Fluctuations of a Stationary Nonequilibrium Interface

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We study properties of interfaces between stationary phases of the two-dimensional discrete-time Toom model (north-east-center majority vote with small noise): phases not described by equilibrium Gibbs ensembles. Fluctuations in the interface maintained by mixed boundary conditions grow with distance much slower than in equilibrium systems; they have exponents close to $\frac{1}{4}$ or $\frac{1}{3}$, depending on symmetry, rather than $\frac{1}{2}$, and have long-range correlations reminiscent of self-organized critical behavior. Approximate theories reproduce this behavior qualitatively and lead to novel nonlinear partial differential equations for the asymptotic profile.

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There is much current interest in dynamical models that do not satisfy the condition of detailed balance [1-3]. In this Letter we report on the statistical properties of interfaces between stationary states of such systems as they occur in the Toom model [4].

The Toom model consists of Ising spins ($\sigma_{i,j}=\pm 1$) located on a square lattice. At each time step, all spins are simultaneously updated: $\sigma_{i,j}(t+1)$ becomes equal to the majority of itself and of its northern and eastern neighbors at time t with probability $1-p-q$, to $+1$ with probability p, and to -1 with probability q. Nonzero p and q represent the effect of noise; $p\neq q$ introduces a bias. It has been proved by Toom that for p and q sufficiently small, but otherwise unrestricted, two phases exist, in which the spins are predominantly $+1$ or -1, respectively; see also Ref. [5].

Our main interest in this Letter is the nature of fluctuations in the stationary interface separating the two phases. Since information travels southwest, we consider the Toom model in the third quadrant only and impose boundary conditions at the north and east: We require spins along the negative x axis to be $+1$ and along the negative y axis to be -1 at all times. At low (but nonzero) noise level the system establishes a unique steady state in the long-time limit, in which a fairly well-defined interface making some angle α with the x axis separates the $+$ phase in the upper portion of the quadrant from the $-$ phase in the lower portion; see Fig. 1. We refer to this interface as *anchored*, since it is pinned at the origin.

To understand the behavior of this interface we consider first the time-dependent behavior of a straight interface dividing the whole plane. At zero noise, such an interface, which makes an angle θ with the x axis, is stationary if $\theta\in[0,\pi/2]$, and moves to the southwest with constant velocity if $\theta\in[\pi/2,\pi]$. At low noise level, simulations and heuristic reasoning show that the interface remains flat on a macroscopic scale and has a velocity $v(\theta)$ which is generically nonzero except at some angle θ_0. The dependence of θ_0 (which equals the angle α defined above) on p and q will be determined in the next section; for $p=q$, $\theta_0=\pi/4$ by symmetry and $v(\theta)$ is an odd function of $\theta-\pi/4$. This fact will play an important role in distinguishing the fluctuations in the symmetric and biased cases in the Kardar-Parisi-Zhang (KPZ) approach [6], to be discussed in Sec. (3) after we consider more specific models for the Toom interface in Secs. (1) and (2).

(1) Low-noise approximation.— To study the anchored interface we first note that any stairlike interface configuration starting at the origin and directed along $(-1,-1)$ is invariant for the zero-noise dynamics. If at nonzero noise one starts with such a configuration C, then most fluctuations (for example, a $-$ spin in a sea of $+$ spins) have a short lifetime and the system returns quickly to C. However, if a positive (negative) spin directly west of a vertical (south of a horizontal) portion of the interface flips, then the system will jump rapidly to another zero-noise configuration via the deterministic part of the Toom dynamics. This leads in a suitable limit of low noise to an effective "solid-on-solid"-type model in which only stairlike interfaces occur [7]. The interface is then represented by a one-dimensional system of Ising spins $S_n=\pm 1$, $n=1,2,\ldots$, with $S_n=+1$ (-1) if the nth link along the interface is vertical (horizontal). Since we have no *a priori* measure on this interface, we must compute the stationary state of the one-dimensional model under the inherited dynamics: Each spin is updated independently in such a way that, during a time dt, a $+$ $(-)$ spin is exchanged with the *first* $-$ $(+)$ *spin to its right* with probability $q\,dt$ $(p\,dt)$.

