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Needle models of Laplacian growth

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We consider a simple model to study the competition and screening between branches of anisotropic structures growing in a Laplacian field. Growth conditions are found where the symmetry between the different growing branches is spontaneously broken. In the case of two branches, we obtain an analytic expression for the ratio of the lengths in the stationary regime as a function of the angle between the two needles. We also find that a symmetric pattern with n branches is unstable for $n>6$ in the simplest diffusion-limited-aggregation-like case. These results are generalized to the more general problem of dielectric breakdown.

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I. INTRODUCTION

The different patterns produced by diffusion-limited growth have been the subject of many investigations in recent years [1–3]. Anisotropy in the growth rules has been shown to play a very important role both for the existence of steady states of the classical deterministic equations of motion [1] (dendrites) and in the large-scale shapes of stochastic fractal aggregates [2,3]. In particular, diffusion-limited-aggregation (DLA) clusters [4] slowly evolve into an n-arm star shape when grown in the presence of a low-order n-fold anisotropy [5,6]. This has suggested the approximation of the shape of such clusters by n needles [7–9]. As shown by several authors [7–10], this gives a rather accurate analytic approximation to the real diffusion field. In this paper, we propose to transform the static needle description into a dynamic one and consider a model where n needles of different lengths grow along fixed directions. The growth velocity of a needle is chosen to be proportional to the flux on its tip. This simple model allows us to study analytically the competition and screening between branches and the resulting effect on the shape of the growing cluster. For example, it is shown in Fig. 1 that a threefold anisotropy gives rise to a threefold symmetric structure. Does this remain true for an anisotropy of higher order? Namely, does an anisotropy with n-fold symmetry give rise to an n-fold symmetric structure?

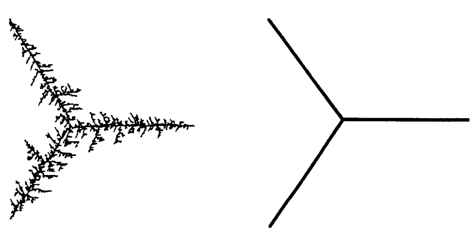

FIG 1. An aggregate of 10^4 particles grown with the antenna method [9,10] (threefold anisotropy) and a simple needle model of the aggregate large-scale structure.

The simplest case is the one where there are only two competing needles. For a different choice of dynamics, a similar problem has already been considered by other authors [11,12] who have shown that the symmetry between the two needles can be spontaneously broken when the angle between them is small enough. For our model, we reach the same conclusion and give, in addition, an analytic expression for the ratio of needle lengths in the spontaneously broken phase.

We then extend this computation to the case of n needles. We recover the result that the symmetry between the different branches is spontaneously broken if the anisotropy is more than sixfold symmetric [13]. These results are then generalized to the case when the velocity of a needle is proportional to an arbitrary power η [14] of the flux at its tip and the critical number of needles for spontaneous symmetry breaking is determined.

II. THE MODEL

We consider n needles growing from a common center along fixed directions in the plane (see Fig. 2). The field ϕ which controls the growth of the needles is supposed to satisfy Laplace's equation

$$\nabla^2\phi=0 \; . \tag{1}$$

(ϕ is meant to represent pressure in viscous fingering, reduced temperature, or impurity concentration in dendritic growth and the probability field of the random walker in DLA [1,2]). The field ϕ is chosen to be zero on the needles and grows logarithmically with r at large distance r from the center. These boundary conditions and Eq. (1) determine ϕ completely for a given geometry of needles. By definition of the model, the velocity of a needle is

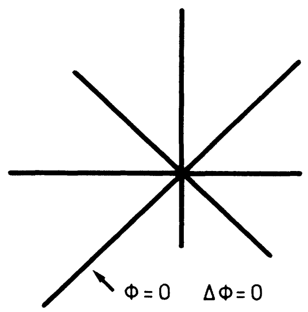

FIG. 2. A simple dynamic model for the growth of the main branches of an anisotropic aggregate. The potential is equal to zero on the needles and satisfies Laplace's equation. The growth of a needle is proportional to the field at its tip, as explained in the text.

chosen to be proportional to the gradient of ϕ at its tip. Since the gradient of ϕ is infinite at the geometrical tip of a needle, this definition requires some elaboration.

