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DIRECTED POLYMERS IN A RANDOM MEDIUM

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The problem of a polymer in a random medium can be formulated as follows: consider a regular lattice with on each bond ij a random energy ϵ_{ij}. The energy E_w of a self avoiding walk w on this lattice is the sum of the energies of all the bonds visited by the walk. Then the problem is to compute the partition function $Z = \Sigma_w \exp(-E_w/T)$. Several recent results will be presented: (1) The mean-field version of this problem can be solved exactly and one observes a low-temperature phase similar to the spin-glass phase. (2) For finite-dimensional lattices, one can derive bounds on the transition temperature in high enough dimension. (3) For hierarchical lattices, the problem can be reduced to finding stable laws when one combines random variables in a non-linear way. For example: $Z = Z_1Z_2 + Z_3Z_4$.

1. Introduction

The problem of directed polymers in a random medium is a problem in the theory of disordered systems in which a lot of progress has been made recently [1–6]. The problem in its lattice version can be formulated as follows [7]: consider a regular lattice with a random energy (or potential) ϵ_{ij} on each bond ij of the lattice. These energies are statistically independent and they are distributed according to a given probability distribution $\rho(\epsilon_{ij})$. On this regular lattice, we consider all the directed walks of length L starting at a fixed origin O. By definition, the energy E_w of a walk w is

$$E_w = \sum_{ij} \epsilon_{ij} , \tag{1}$$

where in (1) the sum runs over all the bonds ij visited by the walk (a directed polymer is, by definition, a walk stretched along a single longitudinal direction with fluctuations only in the transverse directions: the simplest example of a directed walk in dimension $d+1$ is to consider a usual random walk on a hypercubic lattice and to take the time direction for the longitudinal direction).

Three kinds of properties can be studied for such a system: thermal properties, overlaps and geometrical properties.

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1.1. Thermal properties

Thermal properties are those which can be obtained from the partition function Z,

$$Z = \sum_w \exp(-E_w/T) . \tag{2}$$

In (2), the sum includes all the directed walks of length L starting at a fixed origin O, and T is the temperature.

As usual, from the knowledge of Z, one can calculate the energy and the specific heat as a function of the temperature and one can study the phase diagram. In sections 2–4, we will see that, in the thermodynamic limit ($L \to \infty$), one can observe phase transitions between a high-temperature phase where the effect of disorder is weak and a low-temperature phase very similar to a spin-glass phase.

At zero temperature, the problem becomes an optimization problem: the problem of finding the walk of lowest energy. In addition to the value of the ground-state energy, one can study its fluctuations [3–5] which are characterized by an exponent ω which depends on the dimension of the lattice,

$$\langle E_{\mathrm{GS}}^2 \rangle - \langle E_{\mathrm{GS}} \rangle^2 \sim L^{2\omega} , \tag{3}$$

where $\langle \; \rangle$ means an average over disorder.

1.2. Overlaps [7, 8]

Because one can think of the low-temperature phase as a phase where the directed walks get trapped in deep valleys, it is useful in order to study the statistical properties of the landscape to define the overlaps between two or more walks and the probability distribution of these overlaps.

By definition, the overlap $0 \leq q_2(w, w') \leq 1$ between two directed walks is the fraction of their length that they spend together. The average overlap between two walks is then given by

$$q_2 = \left\langle Z^{-2} \sum_w \sum_{w'} q_2(w, w') \exp\left(- \frac{E_w + E_{w'}}{T} \right) \right\rangle . \tag{4}$$

One can define in a similar way the overlap between m walks $w^{(1)}, \cdots, w^{(m)}$ and an average overlap q_m.

As for spin glasses [9, 10], we will see that in the mean-field case, the overlap is not self-averaging and that its probability distribution $P_2(q)$, defined by

$$P_2(q) = \left\langle Z^{-2} \sum_w \sum_{w'} \exp\left(-\frac{E_w + E_{w'}}{T}\right) \delta(q - q_2(w, w'))\right\rangle , \tag{5}$$

remains broad even in the thermodynamic limit ($L \to \infty$).

1.3. Geometrical properties

The effect of disorder is to influence the transverse fluctuations of the directed walks. These can be characterized by an exponent ν,.

$$Z^{-1} \sum_w (r(w))^2 \exp(-E_w/T) \sim L^{2\nu} , \tag{6}$$

where $r(w)$ is the projection of the walk on the transverse directions.

