Thermal properties of directed polymers in a random medium

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We calculate, by performing a product of random matrices, the specific heat and its derivative for the problem of directed polymers in a random medium. Our results are consistent with the existence of a phase transition both in 3+1 and 2+1 dimensions and with the absence of a phase transition in 1+1 dimensions. Our finite-size scaling analysis leads to $\alpha = -0.1 \pm 0.1$ in 3+1 and $\alpha = -1.0 \pm 0.2$ in 2+1 dimensions.

I. INTRODUCTION

The problem of directed polymers in a random medium has recently motivated a lot of theoretical work,[1-10] both on the numerical and on the analytic sides. It is one of the simplest problems in the theory of disordered systems which undergoes a phase transition from a high-temperature situation where the landscape in phase space is smooth to a low-temperature phase where a multivalley landscape emerges. It is also at the confluence of several problems in statistical mechanics: spin glasses,[7] growth problems,[11-13] pinning of interfaces by random impurities,[5] and localization.[14]

Most of the work done recently on the problem of directed polymers can be classified in three categories.

(i) The attempts to measure (for the problem of directed polymers or for growth models) the exponents which characterize the low-temperature phase (strong disorder regime). Except for the dimension 1+1 where these exponents are known exactly,[5] their precise dependence[15-17] on dimension is still not known and the question of the existence of a finite upper critical dimension above which the exponents take their mean-field value is still controversial.

(ii) The proof of the existence of both a high- and a low-temperature phase in high enough dimension. For $d+1 > 2+1$, one can show[6,18] that above a certain temperature $T_2(d)$, the annealed and the quenched free energy are equal and obtain bounds on the transition temperature.[18] In dimension $d+1 \le 2+1$, this approach does not predict anything. One can then wonder whether the high-temperature phase is always absent for $d+1 \le 2+1$ or whether in some situations one could observe a transition between a weak and a strong disorder phase.

(iii) Mean-field calculations[7] which predict a low-temperature phase similar to the spin-glass phase of the Sherrington-Kirkpatrick model,[19] which according to a recent $1/d$ expansion,[20] seems to persist in high enough dimension.

Because the low-temperature properties have mostly been studied through some growth models[15,16] where there is not always a clear parameter to vary, which would play the role of the temperature, in the polymer problem, much less is known on the phase transition between the high- and the low-temperature phases. The goal of this paper is to present a numerical study of this phase transition based on transfer matrix calculations. We will see that in dimension 3+1, the numerical results are consistent with a singularity in the specific heat with an exponent $\alpha \simeq -0.10 \pm 0.10$. In dimension 2+1, our results indicate also the existence of a phase transition which is of same nature as the transition recently observed by Yan, Kessler, and Sander[21] in a growth model.

This paper is organized as follows. In Sec. II, we define the model and we recall some of its known properties. In Sec. III, we present our results for the specific heat of the directed polymer constrained to a strip geometry $n^d \times \infty$ of width n. We will see that, in addition to the expected transition in dimension $d+1=3+1$, there is some evidence for a transition in dimension $d+1=2+1$.

II. THE MODEL

We will consider the bond version of a model which was already studied in Sec. IX of Ref. 18. A directed polymer of length L in dimension $d+1$ is represented by a sequence of vectors $\{r(l); 0 \le l \le L\}$ where each vector $r(l)$ is a point of a regular d-dimensional hypercubic lattice $[r(l) \in \mathbb{Z}^d]$. By definition of the model, a polymer which is at point $r(l)$ at times l is only allowed to occupy, at time $l+1$, any of the $2d$ neighbors of $r(l)$ on the hypercubic lattice of dimension d.

Therefore the lattice consists of a sequence of d-dimensional lattices. Each point r of the lth hypercubic lattice is connected to all the points $r+e_i$ of the $(l+1)$th hypercubic lattice (where $e_i, 1 \le i \le 2d$, is one of the unit vectors in dimension d).

On each bond $(r(l), r(l)+e_i)$ between two consecutive hypercubic lattices, there is a random energy ε chosen according to a given distribution $\rho(\varepsilon)$. Here we will use only a Gaussian distribution

$$\rho(\varepsilon) = \frac{1}{\sqrt{2\pi}} \exp(-\varepsilon^2/2) . \tag{1}$$

However, all the calculations reported below could be repeated easily for other distributions.

