J. Phys. A: Math. Gen. 26 (1993) L457–L463. Printed in the UK

LETTER TO THE EDITOR

# Anderson model on a Cayley tree: the density of states

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Received 11 November 1992

**Abstract.** We describe a method of calculating the electronic density of states on an arbitrary tree. We use it for the problem of Anderson localization on a regular tree with random site energies and obtain the first term of a weak disorder expansion of the density of states.

Localization is one of the central problems in the theory of disordered systems (Thouless 1973). A lot of effort has been expended in trying to solve the problem in some mean field limit by considering either trees (Abou-Chacra *et al* 1973, Abou-Chacra and Thouless 1974, Kunz and Souillard 1980, 1983, Chalker and Siak 1990, Mirlin and Fyodorov 1991a, Kim and Harris 1985, Acosta and Klein 1992) or diluted lattices (Fyodorov *et al* 1992). In fact, the infinite cluster of diluted lattices is very similar to a tree in that it is without loops in the thermodynamic limit. The aim of these studies is to compute quantities such as the mobility edge, density of states, localization length and inverse participation ratio. The calculation of, for instance, the density of states for the Anderson model either on a tree or for a diluted lattice is very similar to the calculation of the eigenvalue density of a sparse random matrix (Rodgers and Bray 1988, Rodgers and De Dominicis 1990, Mirlin and Fyodorov 1991b).

A number of results are already known for the problem of localization on the Cayley tree. Abou-Chacra *et al* (1973) obtained a nonlinear integral equation in two variables for the probabilty distribution of the real and imaginary parts of the self-energy and calculated the mobility edge for several distributions of the site energies (Abou-Chacra and Thouless 1974). Kunz and Souillard (1983) were able to reproduce the results of Abou-Chacra *et al* (1973) with a more rigorous approach and to show the existence of delocalized states. Chalker and Sisak (1990) introduced a new simplified model which allowed the study of the eigenstate amplitudes in a more straightforward way. More recently, Mirlin and Fyodorov (1991a, b) used a supersymmetric approach to reproduce the known results, such as the position of the mobility edge, and to calculate the density–density correlations in both the localized and extended regimes.

Here we present a simple method for attacking these problems which allows one to recover the results of more sophisticated supersymmetric (Mirlin and Fyodorov 1991a, Fyodorov *et al* 1992) or replica (Kim and Harris 1985) methods. With this simple approach, we show that one can recover in a very direct way the expression of the integrated density of states.

We first show that the integrated density of states can be obtained for any tree structure by an expression similar to the Thouless formula (Thouless 1972) for one-dimensional

0305-4470/93/090457+07$07.50 

systems. Then we show that this approach can be used to calculate the density of states for the Anderson model and obtain a systematic weak disorder expansion by extending a method previously used for the one-dimensional case (Derrida and Gardner 1984). Finally we discuss several possible generalizations.

Our starting point is the Schrödinger equation on a tree

$$E\psi_i = \sum_j \psi_j + \lambda V_i \psi_i. \tag{1}$$

In (1) the sum over $j$ runs over all the neighbours of site $i$. (Here, we do not use the true Laplacian on the lattice, but this could be done easily by just adding to $\lambda V_i$ the number of neighbours of site $i$.)

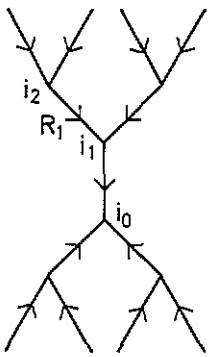


**Figure 1.** A regular tree with coordination $z = 3$.

Our goal is to compute for an arbitrary tree with open boundary conditions the integrated density of states $N(E)$ defined as the number of eigenenergies $E_\alpha$ larger that $E$

$$N(E) = \sum_\alpha \Theta(E_\alpha - E). \tag{2}$$

This can be done by choosing any site $i_0$ to be the central site and drawing arrows on all the bonds of the tree towards that site (see figure 1). Then we can introduce Riccati variables, defined for each branch of the tree as the ratio of the wave function of the site closest to $i_0$ divided by that furthest away. Thus in the figure, $R_1$ is defined by

$$R_1 = \frac{\psi_{i_1}}{\psi_{i_2}}. \tag{3}$$

One can then compute from (1) all the $R_i$ using for the bonds at the boundary of the tree

$$R_i = E - \lambda V_i \tag{4}$$

and for all the other bonds of the tree

$$R_i = E - \lambda V_i - \sum_j \frac{1}{R_j} \tag{5}$$

where $R_i$ is the Riccati variable on the branch which connects site $i$ to site $i_0$ and the summation in (5) is over the other branches $j$ attached to site $i$. So all the $R_i$ of the tree can be calculated by iterating (5) in the direction of the arrows, starting with (4) at the boundary. We are going to show that the integrated density of states $N(E)$ is given by