The height of the interface at a distance L from the origin (measured in number of links) relative to a straight line with inclination $\pi/4$ is given by $M_L=\sum_1^L S_i$. We are interested in the large-L behavior of the averages $\langle M_L\rangle$ and $\langle M_L^2\rangle$ in the steady state of the one-dimensional semi-infinite system: These give, respectively, the asymptotic average angle through $\tan\alpha=(1-\mu)/(1+\mu)$, $\mu=\lim_{L\to\infty}L^{-1}\langle M_L\rangle$, and the fluctuations about the average.

The semi-infinite system has a very unusual property reflecting the behavior of the Toom model in the third

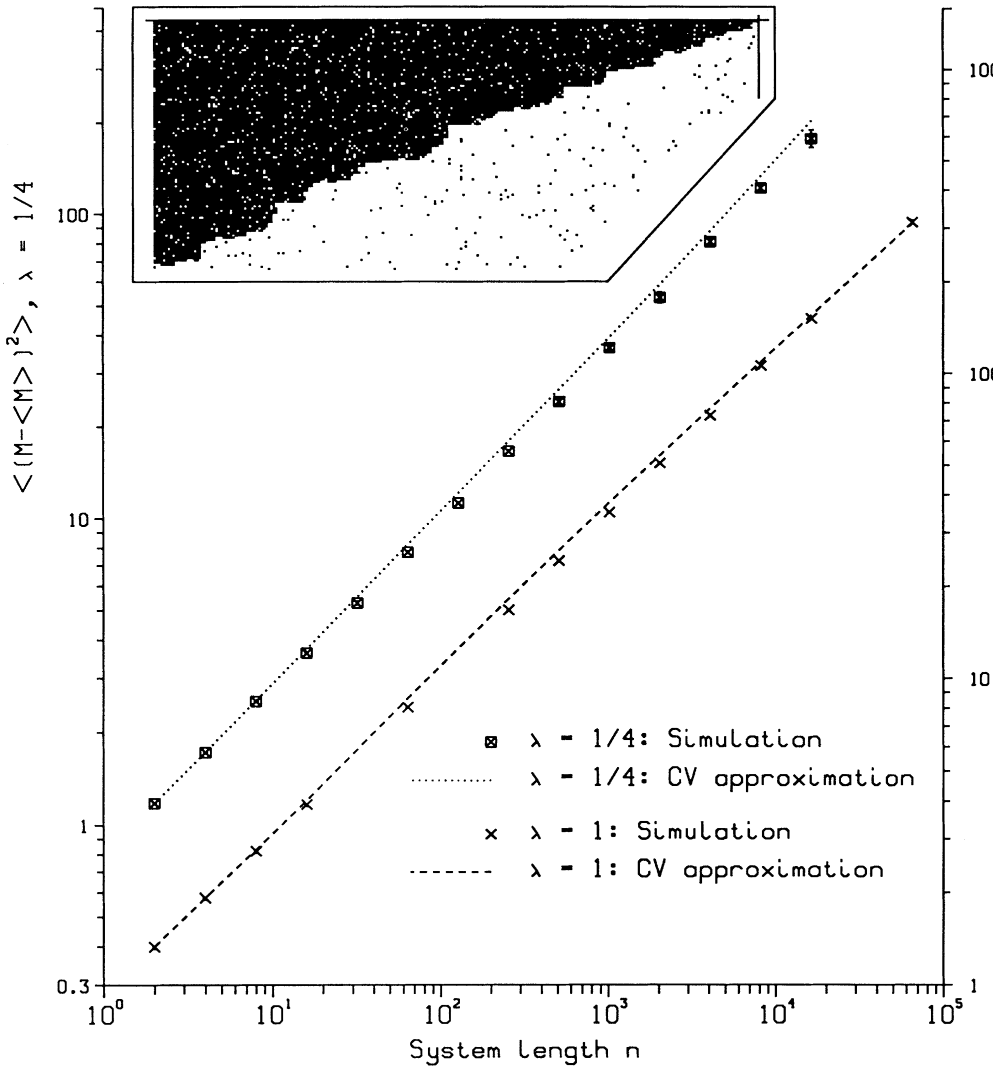

FIG. 1. Mean-square fluctuations in system magnetization for $\lambda=1$ and $\lambda=\frac{1}{4}$, with collective-variable approximations. Inset: Typical anchored Toom interface configuration for $p=0.008$, $q=0.032$.

