \rangle &= \langle \ln(l_1+l_2+\cdots+l_N)\rangle \\ &= \int_0^\infty \frac{1}{t}\{\exp(-t) \\ &\qquad - \langle \exp[-t(l_1+l_2+\cdots+l_N)]\rangle\}dt \\ &= \int_0^\infty \frac{1}{t}\{\exp(-t)-[I(t,\mu)]^N\}dt, \end{aligned} \quad (2.1)$$

where we have defined

$$I(t,\mu) \equiv \mu \int_1^\infty \frac{\exp(-tl)}{l^{\mu+1}}dl. \quad (2.2)$$

The integral $I(t,\mu)$ is positive and finite for all $\mu>0$, and, furthermore, it is exponentially small for large t. Thus, all large contributions to $\langle \ln R_N \rangle$ for large N come from the region of small t. This means that we need to consider the behavior of I for small t only.

Furthermore, we note that, if $\mu>1$, then the step-size average $\langle l \rangle$ is finite and the law of large numbers applies (we are grateful to J. W. Halley and B. D. Hughes for this observation). Thus in this case we have

$$\langle R_N \rangle / N \to \text{const} \quad (\mu > 1), \quad (2.3)$$

as $N\to\infty$ and, trivially, $\nu=1$. This calls for the examination of the integral $I(t,\mu)$ for $0<\mu<1$ and $\mu>1$ separately to obtain the leading and correction behavior for the former region and the correction behavior for the latter region. We need also to examine $I(t,\mu)$ carefully to obtain the behavior at the boundary of the two regions: $\mu=1$. The details of such an analysis are given in Appendix A and here we present the results.

(i) $0<\mu<1$. First, for the region $0<\mu<1$, we find that

$$\langle \ln R_N \rangle = \frac{1}{\mu}\ln N + A_1 - B_1 N^{-\Delta_1} + \cdots , \quad (2.4)$$

which identifies the Flory exponent $\nu=1/\mu$ and the leading correction exponent Δ_1 is the smaller of 1 and $(1/\mu-1)$. The leading exponent is identical to that of the *random* Lévy flights. The two leading correction terms cross at $\mu=\frac{1}{2}$, where the correction exponent is $\Delta_1=1$ (*not* logarithmic).

(ii) $\mu=1$. For the boundary, $\mu=1$, between the long- and short-range regions of the problem, we obtain

$$\langle \ln R_N \rangle = \ln(N \ln N) + A_2 + \cdots . \quad (2.5)$$

Therefore, at $\mu=1$, the leading correction to $\nu=1$ is a multiplicative, logarithmic one with the power of logarithm being 1. The logarithmic nature of the correction could have been anticipated from the fact that the correction exponent Δ_1 in the region (i) goes to zero as $\mu\to1-$.

(iii) $\mu>1$. As we have seen from the law of large numbers, the leading behavior of PALF for $\mu>1$ is short ranged, or put another way, $\nu=1$. The correction behavior, as in

$$\langle \ln R_N \rangle = \ln N + A_3 - B_3 N^{-\Delta_1} + \cdots , \quad (2.6)$$

depends on μ, and we find that $\Delta_1=\mu-1$ if $1<\mu<2$ while $\Delta_1=1$ if $\mu>2$. At the boundary $\mu=2$, the two correction terms are both like N^{-1} and they cross. This is similar to what happens at $\mu=\frac{1}{2}$ discussed for region (i). Although the change in the leading correction behavior is not insignificant, the difference is *not* comparable either to what happens to this system at $\mu=1$ or to a random (non-excluded volume) Lévy flights at $\mu=2$.

III. NODE-AVOIDING LÉVY FLIGHTS IN ONE DIMENSION

First, we review the implications of a mapping of NALF to a spin model given in Ref. 7. In the general d-dimensional case, this mapping considers an n-vector spin model with spins s_i of length $\sqrt{n}$ situated at each site of a d-dimensional hypercubic lattice with the ferromagnetic Hamiltonian,

$$-\beta H = \sum_{(i,j)} K_{ij} s_i s_j , \quad (3.1)$$

between pairs of spins s_i and s_j which are displaced from each other in one of the orthogonal directions of the lattice. The coupling K_{ij} is of a long-range character, falling off with a power law

$$K_{ij} = K / r_{ij}^{1+\mu} , \quad (3.2)$$

where r_{ij} is the distance between the spins. Then, the correspondence is that the two-spin correlation functions can be represented in the limit of $n \to 0$ as the weighted sum (or the *generating function*) of the number of NALF's between sites i and j in the high-temperature expansion.[7] The spin model is then directly relevant to the NALF problem on hypercubic lattices with the step-size distribution given by Eq. (1.1), and in particular, in $d=1$, to our present problem.