Consider a directed walk of L steps which ends at a given point $\mathbf{r}$ of the Lth lattice. By definition, the energy E_W of such a walk $W = \{\mathbf{r}(l), 0 \leq l \leq L\}$ is given by the sum of the energies of the bonds visited by W:

$$E_W = \sum_{l=0}^{L-1} \varepsilon_l(\mathbf{r}(l), \mathbf{r}(l+1)) . \tag{2}$$

One can then define the partition function $Z_L(\mathbf{r})$ of all the walks W of L steps which end at $\mathbf{r}$ at the Lth step by

$$Z_L(\mathbf{r}) = \sum_W e^{-E_W/T} , \tag{3}$$

where in (3) the sum runs over all the $(2d)^L$ directed walks ending at $\mathbf{r}$ after L steps.

Clearly, one can write the following recursion for the $Z_L(\mathbf{r})$:

$$Z_{L+1}(\mathbf{r}) = \sum_{i=1}^{2d} \exp[-\varepsilon_L(\mathbf{r}+\mathbf{e}_i, \mathbf{r})/T] Z_L(\mathbf{r}+\mathbf{e}_i) . \tag{4}$$

So in principle the free energy of directed polymers of ar-bitrary length L can be obtained by iterating (4).

In Sec. III, we will perform our calculations on strips or bars [which are infinite in the L direction but finite in the transverse directions, i.e., such that $\mathbf{r}(l) = (x_1(l), x_2(l), \ldots, x_d(l))$ with $1 \leq x_i(l) \leq n$] with periodic conditions in the transverse directions. For these lattices, the calculation of Z_L by the recursion (4) is the same as the calculation of the largest Lyapunov exponent of a product of L random matrices[22,23] of size $n^d \times n^d$.

We will see that one can estimate the properties of the true system $(n = \infty)$ by looking at the n dependence of physical quantities when n increases. Before doing so, let us recall two important facts about the model studied here.

First, in dimension $d + 1 > 2 + 1$, the free energy is equal with probability 1 to the annealed free energy (in the thermodynamic limit) if the temperature T is high enough.

For the model studied here, one can calculate[18] exactly the ratio $\langle Z_L^2 \rangle / \langle Z_L \rangle^2$. Above a certain temperature $T_2(d)$, which is the solution of

$$\frac{\langle \exp(-\varepsilon/T) \rangle^2}{\langle \exp(-2\varepsilon/T) \rangle - \langle \exp(-\varepsilon/T) \rangle^2} = \frac{d}{2} \int_0^{2\pi} dq_1 \frac{1}{2\pi} \cdots \int_0^{2\pi} dq_d \frac{1}{2\pi} \frac{1}{d^2 - \left[\sum_{\mu=1}^d \cos q_\mu \right]^2} , \tag{5}$$

one can prove that $\langle Z_L^2 \rangle / \langle Z_L \rangle^2$ has a finite limit as $L \to \infty$. [The derivation of (5) is analogous to the derivation of (145) and (146) of Ref. 18: the only difference is that here we treat the bond problem instead of the site problem.] Following the same arguments as in Ref. 18, one can deduce that above the temperature $T_2(d)$ solution of (5), the free energy is given with probability 1 for large L by

$$\frac{\ln Z_L}{L} = \frac{\ln \langle Z_L \rangle}{L}$$

$$= \ln[2d \langle \exp(-\varepsilon/T) \rangle] \quad \text{for } T > T_2(d) . \tag{6}$$

One should notice that (6) might remain true for $T < T_2(d)$ but one cannot prove it by calculating $\langle Z_L^2 \rangle / \langle Z_L \rangle^2$.

For the Gaussian distribution (6) becomes

$$\frac{\ln Z_L}{L} = \ln 2d + \frac{1}{2T^2} = \ln 2d + \frac{\beta^2}{2} \quad \text{for } T > T_2(d) , \tag{7}$$

where as usual $\beta = 1/T$ and $T_2(d)$ are given by (5),

$$T_2(d) = \left[\ln \left[1 + \frac{2}{J_d} \right] \right]^{-1/2} , \tag{8}$$

where

$$J_d = \int_0^{2\pi} dq_1 \frac{1}{2\pi} \cdots \int_0^{2\pi} dq_d \frac{1}{2\pi} \left[d - \sum_{\mu=1}^d \cos q_\mu \right]^{-1} ,$$