Along the kth needle, at a small distance r from its tip, the gradient of ϕ behaves like $r^{-1/2}$

$$|\nabla\phi(r+r_k)| \sim \frac{E_k}{\sqrt{r}} , \tag{2}$$

where r_k denotes the position of the tip of the kth needle. Now, we envisage the physical tip of the needle as a region of extension a, small compared to the length of the needle, and independent of the length of the needle. The total flux falling on the physical tip of a needle is therefore finite (being equal to the integral of $|\nabla\phi|$ over r, between $r=0$ and $r=a$) and proportional to E_k. So we define our model precisely by imposing that the growth rate of the kth needle is proportional to E_k.

$$\frac{dl_k}{dt} = E_k = \lim_{r\to 0}[\sqrt{r}\,|\nabla\phi(r+r_k)|] , \tag{3}$$

where l_k denotes the length of the kth needle. In order to study this model analytically one needs to compute the

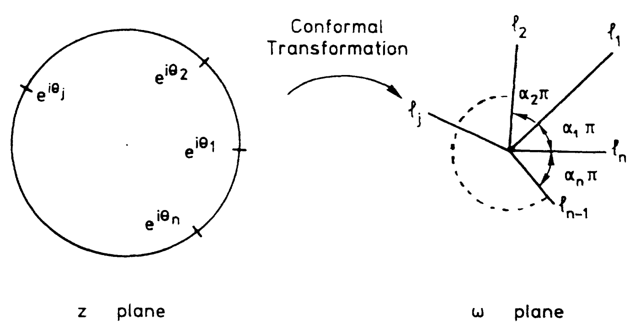

FIG. 3. Sketch of the conformal transformation which maps the exterior of the unit disk in the z plane onto the physical plane (ω plane) cut along the n needles. $e^{i\theta_1},\ldots,e^{i\theta_n}$ are the n points of the unit circle which are mapped onto the star center (origin of the ω plane). $l_1,\ldots,l_n$ are the n lengths of the needles and $\alpha_1\pi,\ldots,\alpha_n\pi$ the angles between them.

field ϕ. This is conveniently done [11,12,15,16] by finding a conformal transformation which maps the exterior of the unit disk (z plane) onto the exterior of the star-shaped object (Fig. 3) formed by the n needles (ω plane). Such a transformation read

$$\omega=f(z)=Az\prod_{j=1}^{n}\left[1-\frac{e^{i\theta_j}}{z}\right]^{\alpha_j}, \quad \sum_{j=1}^{n}\alpha_j=2 . \tag{4}$$

The α_j are fixed parameters given by the angles $\alpha_j\pi$ between the successive needles. In order to fix the orientations of the two complex planes, we set A to be real and $\theta_1=0$.

The remaining n parameters $A,\theta_2,\ldots,\theta_n$ are chosen to obtain the n lengths of the needles in the physical ω plane. The advantage of having written such a conformal transformation is that the field ϕ around the needles can now be easily computed since in the z plane.

$$\phi(z)=\ln|z|=\mathrm{Re}(\ln z) . \tag{5}$$

It is simply the harmonic field which vanishes on the unit circle and grows like the logarithm at large distances. Therefore one obtains

$$\phi(\omega)=\mathrm{Re}\{\ln[f^{-1}(\omega)]\} . \tag{6}$$

The field ϕ being determined, one can write the equations of motion for the needle lengths. The needle tips are located in the z plane at points $z_i=\exp(i\varphi_i)$, $i=1,\ldots,n$ and their lengths l_i are given by the moduli of $f(z)$ at $z=z_i$

$$l_i=4A\prod_{j=1}^{n}\left|\sin\left[\frac{\varphi_i-\theta_j}{2}\right]\right|^{\alpha_j} . \tag{7}$$

Since the tip positions maximize $|f(z)|$ on the unit circle, the φ_i are the n solutions of the equation

$$\sum_{j=1}^{n}\alpha_j\cot\left[\frac{\varphi-\theta_j}{2}\right]=0 . \tag{8}$$