If we allow *all* pairs of spins to interact with the coupling

$$K_{ij} = K / r_{ij}^{d+\mu} , \quad (3.3)$$

then we still get the correspondence to NALF described by Eq. (1.1) which, however, allows steps to be in directions other than the coordinate directions. We assert that the latter model should be in the same universality class (i.e., described by the same critical exponents in the asymptotic limit of large number of steps N) as the original NALF with steps in orthogonal directions only, since both models are asymptotically d dimensional. The spin model defined by Eq. (3.3) can be transformed using standard techniques to a continuum, field-theoretic model of Fisher *et al.*[10] and of Sak,[11] and their renormalization group solutions should then be applicable to our NALF problem.

The results of the spin model imply that the upper marginal dimension d_c is 2μ up to $\mu=2$, and that for any $0<d<4$ there are three distinct regimes: (a) classical Lévy flight (CLF) ($\mu<d/2$), (b) node-avoiding Lévy flight (NALF) ($d/2<\mu<\mu_0$), and (c) self-avoiding walk (SAW) ($\mu>\mu_0$) regimes. Fisher *et al.*[10] gave expansions for the exponents γ (for susceptibility), ν (for scaling the correlation function), and η in powers of $\epsilon=2\mu-d$, to the second order in ϵ. The result for η was further conjectured to be valid to all orders in ϵ; i.e., $\eta=2-\mu$.

In addition, Sak[11] later used an expansion about the point $\mu=2$, $d=4$ to argue that the boundary between regions (b) NALF and (c) SAW occurs at

$$\mu_0 = 2 - \eta_{\mathrm{SAW}} , \quad (3.4)$$

where η_{SAW} is the value of η in the corresponding short-ranged model. Although this result was obtained near the point where $\epsilon=0$ and $\eta_{\mathrm{SAW}}=0$, the relationship (3.4) was conjectured to be valid to all orders in ϵ and thus even far from the point about which the expansion was made. While no one has raised objections to the boundary between (a) and (b), this second boundary at μ_0 has been the object of a controversy.[8]

Thus, the main spin model predictions[7,10,11] of NALF for $d=1$ are as follows.

(a) $\mu<\frac{1}{2}$: $\nu=1/\mu$, $\gamma=1$, $\eta=2-\mu$;

(a′) $\mu=\frac{1}{2}$: logarithmic correction with $\Theta=\frac{1}{2}$ [cf. Eq. (1.4)];

(b) $\frac{1}{2}<\mu<1$: $\nu=\dfrac{1}{\mu}[1+(1/4\mu)\epsilon+O(\epsilon^2)]$, (conjecture) $\gamma=\mu\nu$, $\eta=2-\mu$.

Also, $\Delta_1=\dfrac{\epsilon}{\mu}[1+(1/4\mu)\epsilon+O(\epsilon^2)]$.

(b′) $\mu=1$: A logarithmic correction similar to (a′) is expected if Sak's calculations[11] can be extended away from $d=4$ since they suggest for $\mu<1$ a leading correction exponent Δ_2 which goes to zero as $\mu \to 1-$.

(c) $\mu>1$: $\nu=1$, $\gamma=1$, $\eta=1$.

By comparing these predictions with the PALF results, we note that, while both in the classical region (a) and the short-range region (c) the leading exponents are the same, the NALF predictions for ν and γ are singular at the boundaries $\mu=\frac{1}{2}$ (a′) and $\mu=1$ (b′) but those for PALF are singular only at the latter boundary. In particular, for the NALF,

$$\left[\frac{d\nu}{d\mu} \right]_{\mu=0.5+} = -2 \quad \text{(NALF)}, \quad (3.5)$$

while for the PALF,

$$\left[\frac{d\nu}{d\mu} \right]_{\mu=0.5+} = -4 \quad \text{(PALF)}. \quad (3.6)$$

This implies that ν is larger for the NALF than for the PALF as μ approaches $\frac{1}{2}$ from above, and therefore $\langle \ln R_N \rangle$ is larger for the NALF there. Since PALF is fully extended while NALF contains overlaps, this prediction was quite unexpected. Yet, as will be seen below, this is indeed the correct sense of inequality in all regions of μ according to our own Monte Carlo simulations.