quadrant: There are no finite-size effects. Specifically, if we restrict our attention to the subsystem $1 \le n \le L$ and define the dynamics so that a + (−) spin with no spin of opposite sign to the right is simply flipped with probability $q\,dt$ ($p\,dt$), then the dynamics of the subsystem is identical to the dynamics induced from the $L=\infty$ system, as long as that system contains an infinite number of spins of both signs. This permits in theory the exact calculation of any correlation function; in practice, however, this does not help much in obtaining rigorous results about M_L when $L \gg 1$. The lack of finite-size effects does help greatly with the computer simulations, as does the fact that the relaxation time for a system of size L may be shown to be proportional to L [7].

On the other hand, we can easily determine the steady state on a ring with periodic boundary conditions; in this case, a spin exchanges with the first spin of opposite sign in a counterclockwise direction, and in the steady state for any fixed total magnetization μL (which is conserved on the ring) every spin configuration has equal weight [7]. Letting the size of the ring go to infinity gives for the model on the infinite line a stationary state which is just a product measure with $\langle S_j \rangle = \mu$ for all j. This state corresponds to an interface at an inclination $\theta = \tan^{-1}(1-\mu)/(1+\mu)$. Its normal velocity $v(\theta)$ may be computed through the formula $v(\theta) = J(\mu)/[2(1+\mu^2)]^{1/2}$, where the "current" $J(\mu)$ is defined as the number of + spins less the number of − spins crossing any bond per unit time. An easy computation gives [7] $J(\mu) = 2[q(1+\mu)^2 - p(1-\mu)^2]/(1-\mu^2)$.

It is reasonable to assume, and computer simulations confirm, that the stationary state of the semi-infinite system approaches, far from the origin, a product measure with average magnetization μ determined by setting $J(\mu)$

equal to zero. This yields $\langle M_L\rangle \simeq \mu L$ with $\mu=(1-\sqrt{\lambda})/(1+\sqrt{\lambda})$, where $\lambda=q/p$, and $\tan\alpha=\sqrt{\lambda}$.

The asymptotic behavior of the steady-state measure suggests, at first sight, that we should have $\langle M_L^2\rangle - \langle M_L\rangle^2 \sim L^{2\nu}$ with $\nu=\frac{1}{2}$. This, however, turns out to be wrong due to the very slow approach of the stationary state to its asymptotic independence, a fact reminiscent of generic nonequilibrium or self-organized critical behavior [1]. This leads to a great suppression of the fluctuations as seen in Fig. 1, where we show the results of Monte Carlo simulations for two values of λ. The results for systems of sizes of 65 536 and 16 384 sites, respectively, indicate $\nu=0.265$ if $\lambda=1$ and $\nu=0.285$ if $\lambda=\frac{1}{4}$. We shall discuss later two approximate treatments which suggest that $\nu=\frac{1}{4}$ for $\lambda=1$ and $\nu=\frac{1}{3}$ for $\lambda\neq 1$.

To get an analytic handle on the measure we consider the stationarity conditions obtained by choosing some function $\Psi(\mathbf{S})$ of the spin configuration $\mathbf{S}=\{S_1,\ldots,S_L\}$ and setting $d\langle\Psi\rangle/dt=0$. For simplicity, we consider here only the unbiased case $\lambda=1$. Since our primary interest is in the growth of $\langle M_n^2\rangle$, we take Ψ to be a function of the magnetization M_n and argue as follows: During a time dt, M_n remains unchanged with probability $1-K_n\,dt$ and M_n becomes M_n-2S_n with probability $K_n\,dt$, where K_n is the number of successive spins directly preceding S_n (including S_n itself) which have the same sign as S_n. Therefore, stationarity requires that for *arbitrary* functions $\Psi(m)$