The exponent ν depends on whether the system is in its weak- or its strong-disorder phase.

The exponent ν is $\frac{1}{2}$ in absence of disorder or in the weak-disorder phase [6], whereas larger values of ν have been predicted and measured in the strong-disorder regime [3–5]. It is in dimension $1 + 1$ (1 longitudinal + 1 transverse direction) that the problem is best understood. Analytic predictions based on several approaches [3–5] as well as numerical simulations give $\nu = \frac{2}{3}$ and $\omega = \frac{1}{3}$. In dimension $d + 1 > 3$, it has been shown that at high enough temperature [6] $\nu = \frac{1}{2}$ whereas below a certain temperature, disorder is relevant and $\nu > \frac{1}{2}$ and numerical simulations indicate that ν changes slowly with dimension [4].

In the present talk, only a few aspects of this problem will be discussed. First, the solution and the properties of the mean-field model [7] will be described (analogy with travelling waves, a low-temperature phase similar to a spin-glass phase with a broken symmetry of replica). Second, the case of finite-dimensional lattices will be considered. We will see that in dimension $d + 1 > 3$, there exists a phase transition and that one can obtain bounds for the transition temperature [3].

Lastly, the model on a hierarchical lattice will be discussed. We will see that that problem can be reduced to finding stable laws when one combines random variables in a non-linear way [8–11]. Several results obtained recently for hierarchical lattices will be summarized.

2. Polymers on a tree and travelling waves

There exists a mean-field version of the problem of polymers in a random medium which can be solved exactly. This version consists of choosing a tree for the regular lattice. Several methods give the free energy as a function of the temperature: the analogy with travelling waves, the replica approach, the results known on the GREM [12]. In this section, I will describe the analogy with the problem of travelling waves and I will summarize the results [7]. The replica approach is briefly presented in appendix A.

Consider a tree [or more precisely the branch of a tree (see fig. 1)]. On each bond ij of this lattice, there is a random energy ϵ_{ij} chosen according to a given probability distribution $\rho(\epsilon_{ij})$. If $Z_{L+1}(T)$ is the partition function of the 2^{L+1} walks of $L+1$ starting at O, one can write a recursion relation,

$$Z_{L+1}(T) = \mathrm{e}^{-\epsilon_{\mathrm{OA}}/T} Z_L^{(\mathrm{A})}(T) + \mathrm{e}^{-\epsilon_{\mathrm{OB}}/T} Z_L^{(\mathrm{B})}(T) , \tag{7}$$

where ϵ_{OA} and ϵ_{OB} are the energies on the bonds OA and OB and $Z_L^{(\mathrm{A})}$ and $Z_L^{(\mathrm{B})}$ are the partition functions of walks of L steps on the branches starting at A or at B. Since the energies ϵ_{ij} are random, the partition function is itself random. It is then natural to consider the probability distribution or a generating function of $Z_{L+1}(T)$. It turns out that there exists a generating function $G_{L+1}(x)$, defined by

$$G_{L+1}(x) = \langle \exp(-\mathrm{e}^{-x/T} Z_{L+1}(T)) \rangle , \tag{8}$$

for which recursion (7) takes a simple form: by replacing Z_{L+1}, given by eq. (7), in (8) and using the fact that ϵ_{OA}, ϵ_{OB}, $Z_L^{(\mathrm{A})}$ and $Z_L^{(\mathrm{B})}$ are independent random variables, one can show that

$$G_{L+1}(x) = \left[\int \rho(\epsilon)\, G_L(x+\epsilon)\, \mathrm{d}\epsilon \right]^K , \tag{9}$$

where K is the branching rate of the tree ($K = 2$ in the case of fig. 1).

One should notice that this recursion does not depend on temperature. The only temperature dependence comes from the initial condition

$$G_0(x) = \exp(-\mathrm{e}^{-x/T}) , \tag{10}$$

since the partition function $Z_0 = 1$ for a walk of 0 step.

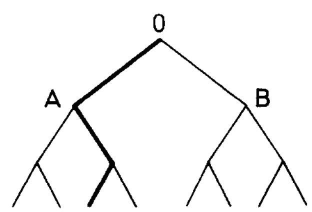

Fig. 1. Self-avoiding walk on a tree for $K = 2$.

So the whole knowledge of the probability distribution of Z is reduced to the study of recursion (9) with initial condition (10).