Let us consider the following argument which attempts to establish $\langle \ln R_N \rangle_{\text{PALF}} \geq \langle \ln R_N \rangle_{\text{NALF}}$ (contrary to the above observations). For large N, the distribution of the step size for any given walk of N steps should be essentially as in Eq. (1.1) with probability one. For PALF, the $\ln R_N$ for the given walk is just the sum of these step sizes. For NALF, $\ln R_N$ is smaller because some of these same step sizes must occur with a minus sign. The only way to reconcile this type of argument and the opposite prediction from Eqs. (3.5) and (3.6) (and also from the numerical results) appears to be that the ensemble of N-step NALF is an extremely unusual one. It must be that the only surviving NALF's are those with the distribution of step sizes within each walk not like that of Eq. (1.1), but those that would occur with zero probability if steps were generated with probability as in Eq. (1.1) with no constraints.

A physical picture might be as follows: since PALF grows along one direction, every walk contains many short steps as expected from Eq. (1.1). On the other hand, as NALF chooses the direction of each step randomly, allowing back-tracking, many of the walks with the natural distribution as in Eq. (1.1) are rejected from the ensemble and the surviving ones are those that contain sufficiently long steps to avoid the multiple occupancy of the same sites. This leads to larger moments of the end-to-end distance when they are finite. We take note that, if this is the case, then the surviving ensemble of the NALF's generated algorithmically in this way do *not* necessarily follow the Lévy distribution Eq. (1.1). Thus we may expect the NALF problem to be a rather difficult mathematical one, even in *one* dimension.

IV. MONTE CARLO SIMULATIONS

We have studied the one-dimensional NALF in the region of $\frac{1}{2} \leq \mu \leq 2$ by Monte Carlo simulations using an extension of a standard enrichment method.[12] In addition, we have also performed some simulations of the PALF in one dimension in order to obtain the convergence behavior toward the asymptotic limits given in Sec. II. In contrast to Ref. 8, which evidently allowed step-size increment of one lattice constant, we use step sizes l corresponding to an integer power of a chosen size b (we use $b=2$ lattice constants) with the (initial) probability for $l=b^n$ given by

$$p(l)=\frac{a-1}{a}\sum_{n=0}^{\infty} a^{-n}\delta_{l,b^n}, \tag{4.1}$$

where the index μ of the Lévy flight is related to a and b by

$$a=b^{\mu}. \tag{4.2}$$

In practice, the step size must be cut off at some finite value, which may possibly cause some problems. This point is discussed in Appendix B together with other details of the simulations.

For the NALF, an attempt is made to generate each step with equal probability in each direction and with a step size chosen according to Eq. (4.1). If such a step would result in the violation of the node-avoiding constraint, then the current *stage* of the walk is discarded and a fresh attempt is made to generate the stage, as in the standard enrichment technique. Although we have also used simple sampling without enrichment, there is too much attrition for this problem for simple sampling to give useful statistics. For the PALF, we simply grow the walk in a fixed, chosen direction with the step size chosen according to (4.1). Thus for the latter problem, there is *no* attrition.

We have generated NALF's for several different values of μ between $\frac{1}{2}$ and 2, and calculated $\langle \ln R_N \rangle$ for N of up to 300. The number of walks used range from 6.1×10^3 to 1.15×10^5 for each μ. Our results on ν are expressed in terms of its *effective* value up to N steps, denoted by ν_N, defined as

$$\nu_N=\frac{2N\exp(\langle \ln R_N \rangle)}{\exp(\langle \ln R_1 \rangle)+\exp(\langle \ln R_N \rangle)+2\sum_{i=2}^{N-1}\exp(\langle \ln R_i \rangle)}-1. \tag{4.3}$$

It is easy to show that, given the power-law correction-to-scaling behavior of $\langle \ln R_N \rangle$ as in Eq. (1.3), the coefficient B in (1.3) is related to C in the asymptotic behavior of ν_N,

$$\nu_N=\nu+C\,N^{-\Delta_1}+\cdots, \tag{4.4}$$

by

$$C=\frac{B\Delta_1(\nu+1)}{\nu-\Delta_1+1}. \tag{4.5}$$

For the case of logarithmic correction (1.5), ν_N must behave as

$$\nu_N=\nu+\frac{\Theta}{\ln N}+\frac{\Theta}{(\nu+1)(\ln N)^2}+\cdots, \tag{4.6}$$

independent of the overall amplitude A in Eq. (1.4).