At the ith tip, $f'(z_i)=0$ and for z close to z_i, one has

$$f(z)\simeq f(z_i)+\frac{f''(z_i)}{2}(z-z_i)^2 . \tag{9}$$

Consider a point z close to z_i such that

$$z=z_i(1+\rho e^{i\theta}) \tag{10}$$

where ρ is small and real. The potential ϕ at this point z in the z plane is

$$\phi=\ln|z|\simeq\rho\cos\theta \tag{11}$$

and this value of ϕ is also the potential at point $\omega=f(z)$ in the ω plane. From (9), one has

$$\rho e^{i\theta}=\left[2\frac{\omega-f(z_i)}{z_i^2 f''(z_i)}\right]^{1/2} . \tag{12}$$

When $\theta\to-\pi/2$ or $+\pi/2$, since ω has the same phase as $f(z_i)$ (because they are both on the same needle), one knows that $z_i^2 f''(z_i)$ has the same phase as $f(z_i)$. Therefore, using the fact that $|z_i|=1$, one can write (12) as

$$\rho e^{i\theta}=\left|\frac{2f(z_i)}{f''(z_i)}\right|^{1/2}\left[\frac{\omega}{f(z_i)}-1\right]^{1/2} \tag{13}$$

and from (11) one gets

$$\phi(\omega)=\left[\frac{2}{|f''(z_i)|}\right]^{1/2} \times \mathrm{Re}\left[|f(z_i)|\left[\frac{\omega}{f(z_i)}-1\right]\right]^{1/2} . \tag{14}$$

This leads to

$$\frac{dl_i}{dt}\propto|f''(z_i)|^{-1/2}=\left[A\prod_{j=1}^{n}\left|\sin\left[\frac{\varphi_i-\theta_j}{2}\right]\right|^{\alpha_j}\sum_{j=1}^{n}\frac{\alpha_j}{\sin^2\left[\dfrac{\varphi_i-\theta_j}{2}\right]}\right]^{-1/2} . \tag{15}$$

Equation (15) defines implicitly the time dependence of $A(t),\theta_2(t),\ldots,\theta_n(t)$ through Eqs. (7) and (8). In the following two sections we study the dynamics of this model in two particular cases.

III. TWO COMPETING NEEDLES

We begin by considering the simplest case of competition, the case of two needles of lengths $l_1(t),l_2(t)$ growing at an angle $\alpha\pi$ with each other. If one needle is longer than the other at time zero, then its velocity of growth will be greater, being proportional to the gradient of the diffusion field ϕ. It is therefore clear that $l_1=l_2$ is an unstable situation and that the difference $|l_1-l_2|$ grows without bound as time passes. This is a well-known instability in Laplacian growth [17,18]. Here, we want to consider a slightly different question. The symmetric visual appearance of the three branches in Fig. 1 is clearly due to the fact that their length ratio is close to one while their length difference can be arbitrarily large. So in our simple model we are going to compute the asymptotic value of the length ratio l_1/l_2 of the two needles in the long-time limit. The time evolution of l_1/l_2 is given by [using Eq. (3)]

$$\frac{d}{dt}\left[\frac{l_1}{l_2}\right]=\frac{E_2}{l_2}\left[\frac{E_1}{E_2}-\frac{l_1}{l_2}\right] . \tag{16}$$

Since Laplace's equation has no length scale, E_1/E_2 depends only on the length ratio $\lambda=l_1/l_2$

$$\frac{E_1}{E_2}=F(\lambda) . \tag{17}$$

Therefore the fixed points of (16) and their stability can be obtained once the function $F(\lambda)$ is known. In the case of two needles, Eq. (8), giving the positions of the needle tips,

$$\alpha\cot\left[\frac{\varphi}{2}\right]+(2-\alpha)\cot\left[\frac{\varphi-\theta_2}{2}\right]=0 . \tag{18}$$