$$N(E) = \frac{1}{\pi} \lim_{\epsilon \to 0} \operatorname{Im} \left\{ \log \left[ \left( E - \lambda V_{i_0} - \sum_j \frac{1}{R_j} \right) \prod_i R_i \right] \right\} \tag{6}$$

where in (6) the sum is over all the branches $j$ attached to the central site $i_0$ and the product is over all the sites $i$ (or equivalently all the bonds) of the lattice. In (6), $\epsilon$ is a small positive imaginary part added to the energy $E$, and it is easy to check that all the $R_i$ have a positive imaginary part which vanishes as $\epsilon \to 0$. Consequently $N(E)$ is equal to the number of negative $R_i$ on the lattice plus 1 when the first term in the argument of the log is itself negative.

One can derive (6) by using Green functions. However, here we use a slightly simpler method. If (5) is iterated with the boundary condition (4), the Schrödinger equation (1) is automatically satisfied at every site of the lattice except site $i_0$. If we also require (1) to be satisfied at site $i_0$, we obtain a condition on the energy given by

$$E - \lambda V_{i_0} - \sum_j \frac{1}{R_j} = 0. \tag{7}$$

where the sum is over all the bonds attached to the central site $i_0$. The values $E_\alpha$ of $E$ which satisfy this equation are the eigenenergies of the system.

The difficulty with (7) is that it is not a polynomial in $E$. It is easy to check that the product of all the $R$'s in any given branch is a monic polynomial in $E$ of degree the number of sites in the branch (a polynomial is monic when the coefficient of its highest degree term is one). Therefore if one multplies (7) by the product of all the $R_i$ in the system, one obtains a monic polynomial of degree the number of lattice sites. The roots $E_\alpha$ of this expression are all the eigenenergies, and therefore one has

$$\left( E - \lambda V_{i_0} - \sum_{j=1}^{z} \frac{1}{R_j} \right) \prod_i R_i = \prod_\alpha (E - E_\alpha) \tag{8}$$

where $\alpha$ runs from one to the number of lattice sites, i.e. over all the eigenstates of the system. Then taking the log one ends up with (6). This result holds for any tree structure, regular or irregular. It is easy to check that altering the position of $i_0$ leaves the result unchanged. In the one-dimensional case, i.e. when each site has at most two neighbours, it is a version of the well known Thouless formula (Thouless 1972).

We now consider the Anderson model on a regular tree of coordination number $z$ where each site $i$ has a random site potential $V_i$ drawn from a probability distribution $\rho(V)$. We choose the site $i_0$ to be the centre of the tree. Consequently, the $R_i$ are random variables with probability $P_k(R)$ which depends on the distance $k$ from the bond to the boundary of the tree. A tree of depth $n$ has $z(z-1)^{n-1}$ sites at a distance 1 from the boundary, $z(z-1)^{n-2}$ at a distance 2, ... and one site at the centre. All the $R_i$ at a given distance $k$ to the boundary have the same probability distribution $P_k(R)$ which can be computed by (4), (5)

$$P_1(R) = \int \rho(V)\, \mathrm{d}V \delta(E - \lambda V - R) = \lambda^{-1} \rho\left( \frac{E - R}{\lambda} \right) \tag{9}$$

$$P_{k+1}(R) = \int \prod_{j=1}^{z-1} P_k(R_j)\, \mathrm{d}R_j \int \rho(V)\, \mathrm{d}V\, \delta\left( R - E + \lambda V + \sum_{j=1}^{z-1} \frac{1}{R_j} \right). \tag{10}$$

Then the total integrated density of states $N(E)$ is given by

$$N(E) = \frac{1}{\pi} \lim_{\epsilon \to 0} \operatorname{Im} \left\{ \sum_{k=1}^{n} z(z-1)^{n-k} \int P_k(R)\, \mathrm{d}R\ \log R \right.$$
$$\left. + \int \prod_{j=1}^{z} P_n(R_j)\, \mathrm{d}R_j \int \rho(V)\, \mathrm{d}V\ \log\left(E - \lambda V - \sum_{j=1}^{z} \frac{1}{R_j}\right) \right\}. \tag{11}$$