$$J_d = 0.505\,46 \quad \text{for } d = 3 , \tag{9}$$

$$J_d = \infty \quad \text{for } d = 2 ,$$

so that

$$T_2(d) = 0.790 \quad \text{for } d = 3$$

$$T_2(d) = \infty \quad \text{for } d = 2 . \tag{10}$$

The second claim one can make about the model is that it should undergo a phase transition at a transition temperature T_c which satisifies the following bounds:[18]

$$T_0(d) \leq T_c \leq T_2(d) , \tag{11}$$

where $T_0(d)$ is the temperature where the expression [Eqs. (6) and (7)] of the free energy valid above $T_2(d)$ would give a zero entropy. For the Gaussian distribution (1), $T_0(d)$ is given by

$$T_0(d) = (2 \ln 2d)^{-1/2} ,$$

$$T_0(d) = 0.528 \quad \text{for } d = 3 , \tag{12}$$

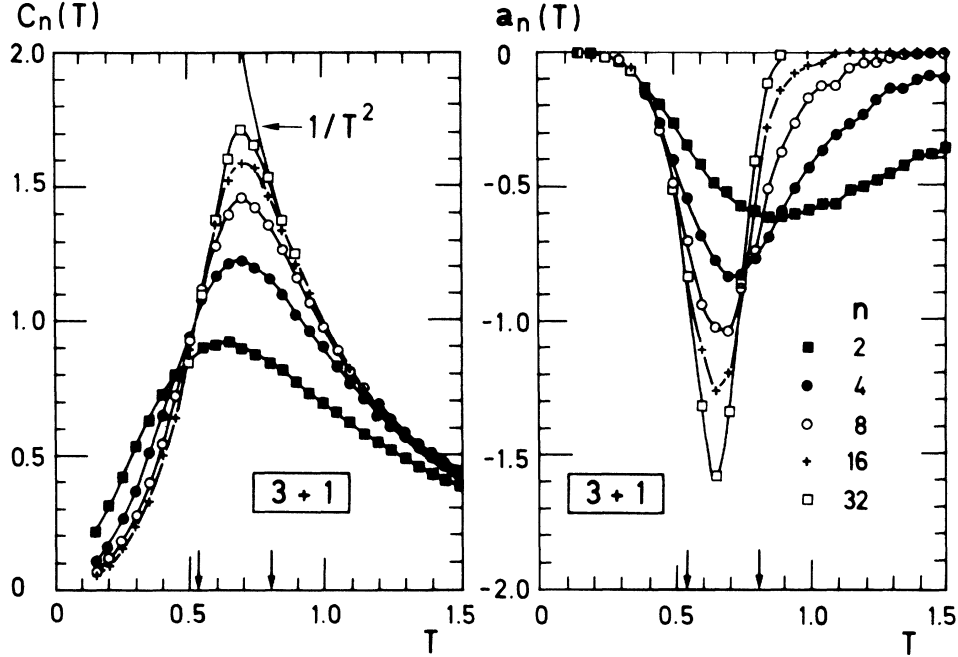$$T_0(d) = 0.601 \quad \text{for } d = 2 .$$

FIG. 1. (a) and (b) Dimension $3+1$: the specific heat $c_n(T)$ and the third derivative $a_n(T)$ defined by (15) and (16) vs the temperature T for different transverse sizes $n = 2,4,8,16,32$. The length L is 10^4. The arrows correspond to the bounds (13) for T_c.

Therefore, in dimension $3+1$, we know that the free energy is given by (7) above a transition temperature T_c which satisfies (10)–(12),

$$0.528 \le T_c \le 0.790 \quad \text{for } d = 3 \ . \tag{13}$$

In dimension $2+1$, because $T_2(d)$ is infinite one does not know if (7) is valid at any temperature. If one could be sure that (7) is true at high enough temperature, then one could conclude that there must be a transition temperature T_c which would satisfy

$$0.601 \le T_c \ . \tag{14}$$

Otherwise, if (7) is valid nowhere except at $T = \infty$, one cannot predict from the above arguments anything about the absence or the existence of a transition in dimension $2+1$.

III. THE SPECIFIC HEAT

As mentioned in Sec. II, the iteration (4) reduces to a product of random matrices of size $n^d \times n^d$ for systems which are finite (of linear size n) in the transverse directions. The aim of this section is to show the results obtained for several n in dimensions $d+1=3+1$, $2+1$, and $1+1$ and to extract from the n dependence of these results the behavior of the system for $n = \infty$.

