SPIN GLASSES, RANDOM BOOLEAN NETWORKS AND SIMPLE MODELS OF EVOLUTION

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The organization of many systems in physics and in biology appears random and non-self-averaging even at a macroscopic level. In statistical physics, phase space of spin glasses is a random energy landscape which can be decomposed into valleys, the weights of which remain sample dependent even in the thermodynamic limit. Similarly, in biology, the genetic distances between the individuals of a same population allow to decompose this population into families whose relative sizes fluctuate in time. The goal of this talk[a] is to show that non-self-averaging effects are present in a large variety of systems and that similar quantities can be used to describe qualitatively and quantitatively these fluctuations.

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1 Introduction

Replica theory predicts the existence of non-self-averaging effects in the low temperature of spin glasses [1,2,3]. This means that for mean field spin glasses, phase space can be decomposed into valleys (or pure states), the weights of which remain sample dependent even in the thermodynamic limit (i.e. when the system size becomes infinite). For spin glasses, the weight W_α of a valley α is in principle the probability, at thermal equilibrium, of finding the system in this valley. Unfortunately, a precise definition of the W_α is rather difficult, because, due to thermal fluctuations, the notion of valley in a random landscape is rather vague. There exists, however, a precise and simple prescription (see (7) below) to calculate the moments of the W_α

$$Y_p = \sum_\alpha (W_\alpha)^p \tag{1}$$

where in (1) the sum runs over all the pure states α. The fact that these moments Y_p have non-trivial probability distributions, even in the thermodynamic limit, appears as a signature of the presence of non-self-averaging effects in the spin glass phase. The whole probability distribution of the Y_p was first calculated for mean field spin glasses [4,5] by using the replica approach [2,3,6] and the results can be confirmed by a direct calculation for simple enough spin glass models like the random energy model [7,8].

One can define quantities similar to the Y_p in many systems other than spin glasses. Each time a structure looks random and fluctuates with the sample considered, one can try to define the weights W_α of the components which compose the structure and study the statistical properties of the Y_p defined by (1). We will see below that the W_α can be the relative sizes of the basins of attraction for the evolution of some complex system or the relative sizes of families in some simple model of an evolving population.

2 Mean field spin glasses

The most studied example of a mean field spin glass is the Sherrington Kirkpatrick model[4,5] for which the replica approach was initially designed[1,2,3].

The Sherrington Kirkpatrick model describes a system of N Ising spins $S_i = \pm 1$ which interact with random long-range interactions. The system has 2^N possible spin configurations and the energy E_a of a configuration $a \equiv \{S_i^a\}$ is given by

$$E_a = - \sum_{1 \le i < j \le N} J_{ij} \; S_i^a S_j^a \; . \tag{2}$$

In the Sherrington Kirkpatrick model, for each pair ij of spins, there is a random interaction J_{ij} chosen according to

$$\rho(J_{ij}) = \sqrt{\frac{N-1}{2\pi}} \exp\left(- \frac{(N-1) \; J_{ij}^2}{2} \right) \; . \tag{3}$$

A given sample corresponds to a random choice of the interactions $\{J_{ij}\}$ and the partition function of each sample is given by

$$Z(\{J_{ij}\}) = \sum_{\alpha=1}^{2^N} \exp\left(- \frac{E_a}{T} \right) \tag{4}$$

where T is the temperature.

One difficulty with spin glasses as well as with many other strongly disordered systems is that the landscape is both random and rugged. The simplest way to deal with such a landscape is to compare the properties of several copies (or replica) of the system evolving in the same landscape.