$$\langle K_n[\Psi(M_{n-1}+S_n)-\Psi(M_{n-1}-S_n)]\rangle=0\,, \qquad (1)$$

where we have written $M_n=M_{n-1}+S_n$. For $\Psi(M)=M^2$, this becomes $\langle K_nS_nM_{n-1}\rangle=0$ or, using $K_n=1+(1+S_nS_{n-1})K_{n-1}/2$,

$$2\langle S_nM_{n-1}\rangle+1 = \tfrac{1}{2}\langle S_n(2-K_{n-1})M_{n-1}\rangle+\tfrac{1}{2}\langle 2-K_{n-1}\rangle\,, \qquad (2)$$

where the left-hand side is just $\langle M_n^2\rangle-\langle M_{n-1}^2\rangle$. Now, in the product measure, the first terms on the right-hand and left-hand sides of (2) vanish while the second term on the right is of order 2^{-n}. Hence our steady state can approach a product measure only very slowly. In fact, computer simulations [7] give $\langle S_nS_{n-j}\rangle \sim c_j/\sqrt{n}$ for $n\gg 1$. They, as well as some approximate analysis based on the stationary conditions $d\langle S_iS_j\rangle/dt=0$, also indicate that there is an asymptotic scaling $\langle S_nS_{n-j}\rangle \simeq -(1/\sqrt{12\pi n})\exp[-j^2/12n]$ for $n\gg 1$.

(2) Collective-variable approximation.— The steady state of our system will of course satisfy many stationarity relations other than (1). In this section we construct an approximate measure (the *collective-variable* measure) on which we impose only the relations (1) but for which we make the strong simplifying assumption (we can actually get away with considerably less [7]) that the expectation of S_{n+1}, given all the previous spins, depends only on their sum M_n:

$$\langle S_{n+1}|S_1,\ldots,S_n\rangle=\langle S_{n+1}|M_n\rangle\,. \qquad (3)$$

This is in spirit a mean-field approximation since only the collective variable M_n matters.

Our starting point is the recursive equation for the probability $W_n(m)$ that $M_n=m$,

$$W_{n+1}(m)=\tfrac{1}{2}[1+H_n(m-1)]W_n(m-1)+\tfrac{1}{2}[1-H_n(m+1)]W_n(m+1)\,, \qquad (4)$$

where $H_n(m)\equiv\langle S_{n+1}|M_n=m\rangle$. Equations (1) and (4), together with the assumption (3), lead (after some manipulation) to the recursion relations

$$H_n(m)=[U_n(m+1)-U_n(m-1)]/[U_n(m+1)+U_n(m-1)+2W_n(m)] \qquad (5)$$

and

$$U_{n+1}(m)=[W_n(m)+U_n(m+1)][W_n(m)+U_n(m-1)]/[U_n(m+1)+U_n(m-1)+2W_n(m)]\,, \qquad (6)$$

where

$$U_n(m)\equiv\langle \tfrac{1}{2}(1+S_n)K_n|M_n=m+1\rangle W_n(m+1)=\langle\tfrac{1}{2}(1-S_n)K_n|M_n=m-1\rangle W_n(m-1)\,, \qquad (7)$$

with the second equality in (7) a consequence of (1). Equations (3), (5), and (6) and their counterparts for the biased case, $\lambda\neq 1$, determine H, U, and W recursively. This approximation is compared in Fig. 1 with the results of the Monte Carlo simulations. We see that the agreement is rather good, particularly in the unbiased case. The figures suggest, and the scaling relations discussed below confirm, that the asymptotic value of the exponent ν in the collective-variable approximation is $\frac{1}{4}$ for $\lambda=1$ and $\frac{1}{3}$ for $\lambda=1$.