Our results shown in Fig. 2 are not very different from those of Ref. 8 up to the maximum number of steps used in that reference. However, beyond that maximum (mostly 50 steps; for some μ, 100 steps), all of our results begin to show apparent decreases in ν_N, indicating that corrections to scaling are indeed important in this problem. At the boundary of the classical Lévy flight and the NALF regions at $\mu=\frac{1}{2}$, our data can be fitted with a very sharp logarithmic correction in agreement with the theoretical prediction[7,10] discussed in Sec. III, although the power Θ of logarithm that best fits the data (indicated by a solid line in Fig. 2) is about 0.60 in contrast to the theoretical value of $\Theta=0.5$. Since the renormalization calculations at the classical boundary $\epsilon\equiv 2\mu-1=0$ must be exact, this discrepancy could be due to the truncation of higher-order terms in Eq. (4.6) and the mixing from additional higher-order corrections. In any event, Monte Carlo estimations of logarithmic corrections are difficult and inaccurate in the best of the cir-

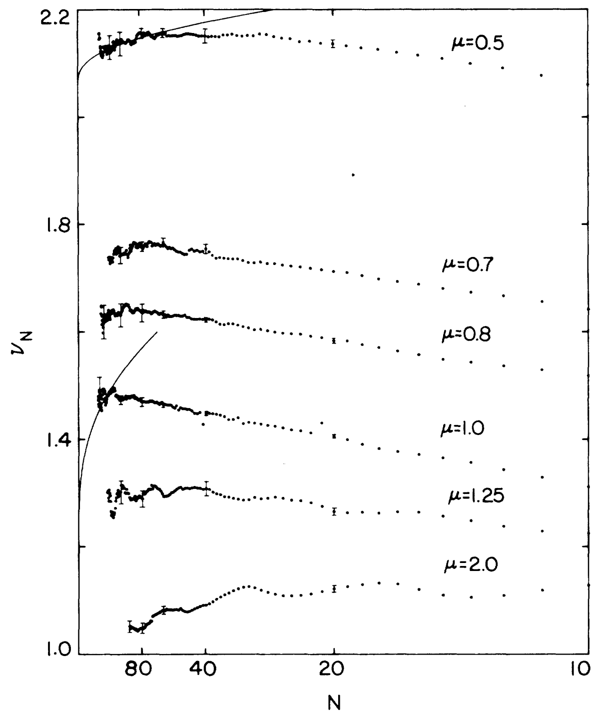

FIG. 2. The effective exponent ν_N of Eq. (4.3) is shown against $1/N$ from the Monte Carlo simulation of node-avoiding Lévy flights. Solid lines indicate numerical fits using the corrections of the form (4.6) with the best-fit values of Θ as discussed in the text, and the error bars were obtained from 6–10 batches of data for each μ.

cumstances,[13] and our data should be taken to corroborate the theoretical predictions rather than to give evidence against them.

The other boundary between the NALF and SAW regions occurs at $\mu=1$ if Sak's results[11] are correct even in one dimension. If this is the case, then there will also be a logarithmic correction at $\mu=1$, although the power Θ of logarithm there has not been predicted. Thus, throughout the region $\frac{1}{2}<\mu<1$, corrections to scaling are expected to be rather large for the NALF. Our Monte Carlo data for $\mu=0.7$ and 0.8 are found to be consistent with such strong corrections. The solid line indicated in Fig. 2 for $\mu=1$ shows a possible logarithmic fit with the numerically fitted exponent of about $\Theta=2.5$.

While this value is large compared with other known powers of logarithm in critical phenomena, a relatively large value of Θ is expected in our case. This is because, if ν for the NALF is larger than $1/\mu$ in this region, then it must come down onto the SAW value of 1 at $\mu=1$, and the steep descent should provide a large discontinuity in the derivative of ν with respect to μ at this point. Such discontinuities are known to produce logarithmic corrections of the type being discussed.

For $\mu>1$, the exponents are expected to assume the usual SAW values. In one dimension, this means $\nu=\gamma=\eta=1$ with no nonanalytic corrections to scaling in the sense of renormalization group.[14] This implies that there may be a correction term which is analytic in $1/N$. Our data at $\mu=2$ appears to have only such analytic correction, suggesting that the boundary between NALF and SAW must indeed satisfy $\mu_0<2$.

Thus, so far all our numerical data can be considered consistent with the theoretical predictions as long as proper accounts of the rather strong corrections to scaling are taken. While the data are not conclusive in establishing, e.g., $\mu_0=1$, or the value of the logarithmic power Θ at either boundary, we have not found any evidence that conclusively violates the theoretical predictions. This contrasts sharply with the interpretation given by Grassberger in a recent Monte Carlo study,[10] which is in fact similar to ours but on a much smaller scale.