From (8) and (10), one can easily check that

$$\begin{aligned} &G_L(x) \to 1, \quad \text{as } x \to +\infty, \\ &G_L(x) \to 0, \quad \text{as } x \to -\infty, \end{aligned} \tag{11}$$

so $G_L(x)$ has the shape of a front (or a kink). Equation (9) belongs to a class of non-linear equations of the diffusion reaction type and has properties similar to the KPP (Kolmogorov, Petrovsky, Piscounov) equation [13],

$$\frac{\partial}{\partial L} G = \frac{\partial^2}{\partial x^2} G + G^2 - G . \tag{12}$$

Let me just summarize here what is expected [7] for the solutions of (9). In the limit $L \to \infty$, these solutions become travelling waves W which move with a velocity v,

$$G_L(x) = W(x - vL - c_L), \quad \text{for large } L , \tag{13}$$

where $c_L/L \to 0$ as $L \to \infty$. The velocity v depends on the initial condition; if the initial condition has the following shape for $x \to \infty$,

$$G_0(x) = 1 - \mathrm{e}^{-\gamma x} \tag{14}$$

(with the restriction that $G_0(x) \to 0$ as $x \to -\infty$ and that $G_0(x)$ is monotonic), then the velocity v is given by

$$\begin{aligned} v &= \frac{1}{\gamma} \log\Big[K \int \rho(\epsilon)\, \mathrm{e}^{-\gamma\epsilon}\, \mathrm{d}\epsilon\Big], && \text{for } \gamma \leqslant \gamma_{\min}, \\ &= v_{\min} = \frac{1}{\gamma_{\min}} \log\Big[K \int \rho(\epsilon)\, \mathrm{e}^{-\gamma_{\min}\epsilon}\, \mathrm{d}\epsilon\Big], && \text{for } \gamma \geqslant \gamma_{\min}, \end{aligned} \tag{15}$$

where $\gamma_{\min}$ is the value of γ which makes v minimum, i.e., such that

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} \left[\frac{1}{\gamma} \log\Big(K \int \rho(\epsilon)\, \mathrm{e}^{-\gamma\epsilon}\, \mathrm{d}\epsilon\Big) \right] \Bigg|_{\gamma_{\min}} = 0 . \tag{16}$$

So the exponential decay γ of the initial condition determines the velocity v. For $\gamma < \gamma_{\min}$ (slow decays), the velocity depends on γ whereas for $\gamma \geqslant \gamma_{\min}$, the velocity remains constant ($v = v_{\min}$).

From (8) and (13), one can translate these properties of travelling waves into properties of the partition function. One can show in particular that $-v$ is the free energy per unit length and, therefore,

$$\lim_{L\to\infty} \frac{1}{L} \log Z_L(T) = \log\left[K \int \rho(\epsilon)\, \mathrm{d}\epsilon \exp(-\epsilon/T)\right], \qquad \text{for } T > T_c\,, \tag{17a}$$

$$= \frac{T_c}{T} \log\left[K \int \rho(\epsilon)\, \mathrm{d}\epsilon \exp(-\epsilon/T_c)\right], \qquad \text{for } T < T_c\,, \tag{17b}$$

where $T_c = 1/\gamma_{\min}$.

From (17), one can show that the entropy vanishes in the whole low-temperature phase. This low-temperature phase is completely frozen. Another consequence of (13) is that the width of the probability distribution of $\log Z_L$ is of order 1. Therefore, the exponent ω (defined in (3)) vanishes in the case of a tree.

From the analogy with travelling waves [7] or other approaches (GREM [12] or replica), one can obtain other properties of interest. For example, the averaged overlaps q_2 between 2 walks and q_m between m walks are given, for $T < T_c$, by

$$q_2 = 1 - T/T_c\,, \qquad q_m = \frac{\Gamma(m - T/T_c)}{\Gamma(m)\Gamma(1 - T/T_c)} \tag{18}$$

and they vanish above T_c. The distribution $P_2(q)$ can also be obtained,

$$P_2(q) = \frac{T}{T_c}\, \delta(q) + \left(1 - \frac{T}{T_c}\right) \delta(q-1)\,. \tag{19}$$

We see that $P_2(q)$ is more complex than a single delta function and this means that the low-temperature phase is analogous to the spin-glass phase [9, 10] of the Sherrington–Kirkpatrick model.