Scaling relations.— For large n the recursive relations (3) and (5)-(7) can be approximated by continuum partial differential equations [7]. For the unbiased case $W_n(m)=w(n,m/n^{1/4})$, where $w(t,x)$ satisfies

$$\partial_t w=-\tfrac{3}{8}\,\partial_x^2[w\,\partial_x^2(\ln w)]\,. \qquad (8)$$

Equation (8) has the Gaussian scaling solution

$$W_G(t,x)=[2\pi\sigma(t)]^{-1/2}\exp[-x^2/2\sigma(t)],\quad \sigma(t)=\sqrt{3t/2}\,. \qquad (9)$$

It implies $\langle M_L^2\rangle \cong \sqrt{3L/2}$, which agrees reasonably well with the numerical data. In the biased case [7] the scaling form $W_n(m)=w(n,(m-\mu n)/n^{1/3})$ yields

$$\partial_t w = \mu C\{\tfrac{4}{3}\partial_x^3 w - \partial_x[w^{-1}(\partial_x w)^2]\}, \tag{10}$$

with $C=\sqrt{\lambda}/[1+\sqrt{\lambda}]^2$. The physically relevant scaling solution of (10) is $w(t,x)=t^{-1/3}g^4(t^{-1/3}x)$ with

$$g(y)=\begin{cases} c\,\mathrm{Ai}[-(16mC)^{-1/3}y], & \text{if } y<(-16mC)^{1/3}z_0, \\ 0, & \text{if } y\geq(-16mC)^{1/3}z_0, \end{cases} \tag{11}$$

where z_0 is the largest zero ($z_0<0$) of the Airy function Ai.

(3) The KPZ approach.—In this section we develop a coarse-grained description of the interface dynamics using the method of Kardar, Parisi, and Zhang [6] (cf. also Ref. [8]). Starting with a straight interface in the plane we assume that, after some suitable local averaging, the interface at time t can be described by a single-valued function $h(t,x)$, where x is the coordinate along a reference line. If $v(\partial_x h)$ denotes the interface velocity normal to the reference line, then on a macroscopic scale the motion of the interface is governed by the equation

$$\partial_t h(t,x) = v(\partial_x h(t,x)). \tag{12}$$

To include fluctuation effects the KPZ approach is to add noise and dissipation to (12) and to expand v up to the relevant order. This yields

$$\partial_t h = v_0 + v_1\partial_x h + \tfrac{1}{2}v_2(\partial_x h)^2 + \tfrac{1}{6}v_3(\partial_x h)^3 + D\partial_x^2 h + \sqrt{\sigma}\xi, \tag{13}$$

where $\xi(t,x)$ is normalized white noise. Simple power counting shows that higher-order nonlinearities are irrelevant on a large scale; the cubic nonlinearity scales as noise and dissipation and is thus marginal.

We first consider the time-dependent broadening of a static ($v_0=0$) infinite straight interface. The presence of a quadratic term in (13) ($v_2\neq 0$) is known to lead to an interface which broadens in time as $t^{1/3}$ [6]. Simulations, or the low-noise approximation to the velocity mentioned earlier, indicate that in the Toom model the only static interface with $v_2=0$ is the one at angle $\pi/4$ in the unbiased case ($p=q$); there $v_2=0$ by symmetry. For this interface the cubic term in (13) should be taken into account, but we do not at present know how to compute its effect. Neglecting this term leads to a linear theory in which the interface broadens as $t^{1/4}$ [7].

The anchored interface is described by stationary solutions $h(x)$ of (13), defined for $x\geq 0$ and satisfying the boundary conditions $h(0)=0$. We can solve this problem only in the linear case, that is, when $p=q$ (and thus $v_2=0$) and we neglect the marginal cubic nonlinearity. We find then $\langle h^2(x)\rangle^{1/2}\simeq x^{1/4}$, so that the exponent is $\nu=\frac{1}{4}$ as before. In the general case we argue heuristically as follows: The coefficient v_1 in (13) is such that excitations along the interface travel outwards from the origin. Thus excitation growth in time as t^ν leads to excitations at position x of order x^ν, through the identification $x=v_1 t$. The argument again gives $\nu=\frac{1}{4}$ for the unbiased model and predicts $\nu=\frac{1}{3}$ in the biased ($v_2\neq 0$) case. The situation is in contrast to that in equilibrium Ising models, for which the interface velocity vanishes for any inclination and excitations on an interface have no translational velocity; in such models, interface fluctuations are characterized by $\nu=\frac{1}{2}$.

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