In this section we shall discuss the results obtained for the specific heat $c_n(T)$ (which is a second derivative of the free energy) and for the third derivative $a_n(T)$ of $\ln Z_L / L$ with respect to $\beta = 1/T$:

$$c_n(T) = \frac{d}{dT} \left[T^2 \frac{d}{dT} \frac{\ln Z_L}{L} \right] , \tag{15}$$

$$a_n(T) = \frac{d^3}{d\beta^3} \frac{\ln Z_L}{L} \ . \tag{16}$$

If there is a high-temperature phase $T > T_c$, where the free energy is given by the annealed free energy (7), $c_n(T)$ and $a_n(T)$ should converge to

$$c_\infty(T) = 1/T^2 \quad \text{for } T > T_c \ , \tag{17}$$

$$a_\infty(T) = 0 \quad \text{for } T > T_c \ . \tag{18}$$

So one can identify the high-temperature phase by looking at the range of temperature where $a_n(T)$ vanishes as $n \to \infty$. Another interest in $a(T)$ is that, as it is a higher derivative with respect to T, the singularities should be easier to see in $a(T)$ than in $c(T)$.

Let us now describe how our numerical calculations are done. For each transverse size n and each temperature, we generate a single random sample of length $L = 10^4$; $L = 10^5$; $L = 10^6$ for systems of transverse size $n = 2,4,8,16,32$; $n = 4,8,16,32,64$; and $n = 4,8,16,32,64,128$ in dimensions $d+1=3+1$; $2+1$; and $1+1$, respectively. To eliminate boundary effects, we discarded the first $L/20$ iterations. Because, for finite n, the system is one dimensional, one knows that for large enough L, all the quantities (i.e., the free energy, and its derivatives) are self-averaging.

To compute the second and third derivatives (15) and (16) of the free energy, we iterate (4) and its derivatives as in Ref. 24, i.e., for the first derivative on iterates
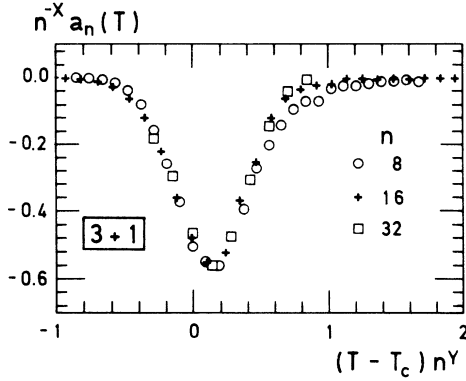
FIG. 2. Finite-size scaling analysis of the third derivative; $a_n(T) = n^x A[n^y(T - T_c)]$ with $x = 0.3$, $y = 0.3$, and $T_c = 0.6$ in dimension $3 + 1$.

$$\frac{dZ_{L+1}(\mathbf{r})}{d\beta} = \sum_{i=1}^{2d} \exp[-\beta \varepsilon_L(\mathbf{r} + \mathbf{e}_i, \mathbf{r})]$$

$$\times \left[\left| \frac{dZ_L}{d\beta}(\mathbf{r} + \mathbf{e}_i) \right. \right.$$

$$\left. \left. - \varepsilon_L(\mathbf{r} + \mathbf{e}_i, \mathbf{r}) Z_L(\mathbf{r} + \mathbf{e}_i) \right| \right] \quad (19)$$

together with (4).

In this way one does not lose accuracy when one takes derivatives. We noticed, however, that, for the higher derivatives, one needs to take a larger L in order to keep the relative error roughly constant (we usually estimated

the error bars by dividing the length L into ten subsystems of lengths $L/10$).

Let us now discuss our results for $c_n(T)$ and $a_n(T)$, which are shown in Figs. 1, 3, and 4. In dimension $3 + 1$ (see Fig. 1) it is clear that at high enough temperature $c_n(T) \to c_\infty(T)$ and $a_n(T) \to 0$. From the specific-heat curves [Fig. 1(a)] it is not easy to estimate the transition temperature above which $c_\infty(T) = 1/T^2$. However, the singularity is much more visible in $a_n(T)$ [see Fig. 1(b), one sees that $T_c \simeq 0.65$ which satisfies the bounds (13)].