3. Finite dimensional lattices

We have seen that the mean-field version of the problem possesses a low-temperature phase similar to the spin-glass phase. One can then wonder whether this phase is still present in finite dimension. We are now going to see that if $d + 1 > 3$, there exists a phase transition [6] at a certain temperature T_c, and that one can give an upper and a lower bound [8] to T_c.

We consider here a site version of the problem [8]: the polymer lies on a $(d+1)$-dimensional lattice. At each time step $L \rightarrow L+1$, the polymers occupies a site $\boldsymbol{r}$ of a regular hypercubic d-dimensional lattice. Suppose that at step L, the polymer is on site $\boldsymbol{r}$. Then at step $L+1$, by definition of the model, the polymer is allowed to occupy any of the $2d$ neighbors of $\boldsymbol{r}$. If $Z_L(\boldsymbol{r})$ is the partition function of a polymer of L steps, which ends at point $\boldsymbol{r}$, one can write the following recursion relation

$$Z_{L+1}(\boldsymbol{r}) = \sum_{j=1}^{2d} \mathrm{e}^{-\epsilon_L(\boldsymbol{r})/T} Z_L(\boldsymbol{r} + \boldsymbol{e}_j) . \tag{20}$$

Using this relation, one can show [8] that there exists a temperature T_2 such that

$$\lim_{L\rightarrow\infty} \frac{\langle Z_L^2(\boldsymbol{r})\rangle}{\langle Z_L(\boldsymbol{r})\rangle^2} \quad \text{is finite, if } T > T_2 . \tag{21}$$

For $T < T_2$, this ratio is exponentially large in L. The temperature T_2 is a solution of the following equation [8],

$$\frac{\langle \mathrm{e}^{-\epsilon/T_2}\rangle^2}{\langle \mathrm{e}^{-2\epsilon/T_2}\rangle - \langle \mathrm{e}^{-\epsilon/T_2}\rangle^2} = \int_0^{2\pi} \frac{\mathrm{d}q_1}{2\pi} \cdots \int_0^{2\pi} \frac{\mathrm{d}q_d}{2\pi} \, \frac{(\Sigma_{\mu=1}^d \cos q_\mu)^2}{d^2 - (\Sigma_{\mu=1}^d \cos q_\mu)^2} , \tag{22}$$

where $\langle\ \rangle$ represents the average over ϵ.

By using and generalizing (21), one can show that, for $T > T_2$, the free energy is equal to the annealed free energy with probability 1,

$$\lim_{L\rightarrow\infty} \frac{1}{L} \log Z_L(\boldsymbol{r}) = \log(2d) + \log\left\langle \exp - \frac{\epsilon}{T}\right\rangle , \quad \text{for } T > T_2 . \tag{23}$$

So T_2 is an upper bound for the transition temperature.

From the free energy given by the right-hand side of (23), one can calculate the entropy as a function of temperature and show that this expression of the entropy (which is correct only in the high-temperature phase) vanishes at a temperature T_0 (and becomes negative below T_0). This implies that (23) cannot remain valid for $T < T_0$. So this proves that there exists a transition at a certain temperature T_c which satisfies

$$T_0 \leqslant T_\mathrm{c} \leqslant T_2 , \tag{24}$$

where T_2 can be obtained by comparing $\langle Z^2\rangle$ and $\langle Z\rangle^2$ (see (21)) and T_0 is the temperature below which the annealed free energy would give a negative entropy.

For $d = 3$ (i.e., for directed polymers in dimension $3+1$), on the lattice described above and with energies ϵ distributed according to a Gaussian distribution of width 1, one gets [8]

$$0.528 \leq T_c \leq 0.963 . \tag{25}$$

The problem of directed polymers in a random medium is analogous, at least in its mean-field version, to spin glasses. In finite dimension, we see that one can go much further in the case of the polymer problem since one has the exact expression of the free energy in the high-temperature phase (23), one can prove the existence of a transition temperature and even obtain bounds on the temperature (25).

4. Hierarchical lattice

Except for the free energy in the high-temperature phase (23), it is hard to obtain analytic expressions for other quantities in the finite-dimensional problem. Even T_c is not known (24).

This led us to consider a problem of intermediate difficulty [11, 8] between the mean-field case (section 2) and the finite-dimensional lattice, the problem of polymers on hierarchical lattices. Hierarchical lattices [14, 15] have been often used to build models in statistical mechanics which can be solved exactly. All kinds of spin models (Ising, Potts, . . .) defined on hierarchical lattices can be reduced to the study of a simple renormalization transformation and properties like the critical temperature, critical exponents can easily be obtained from the dynamical properties of the transformation.