Specifically, it was suggested that the short-range SAW behavior begins at $\mu_0=2$ and that the critical exponent ν agrees better with a Flory-type approximation

$$\nu_F=\frac{3}{\mu+1}, \tag{4.7}$$

all the way up to $\mu=2$. However, we believe that this is probably the result of overestimating the value of ν by the use of a simple linear extrapolation of their Monte Carlo data without properly considering the corrections to scaling. Although this procedure did appear to give the correct value $\nu=2$ at $\mu=\frac{1}{2}$, we believe this was accidental: the set of data points used for this linear fit ap-

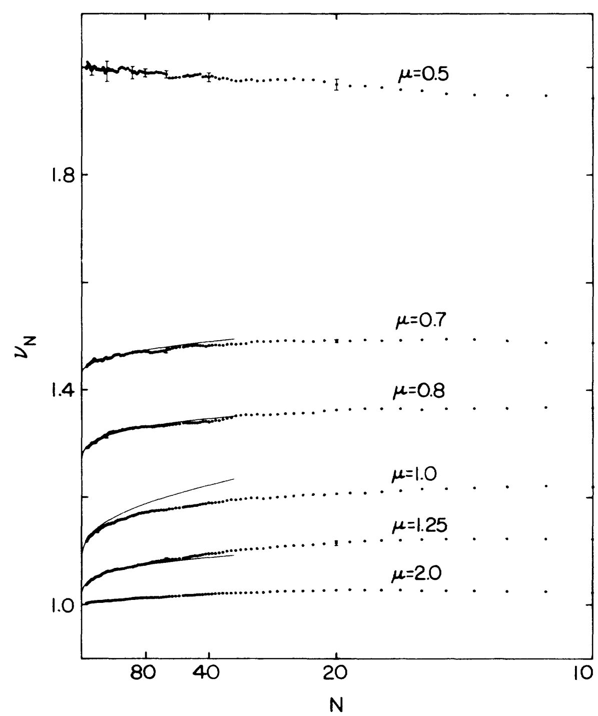

FIG. 3. The effective exponent ν_N of Eq. (4.3) is shown against $1/N$ from the Monte Carlo simulation of path-avoiding Lévy flights. Solid lines indicate numerical fits using the corrections of the form (4.4) with the appropriate values of Δ_1 from Sec. II for $\mu\neq1$, and the error bars were obtained from six batches of data set for each μ. The fit at $\mu=1$ is of the form (4.6) with the best-fit value of Θ which is 0.85.

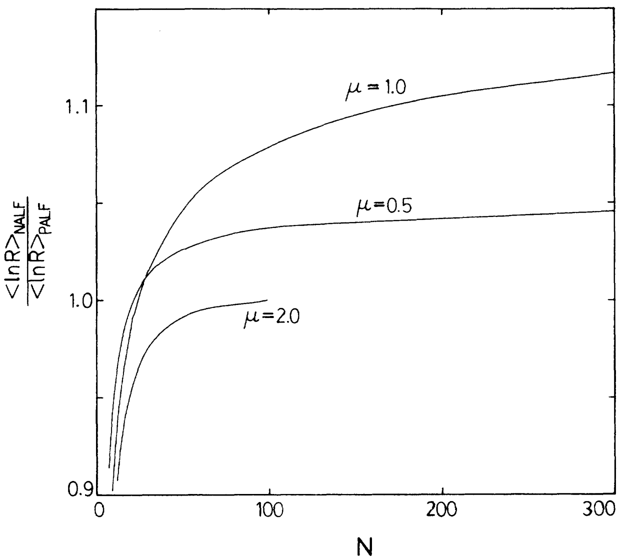

FIG. 4. The ratio of $\langle \ln R_N \rangle$ for the node-avoiding Lévy flight to that for the path-avoiding Lévy flight is shown against N for selected values of μ.

pear to be part of an oscillatory trend (or fluctuation), and the correct fit must be performed with the predicted logarithmic correction on data using larger number of steps.

Another interesting result arises from the direct comparisons of the numerical results for the NALF with those of the PALF. Figure 3 shows ν_N for the PALF as defined by Eq. (4.1) for various μ. The solid lines indicated are the fits obtained using the leading and first correction exponents obtained analytically in Sec. II except at $\mu=1$. At $\mu=1$, where a logarithmic correction with $\Theta=1$ is expected, the best fitted value of Θ is about 0.85, and this latter value is used in the figure. For the other values of μ, the calculated exponents are used with the best fit values of the amplitudes. While these are in principle also calculable, in most cases mixing of correction terms will mask the correct values in any event.