To do so one defines for any pair of configurations $\{S_i^a\}$ and $\{S_i^b\}$ their overlap[9]

$$q_{ab} = \frac{1}{N} \sum_{i=1}^{N} S_i^a S_i^b \tag{5}$$

At thermal equilibrium, for a given realization of the interactions $\{J_{ij}\}$, the distribution $P(q)$ of the overlap q is given by

$$P(q) = \frac{1}{Z^2} \sum_{\{S_i^a\}} \sum_{\{S_i^b\}} \delta(q - q_{ab}) \ \exp\left[- \sum_{i,j} \frac{J_{ij}(S_i^a S_j^a + S_i^b S_j^b)}{T} \right] \tag{6}$$

The replica theory predicts [3] that $P(q)$ fluctuates even in the thermodynamic limit, meaning that

$$\langle P(q) \rangle \neq \langle P(q) \rangle^2$$

where $\langle . \rangle$ denotes an average over the set of interactions $\{J_{ij}\}$.

Overlaps give also a simple prescription to define the moments Y_p of the weights W_α of the valleys. By choosing a reference value Q of the overlap, one can define two configurations a and b to be in the same valley (one should say Q-valley) if their overlap $q_{ab} > Q$. (This definition is, strictly speaking, not transitive in the sense that it is possible to find 3 configurations a, b, c such that $q_{ab} > Q$, $q_{ac} > Q$ but $q_{bc} < Q$. It is believed however, as a prediction of the replica approach [2,3], that the probability of such events vanishes in the thermodynamic limit.)

This allows one to define $Y \equiv Y_2$ by

$$\int_Q^1 P(q) dq = \sum_\alpha (W_\alpha)^2 = Y_2 \tag{7}$$

The left hand side of (7) is, by definition, the probability that, at equilibrium, two configurations belong to the same Q-valley. On the other hand, as $(W_\alpha)^2$ is the probability of finding two configurations in valley α, it is clear that $Y_2 = \sum (W_\alpha)^2$ is also the probability of finding two configurations in the same valley. Thus the definitions (6,7) can in principle be used to calculate $Y \equiv Y_2$ for any system size and any realization of the interactions $\{J_{ij}\}$.

The replica theory predicts [2,3] that in the thermodynamic limit, Y remains a fluctuating quantity distributed according to a distribution $\Pi(Y)$ which belongs to a one parameter family indexed by the first moment $\langle Y \rangle$

$$\langle Y \rangle = \int_0^1 \Pi(Y) \ Y \ dY$$

The parameter $\langle Y \rangle$ is in general a complicated function of the temperature and of the details of the model, but once $\langle Y \rangle$ is known, it determines the whole distribution $\Pi(Y)$. For example [2,3]

$$\langle Y^2 \rangle = \frac{\langle Y \rangle + 2 \langle Y \rangle^2}{3} \tag{8}$$

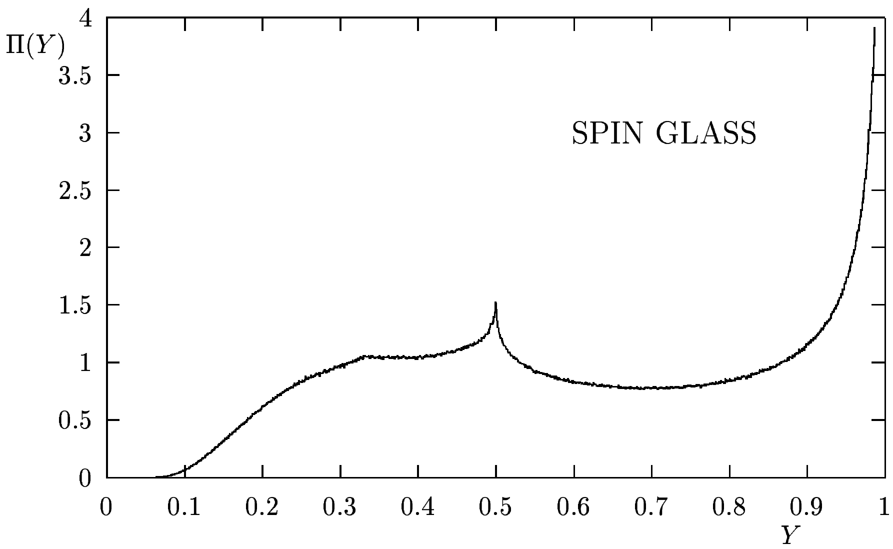

Figure 1: Distribution $\Pi(Y)$ for the spin glass problem when $\langle Y \rangle = \frac{2}{3}$.