θ_2 can be chosen smaller than π without loss of generality. It is useful to note that if φ_1 denotes the tip position of one needle (in the z plane), then the other needle is located at φ_2 such that

$$\varphi_2=\pi+\theta_2-\varphi_1 . \tag{19}$$

It is convenient to introduce the tangents of $\varphi_1/2$ and $\varphi_2/2$

$$t_1\equiv\tan\left[\frac{\varphi_1}{2}\right],\quad t_2\equiv\tan\left[\frac{\varphi_2}{2}\right] \tag{20}$$

which satisfy, using Eqs. (18) and (19),

$$t_1t_2=-\frac{\alpha}{2-\alpha} . \tag{21}$$

The advantage of introducing t_1 and t_2 is that the length ratio λ and the field ratio E_1/E_2 are simple functions of t_1/t_2. From Eq. (7) the needle lengths are

$$\begin{aligned} l_1&=4A\left|\sin\frac{\varphi_1}{2}\right|^{\alpha}\left|\cos\frac{\varphi_2}{2}\right|^{2-\alpha} ,\\ l_2&=4A\left|\sin\frac{\varphi_2}{2}\right|^{\alpha}\left|\cos\frac{\varphi_1}{2}\right|^{2-\alpha} . \end{aligned} \tag{22}$$

With the help of Eqs. (20) and (21), the length ratio can be expressed as

$$\begin{aligned} \lambda&=\left|\frac{t_1}{t_2}\right|^{\alpha}\left[\frac{1+t_1^2}{1+t_2^2}\right]^{1-\alpha}\\ &=\left|\frac{t_1}{t_2}\right|^{\alpha}\left[\frac{2-\alpha-\alpha t_1/t_2}{2-\alpha-\alpha t_2/t_1}\right]^{1-\alpha} . \end{aligned} \tag{23}$$

In the same way, one obtains for the fields, using Eqs. (3) and (15)

$$\begin{aligned} E_1&=\left[\frac{l_1}{4}\left[\frac{\alpha}{\sin^2\frac{\varphi_1}{2}}+\frac{2-\alpha}{\cos^2\frac{\varphi_2}{2}}\right]\right]^{-1/2} ,\\ E_2&=\left[\frac{l_2}{4}\left[\frac{\alpha}{\sin^2\frac{\varphi_2}{2}}+\frac{2-\alpha}{\cos^2\frac{\varphi_1}{2}}\right]\right]^{-1/2} . \end{aligned} \tag{24}$$

The field ratio can then be expressed as

$$\frac{E_1}{E_2}=\left[-\frac{t_1}{\lambda t_2}\right]^{1/2} . \tag{25}$$

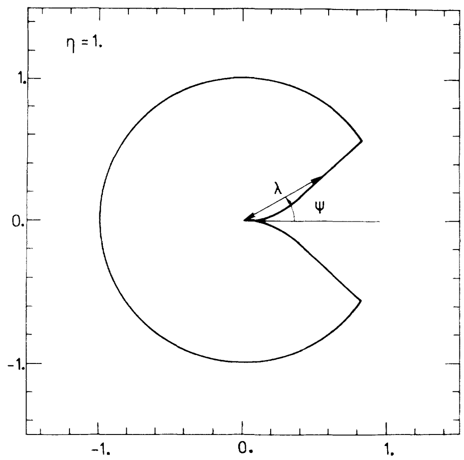

FIG. 4. The asymptotic length of two needles as a function of the angle ψ between their growing directions [Eq. (21)]. The longest needle is represented explicitly and taken as the unit of length. The shortest one has length λ. The position of the tip of the shortest needle is plotted as a function of ψ.

Intuitively, it is clear that the fixed points of the dynamics are such that the field ratio is equal to the length ratio and this can be readily seen from Eq. (16). Using Eqs. (23) and (25) one obtains therefore the equation satisfied by the asymptotic length ratio $\lambda = l_1/l_2$.

$$\lambda^2 \left[\frac{2-\alpha+\alpha\lambda^3}{(2-\alpha)\lambda^3+\alpha} \right]^{1-\alpha} = 1 \ . \tag{26}$$

$\lambda=1$ is obviously always a solution of this equation. This is the only solution for $1-\sqrt{\frac{2}{3}} \le \alpha \le 1+\sqrt{\frac{2}{3}}$. Two new solutions $\lambda(\alpha)$ and $[\lambda(\alpha)]^{-1}$ appear for $\alpha < 1-\sqrt{\frac{2}{3}}$ or $\alpha > 1+\sqrt{\frac{2}{3}}$, both conditions describing the same geometrical situation of two needles of unequal lengths growing at an angle ψ smaller than a critical angle $\psi_c = 180°\,(1-\sqrt{\frac{2}{3}}) \simeq 33.03°$. An analysis of Eqs. (16), (23), and (25) shows that when the new solutions appear, the solution $\lambda=1$ becomes unstable and the bifurcation at $\psi=\psi_c$ is a standard supercritical pitchfork bifurcation. Thus, for $\psi<\psi_c$ the symmetry between the two needles is spontaneously broken. The length ratio as a function of the angle between the two needles is plotted in Fig. 4. For a different dynamics of needle growth, an analogous spontaneous symmetry breaking has been numerically observed in Ref. [11] and the corresponding bifurcation angle has been computed in Ref. [12].