Comparing Fig. 2 (NALF) with Fig. 3 (PALF), we see a confirmation of the surprising prediction that the moments of the end-to-end distance of the NALF are indeed larger than those of the PALF. This is reflected in the figures in the way that ν_N of NALF is for all μ greater than that of PALF in the range of N and μ used in our simulations. More direct comparisons of the size can be made by looking at $\langle \ln R_N \rangle$ itself. The ratio of this quantity for the two types of Lévy flights is plotted in Fig. 4 and it again confirms that NALF is larger than PALF. While we do believe that the leading exponent ν is identical for $\mu > 1$, it may well be that the amplitudes as defined in Eq. (1.3) do not become identical until $\mu \to \infty$.

V. SUMMARY

Considering these results, it is clear that the PALF and NALF are in different universality classes at least in one dimension, in disagreement with a recent real-space renormalization study.[9] Our numerical results for NALF are consistent with theoretical expectations in all respects as long as very large corrections to the asymptotic scaling are allowed for, in contradiction to the interpretations given by an earlier numerical work of Grassberger.[8] In particular, it seems very likely that the short-range SAW region for the NALF starts at a value of μ strictly less than 2 [Sak's prediction would be 1 (Ref. 11)].

It is also clear that NALF is a rather complex object whose properties are difficult to pin down simply by numerical means. In particular, the moments of its end-to-end distance distribution, when they exist, appear to be larger than those of the PALF. The latter result, as surprising as it seems, is not only numerically obtained, but also predicted by analytical results.

While the subject of this paper is confined to the statistical properties of the NALF and PALF in one dimension, some of the possible applications of these ideas to physics[4,5] occur in higher physical dimensions. Thus it is of much interest to investigate if these conclusions apply also to higher dimensions, particularly on the relationship between the two types of excluded-volume Lévy flights. This is left as a promising future work.

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APPENDIX A: SOLUTION OF PALF IN ONE DIMENSION

In this appendix, we present the calculation of the Flory exponent ν and leading correction behavior for the end-to-end distance of the path-avoiding Lévy flights in one dimension. We start from the expression (2.1) for the mean zeroth moment $\langle \ln R_N \rangle$ and note that, after some manipulations involving a change of variables and integration by parts, we arrive at

$$I=(-\mu t^{\mu})\left[\frac{1}{\mu}\int_{\epsilon}^{\infty}\frac{e^{-y}}{y^{\mu}}dy+\int_{\epsilon}^{t}\frac{e^{-y}}{y^{\mu+1}}dy-\frac{e^{-\epsilon}}{\mu\epsilon^{\mu}}\right], \tag{A1}$$

independent of $\epsilon(>0)$. Let us now discuss each case separately:

(i) $0<\mu<1$. In this case, taking the limit of $\epsilon \to 0$, the first integral reduces to $\Gamma(1-\mu)$, and we have

$$I=-t^{\mu}\Gamma(1-\mu)-(\mu t^{\mu})\lim_{\epsilon\to 0}\left[\int_{\epsilon}^{t}\frac{e^{-y}}{y^{\mu+1}}dy-\frac{e^{-\epsilon}}{\mu\epsilon^{\mu}}\right]. \tag{A2}$$

The sum of the last two terms can be expanded in powers of t for small t, and we finally have

$$I = 1 - t^{\mu}\Gamma(1-\mu) + \left[\frac{\mu}{1-\mu}\right]t - \left[\frac{\mu}{4-2\mu}\right]t^2 + O(t^3). \tag{A3}$$

Reexpressing this result in a form suitable to substitute into Eq. (2.1), we write

$$[I(t,\mu)]^N = I_0(1 + a_1 t^{b_1} + a_2 t^{b_2} + \cdots), \tag{A4}$$

where $I_0 \equiv exp[-Nt^{\mu}\Gamma(1-\mu)]$ and $\{b_i\}$ is arranged in the ascending order ($b_1 < b_2 < \cdots$). Each of the correction terms in Eq. (A4) then yields an integral of the form

$$\int_0^{\infty} \frac{a_i I_0}{t^{1-b_i}} dt \propto N^{c_i - b_i/\mu}, \tag{A5}$$