In presence of disorder [16, 17], despite the simplicity of the renormalization transformation which consists in combining random variables in a non-linear way, the problem can only be solved by numerical or perturbative techniques.

The hierarchical lattice [8, 11] that we considered is the so-called diamond lattice represented in fig. 2. The lattice is built by an iterative rule. The first generation consists of one bond and two sites. In the second generation, the bond is replaced by a set of $2b$ bonds ($b = 2$ in fig. 2a), each of which is replaced by a corresponding set to form the third generation, etc. We shall consider directed walks from A to B on such a lattice. On a lattice at the nth generation which consists of $(2b)^{n-1}$ bonds, the length L of these walks and the number Ω_L of different walks are given by

$$L = 2^{n-1}; \qquad \Omega_L = b^{L-1} .$$

The partition function Z is the sum over all these Ω_L walks (2).

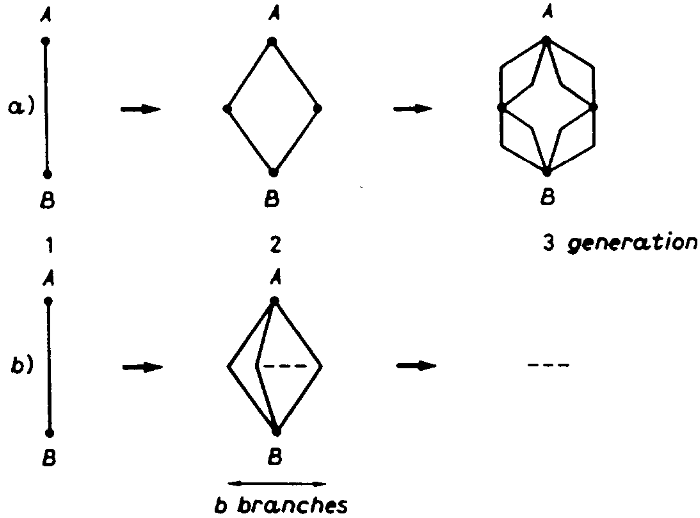

Fig. 2. The iterative construction of the hierarchical lattice (fig. 2a $b = 2$).

Because the lattice at generation $n + 1$ can always be viewed as the combination of $2b$ lattices at generation n, one can write the following recursion for the partition function,

$$Z_{n+1} = Z_n^{(1)} Z_n^{(2)} + \cdots + Z_n^{(2b-1)} Z_n^{(2b)} , \tag{26}$$

where Z_{n+1} is the partition function of a lattice at the $(n + 1)$th generation and $Z_n^{(i)}$ are the partition functions of $2b$ (different) lattices at generation n.

At generation $n = 1$, the partition function Z_1 reduces to

$$Z_1 = \exp(-\epsilon / T) , \tag{27}$$

where ϵ is the random energy (of a single bond on the lattice) distributed according to a given distribution $\rho(\epsilon)$.

Since all the Z_n are random (see (27) and (26)), one has to consider the probability distribution $\Pi_n(\log Z_n)$ of $\log Z_n$.

Although the recursion (26) is very simple, one (or at least we) does not know how to obtain the properties of $\Pi_n(\log Z)$ for n large from those of $\Pi_1(\log Z)$. One can only prove that if $b > 2$ and $\Pi_1(\log Z)$ is narrow enough,

$$\langle Z_1^2 \rangle < (b - 1) \langle Z_1 \rangle^2 , \tag{28}$$

the distribution $\Pi_n(\log Z)$ becomes a delta function (one can show easily from (26) that if (28) is satisfied, then $\langle Z_n^2 \rangle / \langle Z_n \rangle^2 \rightarrow 1$ as $n \rightarrow \infty$).

If $b<2$ or if (28) is not satisfied, the distribution $\Pi_n(\log Z)$ either becomes a delta function as $n\to\infty$ or becomes broader and broader and takes the form

$$\Pi_n(\log Z) \sim L^{-\omega}\delta^{-1}F_b\left(\frac{\log Z - L\gamma}{L^{\omega}\delta}\right), \tag{29}$$

where γ and δ depend on the initial distribution $\Pi_1(\log Z)$, but the shape F_b and the exponent ω do not. The function F_b is a stable law for recursion (26), since if one chooses $2b$ variables, $\log Z_n^{(i)}$ distributed according to (29) and if one combines them according to (26), then $\log Z_{n+1}$ is again distributed according to (29) with renormalized values of γ and δ ($\delta \to 2^{\omega}\delta$).