IV. COMPETITION BETWEEN n NEEDLES

In this section, we consider the case of n needles growing in directions spaced regularly at relative angles of $2\pi/n$ (i.e., $\alpha_j = 2/n$). As explained before, this is meant to be a simple model of more realistic growth rules with n-fold anisotropy as depicted on Fig. 1 for $n=3$. Our goal is to study the stability of a pattern of n needles of equal lengths. A related problem has previously been addressed by Ball [13] who considered a fractal object with a shape consisting of n fingers and suggested a relation between the maximal number of fingers and the fractal dimension of the growing object.

We consider a system of n needles of almost equal lengths. A conformal mapping which transforms the unit circle in the z plane into a set of n regularly spaced needles of equal lengths L in the ω plane is given by (4) with $\theta_j = 2\pi j/n$

$$\omega = f(z) = L \left[\frac{z^{n/2} - z^{-n/2}}{2} \right]^{2/n} . \tag{27}$$

All the points $z_k = \exp(2i\pi k/n)$ map onto $\omega=0$, corresponding to the n returns to the origin when one follows the shape of the cluster. The points $z = \exp(2i\pi k/n + i\pi/n)$ correspond to the tips of the needles.

Now we want to consider a set of n needles where the lengths of the needles are slightly perturbed, keeping the angles fixed. This can be done by considering the following small change in the conformal mapping (27):

$$\omega = f(z) = L \left[\frac{z^{n/2} - z^{-n/2}}{2} + \frac{\epsilon z^{p-n/2} - \epsilon^* z^{-p+n/2}}{2} \right]^{2/n} , \tag{28}$$

where p is an integer $1 \le p \le n-1$ and ϵ is small (ϵ^* is the conjugate of ϵ).

One reason for choosing this perturbation is that we want to keep the fact that as z moves along the unit circle, ω vanishes n times. This happens to leading order in ϵ since the zeros of ω are given to leading order in ϵ by

$$z = \exp\left[i \left[\frac{2\pi k + i(\epsilon e^{2i\pi pk/n} - \epsilon^* e^{-2i\pi pk/n})}{n} \right] \right] . \tag{29}$$

The other reason for (28) is that by choosing $1 \le p \le n-1$, one can generate any small perturbation of the lengths of the needles. Indeed, the length l_k of the kth needle is given by the kth maximum of the modulus of (28) when z moves on the unit circle. One then obtains

$$l_k = L \left[1 - \frac{1}{n}(\epsilon e^{i\pi(2k+1)p/n} + \epsilon^* e^{-i\pi(2k+1)p/n}) + O(\epsilon^2) \right] . \tag{30}$$

The fields E_k can also be computed from (28) using the fact (15) that $E_k \propto |f''(z)|^{-1/2}$ evaluated at the kth maximum of $|f(z)|$. The expression of $|f''(z)|$ simplifies somewhat because, at the maximum of $|f(z)|$, $f'(z)$ vanishes and one gets to first order in ϵ:

$$E_k \propto l_k^{-1/2} \left[1 + \frac{1}{4} \left[\frac{(p-n/2)^2}{n^2/4} - 1 \right] \times (\epsilon e^{i\pi(2k+1)p/n} + \epsilon^* e^{-i\pi(2k+1)p/n}) \right] \tag{31}$$

which becomes using (30)

$$E_k \propto 1+\frac{2p(p-n)+n}{2n^2}(\epsilon e^{i\pi(2k+1)p/n}+\epsilon^* e^{-i\pi(2k+1)p/n}) . \tag{32}$$

So we see that for a given mode characterized by p, the perturbation of the lengths l_k is proportional to the perturbation of the fields E_k.