From the results shown in Fig. 1(b), one can estimate, in addition to the critical temperature, the singularity of the specific heat by assuming a finite-size scaling form valid for large n and T close to T_c,

$$a_n(T) \sim n^x A[n^y(T - T_c)] \ . \quad (20)$$

Because our sizes n are limited and our data have statistical errors, there is a range of values of x, y, and T_c which are compatible with our data (see Fig. 2 for a possible choice $x = 0.3$, $y = 0.3$, and $T_c = 0.6$). By varing the parameters x, y, and T_c, it is, however, possible to estimate from the data of Fig. 1(b),

$$x = 0.30 \pm 0.05 \ ,$$

$$y = 0.30 \pm 0.05 \ , \quad (21)$$

$$T_c = 0.60 \pm 0.03 \ .$$

This would lead to the following singular behavior of the specific heat:

$$C_{\text{sing}}(T) \sim (T_c - T)^{-\alpha} \quad \text{for } T < T_c \quad (22)$$

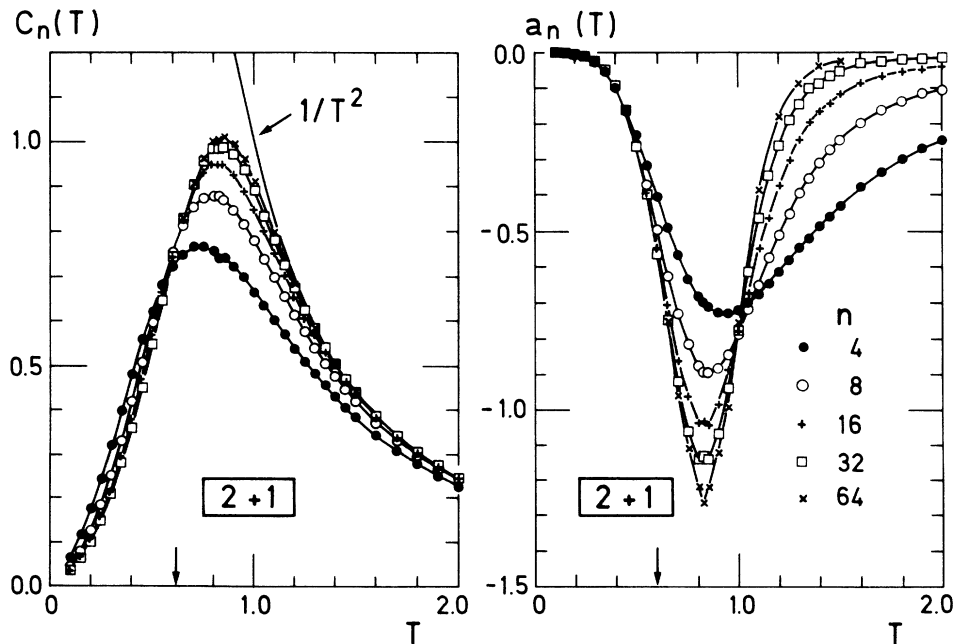with $\alpha = x/y - 1 = -0.10 \pm 0.10$ [this estimate does not

FIG. 3. The same as Fig. 1 in dimension $2 + 1$. The length $L = 10^5$. The arrow represents the lower bound (14).
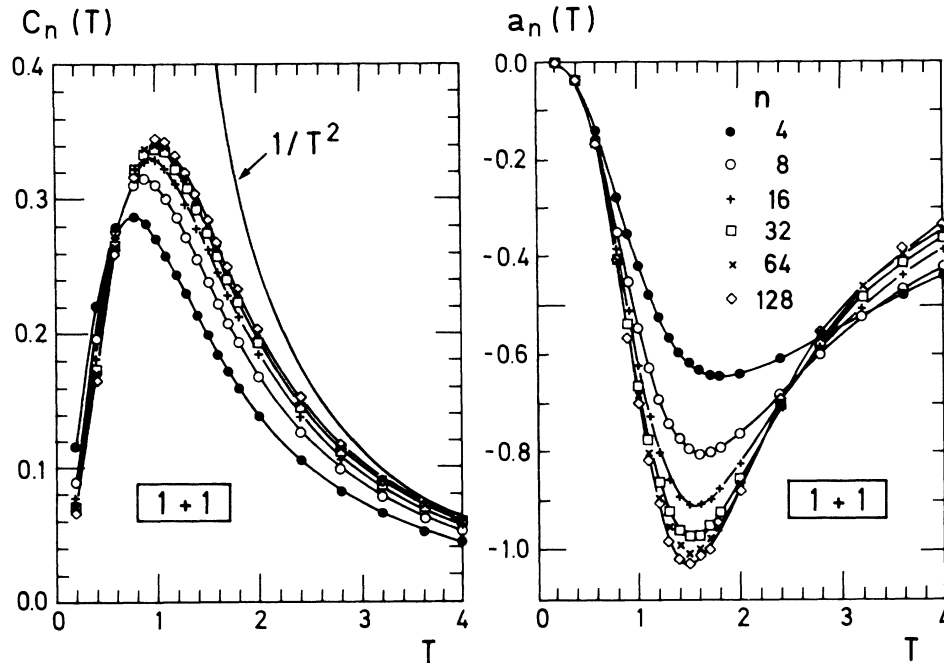
$C_n(T)$