$$\langle Y^3 \rangle = \frac{3\langle Y \rangle + 7\langle Y \rangle^2 + 5\langle Y \rangle^3}{15} \tag{9}$$

The shape of $\Pi(Y)$ predicted by the replica approach [6,2] is shown in figure 1 when $\langle Y \rangle = 2/3$. In fact the moments of all the Y_p defined by (1) can also be calculated from the replica approach, once $\langle Y \rangle$ is known [2,8]. For example

$$\langle Y_3 \rangle = \frac{\langle Y \rangle + \langle Y \rangle^2}{2} \tag{10}$$

$$\langle Y_4 \rangle = \frac{2\langle Y \rangle + 3\langle Y \rangle^2 + \langle Y \rangle^3}{6} \tag{11}$$

and so on.

It has not been possible so far to verify directly the validity of all these predictions (8-11) obtained from the replica approach except in the case of the random energy models [7,8,10] for which they turn out to be correct.

3 Random boolean networks and random map models

In 1986, with Henrik Flyvbjerg, we tried to see whether non-self-averaging effects similar to those predicted of spin glasses could be present in other systems [11].

We considered the problem of random boolean networks which was introduced by Kauffman in the context of cell differenciation [12]. The Kauffman model describes a system of N binary spins ($\sigma_i = 0$ or 1) evolving according to a deterministic rule. The time evolution is determined by N boolean functions f_i of K variables and by K input sites $j_1(i), j_2(i)...j_K(i)$ for each site i (K is a parameter of the model).

In the Kauffman model, for each site i, the input sites $j_1(i), j_2(i)...j_K(i)$ and the boolean functions f_i are choosen at random (each input site is randomly chosen among the N sites and each boolean function is randomly chosen among the 2^{2^K} boolean functions of K variables). Then the time evolution of the spins σ_i is given by

$$\sigma_i(t+1) = f_i\left(\sigma_{j_1(i)}(t), ...\sigma_{j_K(i)}(t)\right)$$

Since the system is deterministic (as the functions f_i and the input sites $j_1(i), j_2(i)..j_K(i)$ do not change with time), the evolution of any configuration ends up by being periodic. One can then define the size Ω_α of an attractor α as the number of initial conditions which converge to this attractor and the weight W_α of the attractor as the fraction of phase space belonging to the basin of the attractor α

$$W_\alpha = \frac{\Omega_\alpha}{2^N}$$

Once the W_α are defined, one can calculate their moments Y_p using (1). Clearly the W_α and the Y_p depend on the particular random realization of the functions f_i and of the input sites $j_1(i), j_2(i)..j_K(i)$. Thus they fluctuate. In [11] we observed numerically that even for large systems, the distribution of the Y_p remains broad. When we tried to measure the moments of the Y_p they even seemed to satisfy the same relations (8-11) as spin glasses. So far, it has not been possible however to develop a theory (similar to the replica theory of spin glasses) allowing to calculate the distribution of the Y_p. Only, in some limit (when the number of inputs $K \to \infty$) of the Kauffman model, the problem becomes a random map model [13] making possible the calculation of the distribution of all the Y_p. For a random map model, one finds [13] for the first two moments of $Y \equiv Y_2$

$$\langle Y \rangle = \frac{2}{3} \quad ; \quad \langle Y^2 \rangle = \frac{52}{105} \tag{12}$$