Since for Laplacian growth, $dl_k/dt \propto E_k$, for the ratio of the lengths to approach 1 in the long-time limit one requires that

$$l_k \Big/ \left[\sum_{k'} l_{k'}\right] \to 1/n \quad \text{as } t\to\infty . \tag{33}$$

The derivative of (33) with respect to time can be written in two different ways: From (30), one gets

$$\frac{d}{dt}\left[\frac{l_k}{\sum_{k'} l_{k'}}\right] = -\frac{1}{n^2}\frac{d}{dt}(\epsilon e^{i\pi(2k+1)p/n}+\epsilon^* e^{-i\pi(2k+1)p/n}) , \tag{34}$$

whereas from the dynamics (30) and (32) one can write

$$\frac{\frac{dl_k}{dt}}{\sum_{k'} l_{k'}} - \frac{l_k \sum_{k'} \frac{dl_{k'}}{dt}}{\left[\sum_{k'} l_{k'}\right]^2} \propto \frac{E_k}{nL} - \frac{l_k \sum_{k'} E_{k'}}{(nL)^2}$$
$$\propto \frac{1}{nL}\left[\frac{2p(p-n)+n}{2n^2}+\frac{1}{n}\right] \times (\epsilon e^{i\pi(2k+1)p/n}+\epsilon^* e^{-i\pi(2k+1)p/n}) . \tag{35}$$

By comparing (34) and (35), we see that the evolution of the perturbation is given by

$$\frac{d\epsilon}{dt} \propto -\frac{n}{L}\left[\frac{2p(p-n)}{2n^2}+\frac{1}{n}\right]\epsilon . \tag{36}$$

Thus for mode p to be damped, one needs that

$$\frac{2p(p-n)+n}{2n^2}+\frac{1}{n}>0 . \tag{37}$$

The analysis is then straightforward. For $n \le 5$, all the modes $1 \le p \le 4$ are damped, whereas for $n \ge 7$, there are always unstable modes (when p is equal to the integer part of $n/2$, the corresponding mode is always unstable). So we conclude that for Laplacian growth, a system of n needles is stable for $n<6$ and is unstable for $n>6$.

For $n=6$, the situation is marginal. One can then show that any shape with alternate lengths $l_1=l_3=l_5=\lambda l_2=\lambda l_4=\lambda l_6$ remains invariant under the dynamics. This can be seen by considering for even n the following conformal mapping:

$$\omega=f(z)=\left[\frac{z^{n/2}-z^{-n/2}}{2}+ia\right]^{2/n} \tag{38}$$

which gives n needles with alternate lengths (a is real and $|a|<1$). The two values of the lengths l_1 and l_2 are then

$$l_1=(1+a)^{2/n} \text{ and } l_2=(1-a)^{2/n} , \tag{39}$$

whereas the two values of the fields $E \propto |f''(z_i)|^{-1/2}$ at the tips are

$$E_1 \propto (1+a)^{1/2-1/n} \text{ and } E_2 \propto (1-a)^{1/2-1/n} . \tag{40}$$

Therefore the dynamics for the ratio $\lambda=l_1/l_2$ is

$$\frac{d\lambda}{dt} \propto (\lambda^{(n-2)/4}-\lambda) \tag{41}$$

which shows that any λ is a fixed point when $n=6$.

V. A DIELECTRIC BREAKDOWN LIKE GENERALIZATION OF NEEDLE DYNAMICS

Witten and Sander's DLA model can be generalized [19] to simulate dielectric breakdown phenomena. In the dielectric breakdown model [14], the growth of the pattern is controlled by a Laplacian field ϕ Eq. (1) as in DLA but the growth velocity is proportional to some power η of the field gradient. The dynamics of the needle growth can be generalized in a similar way by replacing Eq. (3) by

$$\frac{dl_k}{dt}=(E_k)^\eta . \tag{42}$$

It is a simple matter to extend the results of Secs. II and III and obtain their η dependence. In the case of two needles, Eq. (16) is replaced by

$$\frac{d}{dt}\left[\frac{l_1}{l_2}\right]=\frac{E_2^\eta}{l_2}\left[\left[\frac{E_1}{E_2}\right]^\eta-\frac{l_1}{l_2}\right] . \tag{43}$$