$a_n(T)$

$1/T^2$

$1+1$

n
• 4
○ 8
+ 16
□ 32
× 64
◇ 128

$1+1$

FIG. 4. The same as Figs. 1 and 3 in dimension $1+1$. The length $L = 10^6$.

follow directly from (21) because the choices of x and y are not independent]. Of course, above T_c, the specific heat is not singular because (17) is the exact expression of the specific heat.

Figure 3 shows the same quantities $c_n(T)$ and $a_n(T)$ in dimension $2+1$. Here again, it is not easy from the specific-heat data [Fig. 3(a)] to know whether (17) is valid above a certain temperature or not, and whether or not to estimate a transition temperature. The results for $a_n(T)$ [Fig. 3(b)] are more visible. At temperatures higher than 1.2, $a_n(T)$ seems to converge to 0. This would imply that there is a high temperature phase where $a_\infty(T)=0$ and that there exists a transition temperature below which $a_\infty(T)\neq0$. The shape of $a_n(T)$ in the range $0.5 < T < 1.0$ strengthens the idea of a transition in $2+1$ dimensions: There is a clear size dependence of $a_n(T)$ in this region of temperature meaning that there is a characteristic length which is, at least, very large, and probably diverges at some transition temperature.

It is more difficult to estimate T_c and the exponent α in $2+1$ than it was in $3+1$ dimensions. The reason is that, from the data of Fig. 3(b), it is hard to tell whether $a_n(T)$ diverges as $n \to \infty$ or not. If it did not, one would need to compute higher derivatives with respect to temperature to try a finite-size scaling analysis similar to (20). However, we noticed that each new derivative requires more iterations to keep the same level of accuracy, and an accurate determination of $da_n/d\beta$ was beyond our possibilities. One can, however, estimate from the data of Fig. 3(b) that the exponent of $a_n(T)$ is close to zero and therefore that $\alpha \simeq -1.0\pm0.2$. For T_c, we think that the estimate $T_c = 0.9\pm0.1$ would be consistent with all the

possible analyses that we tried.

In dimension $1+1$ (see Fig. 4) we see that $a_n(T)$ does not converge to 0 at any temperature. Moreover, we do not see any strong size dependence at any temperature. So the results are consistent with the absence of a phase transition in this case.[1]

IV. CONCLUSION

We have seen in this paper that transfer matrix calculations and a finite-size scaling analysis of the specific heat and if its derivative $a(T)$ lead to prediction of a phase transition in dimension $3+1$ and $2+1$ for the problem of directed polymers in a random medium. The transition temperature T_c in $3+1$ agrees with bounds which can be derived in the same way as in Ref. 18. In dimension $2+1$, our results indicate a phase transition which is probably the same as the one recently observed for a growth model.[21]

We think that the calculation presented here could be extended in several ways.

(i) First, one could study the size dependence of other Lyapunov exponents of the transfer matrix: usually the difference between the largest two Lyapunov exponents allows one to define a correlation length[25-27] and one could try to study the phase transition by looking for a change of the n dependence of the correlation length.

(ii) Second, one could calculate quantities other than thermal properties: for example, by introducing a chemical potential conjugate to the number of steps in a given direction,[19] one could measure some geometrical properties on strips and try to obtain the behavior of the transverse fluctuations at the transition temperature through a

finite-size scaling analysis.

(iii) Lastly, one could study the temperature dependence of the overlaps.[18,19] This seems *a priori* more difficult because the transfer matrix associated with pairs of walks is $n^{2d} \times n^{2d}$ instead of $n^d \times n^d$. However, this study would certainly be interesting to see whether some of the properties (broad distribution of overlaps) observed in mean-field calculation (7) and in $1/d$ expansion (10) could persist in finite dimension.

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