A typical shape of the function F_b is shown in fig. 3 for $b=5$. The fact that the stable law F_b is the same for a large class of initial distributions has been observed numerically [11] for several initial broad distributions (i.e., in the low-temperature limit), but this has not been proved. It is rather easy to get numerical estimates of the exponent ω which decreases when b increases.

It is much harder to understand the existence of a stable law and to calculate the coefficients γ and δ in (29) by analytic methods. We were only able [8, 11] to develop a perturbation method which is valid for b close to 1,

$$b = 1 + \epsilon . \tag{30}$$

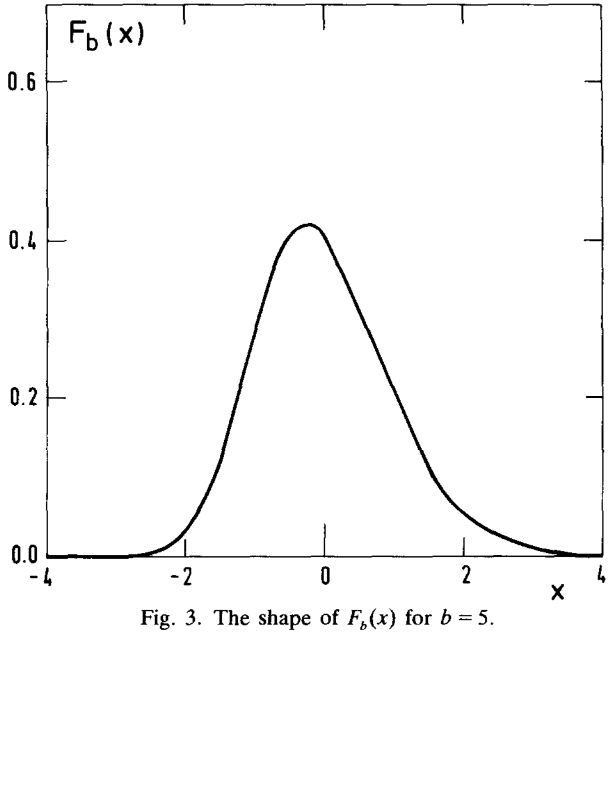

Fig. 3. The shape of $F_b(x)$ for $b=5$.

For ϵ small, the lattice construction is rather questionable since the lattice exists only for integer ϵ (it is, however, possible to build other lattices [8] which depend on a parameter similar to ϵ).

The reason which makes an expansion possible around $b = 1$ is that for $b = 1$, recursion (26) becomes $Z_{n+1} = Z_n^{(1)} Z_n^{(2)}$ and, therefore, the stable law for $\log Z$ is a Gaussian distribution. So the stable distribution F_b can be expanded around a Gaussian shape.

This expansion method [11] allows one to obtain the shape F_b as well as the exponent ω,

$$2^{\omega} = \sqrt{2}\,(1 - \epsilon K_2)\,, \tag{31}$$

where the constant K_2 is given by a complicated integral [11] and is equal to

$$K_2 = 0.29782\ldots. \tag{32}$$

The expansion method can also be used to obtain thermal properties, overlaps or the distribution $P_2(q)$. At low temperatue, one gets [8]

$$\lim_{L\to\infty} \frac{\log Z_L}{L} = \epsilon \frac{K_1}{T} \frac{1}{\sqrt{2}-1} - ((\Gamma'(1))^2 - \Gamma''(1)) \frac{K_1 T}{2(2\sqrt{2}-1)}\, \epsilon + \mathcal{O}(T^2)\,, \tag{33}$$

where the constant K_1 is also given by some integral [11],

$$K_1 = 0.903197\ldots, \tag{34}$$

and [8]

$$q_m = 1 - \epsilon\left(\frac{\Gamma'(m)}{\Gamma(m)} - \Gamma'(1)\right) \frac{K_1 T}{\sqrt{2}-1}\,. \tag{35}$$

From (33), one can see that the specific heat is linear in T at low temperature. This behaviour is also observed in numerical simulations [8] for $b = 2$. By comparing (35) with the mean-field result (18), one can see that at low temperature the dependence of q_m is very similar in the two problems. Lastly, the calculation of $P_2(q)$ to order ϵ gives that $P_2(q)$ is a single delta function, and so for ϵ small, the mean-field property (broken symmetry of replica) does not seem to be present.