Thus the fixed points of the dynamics are such that the field ratio to the power η is equal to the length ratio. Using Eqs. (23) and (25) to relate them, one obtains the equation satisfied by the asymptotic length ratio $\lambda=l_1/l_2$ which generalizes Eq. (26):

$$\lambda^{2/\eta}\left[\frac{2-\alpha+\alpha\lambda^{1+2/\eta}}{(2-\alpha)\lambda^{1+2/\eta}+\alpha}\right]^{1-\alpha}=1 . \tag{44}$$

$\lambda=1$ is always a solution of this equation. Two other solutions appear for $\alpha<1-\sqrt{2/(\eta+2)}$ or $\alpha>1+\sqrt{2/(\eta+2)}$. As before this indicates the dynamical breaking of the symmetry between the two needles when the angle between their growth directions is less than a critical angle $\psi_c=\pi[1-\sqrt{2/(\eta+2)}]$. The length ratio as a function of the angle between the two needles is plotted on Fig. 5 for different values of η.

The case of n needles growing in directions spaced regularly by an angle of $2\pi/n$ can be studied as in Sec. IV. The formulas (30) and (32) are still valid and since the growth is rule is now $dl_k/dt \propto E_k^\eta$, the stability condition (37) becomes

$$\eta\frac{2p(p-n)+n}{2n^2}+\frac{1}{n}>0 . \tag{45}$$

As before the most unstable mode is always $p=n/2$ for

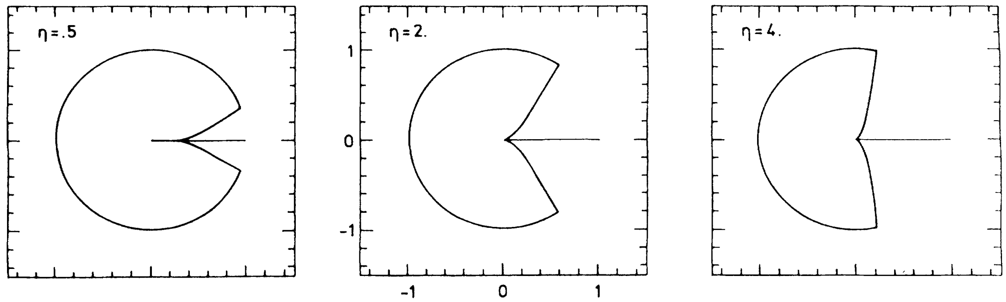

FIG. 5. Same as Fig. 4 for $\eta=0.5$, $\eta=2$, and $\eta=4$.

even n and $p=(n-1)/2$ or $(n+1)/2$ for odd n. This leads to the stability condition

$$\eta < \frac{4}{n-2} \quad \text{for even } n$$
$$\text{and } \eta < \frac{4n}{n^2-2n-1} \quad \text{for odd } n \,. \tag{46}$$

So $n=3$ is unstable for $\eta>6$, $n=4$ for $\eta>2$, $n=5$ for $\eta>10/7$, etc.

VI. CONCLUSION

In this paper we have studied simple models of growing needles with a fixed geometry. We have shown, by the use of conformal transformations, that the competition between the needles can lead to instabilities. In the case of two needles growing at a fixed angle in a Laplacian field, there exists a critical angle below which the two needles grow with a fixed ratio $\lambda \neq 1$ of their lengths. We have obtained an analytical expression for λ and generalized our calculation to the case of a dielectric breakdown model (where the velocity of the tips is an arbitrary power η of the fields at the tips). For a symmetric pattern of n needles, we have shown that for $n>6$, the symmetry between the needles is unstable for a Laplacian field [13] and we have computed the critical value of n as a function of the parameter η which characterizes the dielectric breakdown model.

It would be interesting to investigate the relevance of our results in the problem of growing DLA clusters. One knows that, when special directions are favored (due to lattice effects [2,3,5,6] or to some anisotropy in the sticking probabilities), the structure at the large scale of DLA clusters is a set of branches growing in the preferred directions. These effects can be strengthened by using reduction-of-noise techniques [20,21]. It is reasonable to hope that our predictions could be tested quantitatively in such systems. This, however, might be rather difficult because some additional instabilities could occur due to side branching effects on the main branches.

For off-lattice DLA, recent simulations [22,23] suggest that the angle between branches of different order is dynamically selected and close to 36°. It would be interesting to see if this angle is related to an instability similar to those described here.

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