As is well known, most sites of a tree are close to the boundary. Therefore (11) is dominated by the boundary effects (Eggarter 1974, Müller-Hartmann and Zittartz 1974). However, one can obtain the value of a physical quantity $Q$ for a site far from the boundary by making an appropriate subtraction. If we denote by $Q_n$ the value of some extensive quantity for a tree of depth $n$, one can easily show that $(Q_n - (z-1)Q_{n-1})/2$ will give for large $n$ the value of $Q$ per lattice site far from the boundary. (This can be seen by checking that under the subtraction the contribution of all sites at distance $1, 2, 3, \ldots$ from the boundary has been eliminated, and the factor two is due to the two sites left when one subtracts $z-1$ trees of depth $n-1$ from a tree of depth $n$). In the limit $n \to \infty$, one gets for the average integrated density of states $n(E)$ per site far from the boundary

$$n(E) = \frac{1}{\pi} \lim_{\epsilon \to 0} \operatorname{Im} \left\{ \frac{z}{2} \langle \log R \rangle - \frac{z-2}{2} \left\langle \log\left(E - \lambda V - \sum_{j=1}^{z} \frac{1}{R_j}\right) \right\rangle \right\} \tag{12}$$

where $\langle . \rangle$ denotes an average over $V$ and over the $R$'s, which are all independent. The $R$'s are distributed according to the fixed point distribution $P(R)$ of (10) which is the solution of

$$P(R) = \int \prod_{j=1}^{z-1} P(R_j)\, \mathrm{d}R_j \int \rho(V)\, \mathrm{d}V\ \delta\left(R - E + \lambda V + \sum_{j=1}^{z-1} \frac{1}{R_j}\right). \tag{13}$$

So the problem of calculating the integrated density of states per site far from the boundary is reduced to finding the fixed point distribution $P(R)$ solution of (13). This equation is equivalent to the saddle point equations obtained by replica or supersymmetric methods (Mirlin and Fyodorov 1991a, b, Kim and Harris 1985, Fyodorov *et al* 1992, Rodgers and Bray 1988, Rodgers and De Dominicis 1990). This equivalence can be confirmed by introducing a generating function $g(x)$ defined by

$$g(x) = \int_{-\infty}^{\infty} P(R)\, \mathrm{d}R\ \exp\left(\frac{\mathrm{i}x^2}{2R}\right). \tag{14}$$

Then by using the integral representation of the delta function in (13), one can perform the $R$ integral on the right-hand side and after some algebra, one obtains

$$g(x) = -\int \rho(V)\, \mathrm{d}V \int_0^{\infty} \mathrm{d}y\ J_0(xy) \frac{\partial}{\partial y} \left[ (g(y))^{z-1} \exp\left( -\mathrm{i}(E - \lambda V)\frac{y^2}{2} \right) \right] \tag{15}$$

where the Bessel function $J_0(x)$ is defined by

$$J_0(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}R}{R} \exp\left[ \frac{\mathrm{i}x}{2}\left(R + \frac{1}{R}\right) \right]. \tag{16}$$

This is equivalent to equation (9) of Mirlin and Fyodorov (1991a) apart from the overall minus sign which is probably due to a typographical error.

We are now going to use the recursion (10) and the expression (12) to obtain a weak disorder expansion of the integrated density of states $n(E)$ per site far from the boundary. Here, as before, we integrate the density of states from $E$ to $\infty$. We shall assume that $\langle V \rangle = 0$ (a non-zero average can always be included in the energy $E$). Obviously the leading term in the expansion is obtained by setting $\lambda = 0$. At this stage, it is worth noticing that the limits $k \to \infty$ (when one iterates (10)) and $\epsilon \to 0$ ( where $\epsilon$ is the infinitesimal positive imaginary part of the energy $E$) do not necessarily commute. In fact, if one considers an energy $E$ of the form

$$E = 2\sqrt{z-1}\cos\theta \tag{17}$$

then one can show that if one takes the limit $k \to \infty$ first and then $\epsilon \to 0$, $P(R)$ converges to

$$P(R) = \delta(R - \sqrt{z-1}\mathrm{e}^{\mathrm{i}\theta}) \tag{18}$$

whereas when $\epsilon \to 0$ first, then $R$ remains real and does not converge to any attractive fixed point. Instead, there exists an invariant measure for $R$, which is a solution of (13) for $\lambda = 0$, given by

$$P(R) = \frac{1}{2\pi}\frac{\sqrt{4(z-1)-E^2}}{R^2 - ER + z - 1}. \tag{19}$$

It turns out that if one calculates $n(E)$ in the pure case using either (18) or (19), one obtains a known expression (Kim and Harris 1985)

$$n(E) = \frac{1}{2\pi}\left[z\theta - (z-2)\tan^{-1}\left(\frac{z}{z-2}\tan\theta\right)\right]. \tag{20}$$

One should however remember that the two procedures are different, the correct one being to take the limit $\epsilon \to 0$ first and then the limit $k \to \infty$. Even if the leading order in the weak disorder expansion is not affected by the exchange of limits, it may happen in some cases that expansions obtained by the two methods are different. This is the origin of the anomalies which occur at certain special values of the energy $E$ in the one-dimensional case (Kappus and Wegner 1981, Derrida and Gardner 1984).