5. Conclusion

The problem of directed polymers in a random medium is very similar to the problem of spin glasses, at least in the mean-field limit. It is, however, simpler:

(i) in the mean-field case, the free energy has a simple expression, eq. (17), at all temperature;

(ii) in finite dimension, one can prove the existence of a phase transition and give bounds for the transition temperature.

The problem on hierarchical lattices raises the interesting question of stable laws when one combines random variables in a non-linear way, eq. (26). In some limit ($b \to 1$), it allows one to calculate properties such as the free energy or overlaps perturbatively.

One interesting direction of research would be to investigate whether the low-temperature phase in high, but finite, dimension remains a phase with a broken replica symmetry. This seems to be a question with an attainable answer since one can develop a method which gives the $1/d$ expansion of the free energy. This method and its results will be exposed, I hope, in a forthcoming work.

Appendix A

In this appendix, the free energy (17) and the distribution $P_2(q)$ of the overlaps will be obtained using the replica approach. The calculation is very similar to the section 6 of ref. [12].

We are going to calculate $\langle Z^n \rangle$ using the replica approach. We assume that $\langle Z^n \rangle$ is dominated by sets of n walks which are organized in the following way:

(i) the n walks follow the same path from the origin to a certain length Lq_1;

(ii) from Lq_1 to a length Lq_2, they are grouped into m_1 groups of n/m_1 walks each;

(iii) from Lq_i to Lq_{i+1}, they are grouped into m_i groups of n/m_i walks each.

We have, in general,

$$q_0 = 0 \leqslant q_1 \leqslant q_2 \leqslant \cdots \leqslant q_i \leqslant \cdots \leqslant q_M = 1 , \tag{A.1}$$

$$1 = m_0 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_i \leqslant \cdots \leqslant m_M = n . \tag{A.2}$$

The contribution of these arrangements to $\langle Z^n \rangle$ is

$$\langle Z^n \rangle = \sum_{\{q_i\}} \sum_{\{m_i\}} \exp\left\{ L \sum_{i=1}^{M} (q_i - q_{i-1}) m_i \right.$$
$$\left. \times \left[\log K + \log\left(\int \rho(\epsilon) \exp\left(-\frac{n}{m_i} \frac{\epsilon}{T} \right) \mathrm{d}\epsilon \right) \right] \right\} . \tag{A.3}$$

For large L, $\langle Z^n \rangle$ is dominated by the $\{q_i\}$ and $\{m_i\}$ which maximize (A.3).

In the limit $n \to 0$, following Parisi [9, 10], one has to invert inequality (A.2),

$$1 = m_0 \geqslant m_1 \geqslant \cdots \geqslant m_M = n = 0 , \tag{A.4}$$

and, if one takes a continuous limit ($M \to \infty$), by defining

$$m_i = n / x(q_i) , \tag{A.5}$$

where q_i becomes a continuous variable $0 \leqslant q \leqslant 1$, one gets from (A.3) in the limit $n \to 0$

$$\frac{1}{L} \langle \log Z \rangle = \underset{x(q)}{\text{extremum}} \int_0^1 \frac{\mathrm{d}q}{x(q)} \left[\log K + \log\left(\int \rho(\epsilon) \exp\left(\frac{-\epsilon x(q)}{T} \right) \right) \mathrm{d}\epsilon \right] . \tag{A.6}$$

As usual in the replica approach [9, 10], the function $x(q)$ has to satisfy $0 < x(q) < 1$.

Because there is no explicit q dependence in integral (A.6), $x(q)$ is a constant.

Depending on the temperature T, one finds that the extremum is either

$$\begin{aligned} x(q) &= 1, && \text{if } T > T_c, \quad \text{or} \\ x(q) &= T/T_c = T\gamma_{\min}, && \text{if } T < T_c , \end{aligned} \tag{A.7}$$

where $T_c = 1/\gamma_{\min}$ and $\gamma_{\min}$ is defined by (16). In the replica approach [9], the distribution $P_2(q)$ of overlaps is given by

$$P_2(q) = \frac{\mathrm{d}x(q)}{\mathrm{d}q} , \tag{A.8}$$

which gives two delta functions at $q = 0$ and $q = 1$.

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