for large N where c_i is the asymptotic power of N contained in a_i. The largest correction then derives from the largest exponent $c_i - b_i/\mu$. This is given by $\max(1-1/\mu, -1)$. Substituting Eq. (A4) into (2.1) with Eq. (A5) in mind and carefully dealing with the cancellations of the diverging contributions in each of the two terms of Eq. (2.1), we finally arrive at the expression

$$\langle \ln R_N \rangle = \frac{1}{\mu}\ln N + A_1 - B_1 N^{-\Delta_1} + \cdots, \tag{A6}$$

where A_1 and B_1 are independent of N, and in particular,

$$A_1 = \frac{1}{\mu}\ln\Gamma(1-\mu) + \left[\frac{1}{\mu} - 1\right]\gamma, \tag{A7}$$

where γ is the Euler's constant. From this, we identify that the leading exponent which gives the analog of the Flory exponent for this case is

$$\nu = 1/\mu, \tag{A8}$$

and the leading correction exponent is

$$\Delta_1 = \begin{cases} 1/\mu - 1\ , & \frac{1}{2} < \mu < 1 \\ 1\ , & 0 < \mu < \frac{1}{2}\ . \end{cases} \tag{A9}$$

(ii) $\mu = 1$. At $\mu = 1$, we have for small t,

$$I = t\int_t^{\infty} \frac{e^{-y}}{y^2} dy = 1 + \gamma t + t\ln t + O(t^2). \tag{A10}$$

Thus,

$$\int_{\epsilon}^{\infty} \frac{I(t,1)^N}{t} dt = \int_{\epsilon}^{1} t^{Nt-1}[1 + O(t)]dt + D_0 + \cdots, \tag{A11}$$

where ϵ is asymptotically small as before and D_0 is at most constant for large N. The neglected terms are even smaller. The behavior of the integral on the right can be estimated by considering

$$I' \equiv \int_{\epsilon}^{1} t^{Nt-1} dt = \int_{N\epsilon}^{N} N^{-x} x^{x-1} dx, \tag{A12}$$

whose range of integration can be split in three parts to $(N\epsilon, \delta)$, $(\delta, 1)$, and $(1, N)$. The value of δ is chosen to be a constant such that $N\epsilon \ll \delta \ll 1$. The contributions from the second and third ranges are then at most constant for large N, and the contribution from the first range $(N\epsilon, \delta)$ for large N is approximately $-\ln(\epsilon N \ln N) - \gamma$. This gives

$$I' = -\ln(\epsilon N \ln N) - \gamma + \cdots, \tag{A13}$$

and we conclude that

$$\langle \ln R_N \rangle = \ln(N \ln N) + \cdots. \tag{A14}$$

(iii) $1 < \mu < 2$. Next we consider the case $1 < \mu < 2$. In this case, we can write

$$I(t,\mu) = e^{-t} - t\int_1^{\infty} \frac{e^{-tx}}{x^{\mu}} dx = e^{-t} - \frac{t}{\mu - 1} I(t, \mu - 1), \tag{A15}$$

and thus partly reduce the problem to the first case. We then obtain

$$I = \exp\left[-\frac{\mu}{\mu-1}t\right]\left[1 + \frac{\Gamma(2-\mu)}{\mu-1} t^{\mu} + O(t^2)\right]. \tag{A16}$$

Substituting (A16) into (2.1) then finally yields

$$\langle \ln R_N \rangle = \ln N + A_2 - B_2 N^{1-\mu} + \cdots, \tag{A17}$$

where the constant term is

$$A_2 = \ln\left[\frac{\mu}{\mu-1}\right]. \tag{A18}$$

This identifies the Flory exponent of

$$\nu = 1, \tag{A19}$$

as expected from the law of large numbers, and the leading correction exponent is

$$\Delta_1 = \mu - 1. \tag{A20}$$

(iv) $\mu \geq 2$. For $\mu > 2$, we can apply the transformation used in (A15) successively until $I(t,\mu)$ is reduced to a form involving $I(t,\lambda)$ where $0 < \lambda \equiv \mu - k < 1$ with integer $k \geq 2$. Thus we obtain

$$I(t,\mu) = 1 - \left[\frac{\mu}{\mu-1}\right]t + \frac{\mu}{2(\mu-2)}t^2 + O(t^3) + O(t^{\mu}), \tag{A21}$$

which can be rewritten as

$$I = \exp\left[-\frac{\mu}{\mu-1}t\right]\left[1 + \frac{\mu}{2(\mu-1)^2(\mu-2)}t^2 + \cdots\right]. \tag{A22}$$

Substituting (A22) into (2.1), we obtain

$$\langle \ln R_N \rangle = \ln N + A_4 - B_4 N^{-1} + \cdots, \tag{A23}$$

identifying $\nu = 1$ as expected and the correction exponent

$$\Delta_1 = 1. \tag{A24}$$

As $\mu \rightarrow 2$ limit is taken from either above or below, the two leading correction terms N^{-1} and $N^{1-\mu}$ cross, and even though each is associated with a diverging amplitude, these apparent divergences actually cancel each other and leave one with a simple N^{-1} correction there.