Although one should in principle start the expansion with (19), it is much easier to have at zeroth order (18) since the distribution $P(R)$ is a $\delta$-function. In the one-dimensional case, the two expansions are identical (Derrida and Gardner 1984) except at certain energies where anomalies are present and for which the expansion presents some small denominators. So here we shall assume that we can exchange the two limits and leave for a future analysis the study of possible anomalous energies.

Starting from this zero-order result, we can proceed to obtain a systematic weak disorder expansion of $n(E)$ by extending the simplest weak disorder approach of Derrida and Gardner (1984). Instead of solving directly the steady state equation (13) for $P(R)$, we can expand in powers of $\lambda$ all the averages which we need to compute $n(E)$ to a given order in $\lambda$. We know that if we have $z-1$ variables $R_j$ distributed according to the steady state distribution $P(R)$, then the variable $R$ constructed by

$$R = E - \lambda V - \sum_{j=1}^{z-1}\frac{1}{R_j} \tag{21}$$

is also distributed according to $P(R)$, when the $R_j$ and the $V$ are independent. Therefore if one writes

$$R = A\exp(\lambda B + \lambda^2 C + \lambda^3 D + \cdots) \tag{22}$$

and

$$R_j = A\exp(\lambda B_j + \lambda^2 C_j + \lambda^3 D_j + \cdots) \tag{23}$$

where the $B, C, D, \ldots$ are random variables independent of $\lambda$, one obtains by equating both sides of (21) order by order in $\lambda$

$$\begin{aligned} A^2 &= AE - (z-1) \\ A^2 B &= -AV + \sum_j B_j \\ A^2\left(C + \frac{B^2}{2}\right) &= \sum_j \left(C_j - \frac{B_j^2}{2}\right) \\ &\vdots \end{aligned} \tag{24}$$

At the fixed point all the averages are equal and one finds that if $\theta$ is defined as in (17), then

$$\begin{aligned} A &= \sqrt{z-1}\,\mathrm{e}^{\mathrm{i}\theta} \\ \langle B\rangle &= 0 \\ \langle B^2\rangle &= \frac{A^2}{A^4 - (z-1)}\langle V^2\rangle \\ \langle C\rangle &= -\frac{A^2}{A^4-(z-1)}\frac{A^2+(z-1)}{A^2-(z-1)}\frac{\langle V^2\rangle}{2} \\ &\vdots \end{aligned} \tag{25}$$

Then by using these expressions in (12), and once more the fact that the $R$'s and $V$ are independent, one finds for $n(E)$

$$n(E) = \frac{1}{\pi}\mathrm{Im}\left\{(z-1)\log A - \frac{z-2}{2}\log(A^2-1) - \lambda^2\langle V^2\rangle\frac{A^2}{2(A^2-1)^2} + \mathrm{O}(\lambda^3)\right\} \tag{26}$$

which gives in terms of the variable $\theta$ defined in (17)

$$\begin{aligned} n(E) = \frac{1}{2\pi}\Bigg[&z\theta - (z-2)\tan^{-1}\left(\frac{z}{z-2}\tan\theta\right) \\ &+ \lambda^2\langle V^2\rangle\frac{z(z-1)(z-2)\sin(2\theta)}{[(z-1)^2+1-2(z-1)\cos(2\theta)]^2}\Bigg] + \mathrm{O}(\lambda^3). \end{aligned} \tag{27}$$

Several extensions and generalizations of the present work seem possible. First, one could try to push our weak disorder expansion to higher orders in $\lambda$ to see whether this expansion is singular (due to small denominators) at certain special values of the energy, as it is known to be the case in one dimension (Derrida and Gardner 1984).

One could also try to consider the case where the randomness is on the bonds rather than on the sites, in which case the Schrödinger equation would become

$$E\psi_i = \sum_j t_{ij}(\psi_j - \psi_i) \tag{28}$$

with random $t_{ij}$.

The other type of mean field model for which our approach can work is the case of diluted lattices where any two of the $N$ sites are connected with probability $p/N$ and disconnected with probability $1 - p/N$. This model can be dealt with using the same approach as above, except that the equation (13) should be replaced by

$$P(R) = \sum_{n=0}^{\infty} \frac{p^n e^{-p}}{n!} \int \prod_{j=1}^{n} P(R_j)\,dR_j \int \rho(V)\,dV\,\delta\left(R - E + \lambda V + \sum_{j=1}^{n} \frac{1}{R_j}\right) \tag{29}$$

where for $n = 0$, the summation within the $\delta$-function should be set to zero.

Lastly, let us mention that we would like to extend our approach to calculate other properties of interest for the localization model, for example the mobility edge or the inverse participation ratio.

GJR would like to thank the Service de Physique Théorique at Saclay for their hospitality while this work was started. BD thanks the Department of Theoretical Physics at Oxford University for their kind hospitality while it was completed.

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