APPENDIX B: MONTE CARLO METHOD

In this appendix, we discuss some details of the numerical methods whose results are discussed in Sec. IV. We also present the summary of the simulation parameters in Tables I and II. The main simulation method is a simple extension of the standard enrichment technique[12] whose parameters are also listed in Table I. We will not discuss the details of this method as they are well known.

The main point we wish to discuss is the possible effects of cutting off the maximum step size $l_{\max}$ in the ideal distribution of (4.1). In principle it may have a significant effect on the statistics as the special properties of Lévy flights are caused by rare but very long steps. Since we use Fortran integer arithmetic on various computers, the maximum size of an integer word on a particular machine limits the largest possible step we can allow. While we could use floating-point arithmetic instead with some loss of efficiency, eventually a similar problem would occur there as well and, in addition, we would then be faced with a round-off error problem in applying the node-avoiding constraints. This problem is clearly more significant for smaller μ as the probability of long steps is larger there. On the other hand, for larger μ, even moderately long steps occur with very small probabilities and the generation and comparison of the floating point numbers to distinguish one step size from another becomes a limiting factor. Thus, our simulation faces possible difficulty at both ends of μ.

In practice, we used machines with 32-bit Fortran integers (Masscomp, Digital Equipment Corporation VAX 11/750 and PDP 11/44) and a machine with 48-bit Fortran integers (Control Data Corporation Cyber-205) for the Monte Carlo generation of the node-avoiding Lévy flights. For the 32-bit case, we used a linear congruential random number generator (with an offset of one each time) while for the Cyber-205 computer, we simply used the built-in generator which is a 48-bit linear-congruential one. In the former, the maximum step size allowed was 2^{30} and in the latter, 2^{46}. We generally truncated all steps which would have step sizes greater than these according to (4.1) down to the maximum allowed size.

TABLE II. Summary of data used for path-avoiding Lévy flights.

μ	Number of steps	Number of walks
2.00	1000	50 000
1.25	1000	100 000
1.00	1000	110 000
0.80	1000	300 000
0.70	1000	300 000
0.50	1000	300 000

Typically, for the 32-bit case, the accuracy limitation of the single-precision floating-point numbers is the primary limiting factor for $\mu > 0.75$ or so, while the maximum word size is the limiting factor for $\mu < 0.75$ or so. In the former case, the step-size is effectively cut off much earlier than the theoretically available maximum; in either case, we are cutting off the step-size distribution at a finite value. For the 48-bit case, the border line value of μ for the changeover in the primary cause of this cutoff moves up to about 1.1.

To test the effect of truncating step sizes, NALF's were generated at $\mu = 0.5$ on Cyber 205 both (a) in the usual way as described above using 48-bit integers for the step sizes and single-precision floating-point arithmetic for the probabilities; and also (b) using double-precision floating-point arithmetic for the step sizes and single-precision floating-point arithmetic for the probabilities. The first method (a) gave results that are already quite comparable with those from the usual method on 32-bit machines. The second method (b) was easily capable of keeping track of steps of sizes 2^{47} that were truncated in method (a). The two methods gave raw data for $\langle \ln R \rangle$ for up to 300 steps that were identical to at least three digits (and generally much more). Therefore, we conclude that, even though truncated steps are individually extremely large, the truncation does not affect the overall results as far as the zeroth moment of the end-to-end distance.

TABLE I. The parameters used in the Monte Carlo simulations of the node-avoiding Lévy flights. The number of walks generated is indicated at the first and last steps, and the number of stages, length of each stage, and the number of allowed trials per stage refer to the parameters of a standard enrichment technique (Ref. 12).

μ	Number of stage	Length of stage	Number of trials per stage	Number of walks First step	Number of walks Last step
2.00	10	10	449	10 275 365	8 526
1.25	20	10	113	5 134 042	6 096
1.00	30	10	40	57 344 560	11 261
0.80	30	10	16	11 100 384	18 275
0.70	30	10	10	5 085 360	23 209
0.50	30	10	4	6 267 932	115 057

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