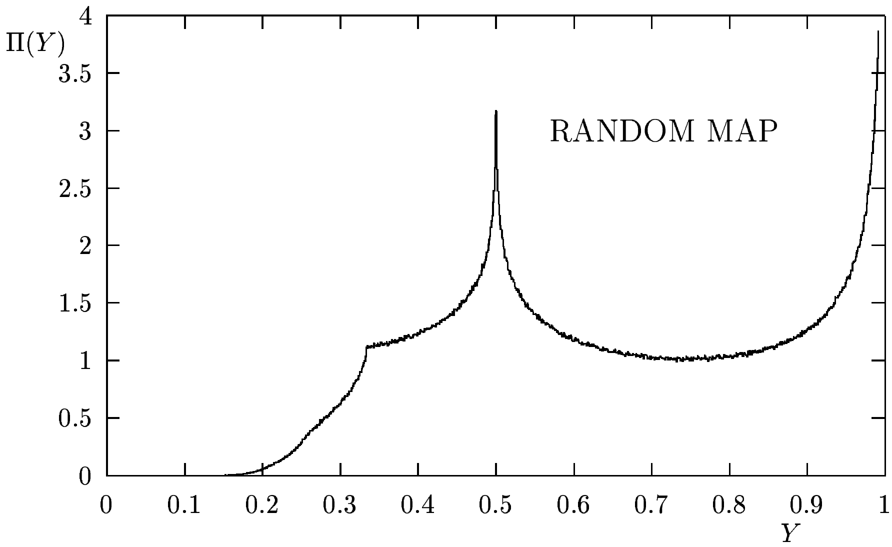

Figure 2: Distribution $\Pi(Y)$ for the random map model.

and for the Y_p with $p > 2$

$$\langle Y_3 \rangle = \frac{8}{15} \quad ; \quad \langle Y_4 \rangle = \frac{16}{35} \tag{13}$$

These results do not satisfy the relations (8-11) predicted by the replica approach for spin glasses. The random map model being a special case ($K = \infty$) of the Kauffmann model, the relations (8-11) have therefore no reason to be valid for other values of K in the Kauffman model.

In fact the difference can be seen clearly by comparing the distributions $\Pi(Y)$ of the random map model (figure 2) and of the mean field spin glass (figure 1 with the same first moment $\langle Y \rangle = 2/3$).

For $K \neq \infty$ (and $K \neq 1$ [14]), the moments Y_p have so far been only determined by numerical simulations. Finding an analytic approach to determine the distribution of these Y_p for general K in the Kauffman model remains a challenging theoretical problem.

4 A simple model of evolution

A third example of systems for which the distribution $\Pi(Y)$ is non-trivial are simple models of an evolving population. With Luca Peliti, we considered a very simple model [15] for the evolution of the genetic diversity of a population. The size of the population remains fixed in time (M individuals) and each individual a (with $1 \leq a \leq M$) at time t has a genome of N binary genes: $S_1^a(t) = \pm 1, ... S_N^a(t) = \pm 1$. The rule of evolution of the population is that, at each generation $t \to t+1$, all the M individuals are replaced by M new individuals and each individual a at time $t+1$ has its parent $G(a)$ chosen at random among the M individuals of the previous generation (i.e. in the population at time t). Then the genes of a at time $t+1$ are idendical to those of its parent $G(a)$ except for mutations wich occur with a small probability

$$S_i^a(t+1) = \begin{cases} S_i^{G(a)}(t) & \text{with probability } (1+e^{-2\mu})/2 \\ -S_i^{G(a)}(t) & \text{with probability } (1-e^{-2\mu})/2 \end{cases} \tag{14}$$

(the parameter μ is small).

The model is a model of asexual reproduction in a flat fitness landscape. Nevertheless, we will see that it leads to non-self-averaging effects similar to those of spin glasses.

Clearly, the evolution rule gives rise to a random genealogical tree. The random structure of this tree can be analysed by considering at each generation the weights W_α of the families: by choosing a time interval T of reference and by saying that two individuals at time t belong to the same family if they had a common ancestor at time $t-T$, one can decompose the population, at time t, into families and define the weight W_α as the fraction of the total population having the same ancestor at time $t-T$. Obviously, the weights W_α depend on the structure of the random genealogical tree produced by the evolution and on the choice of the time interval T.

It turns out that, in the limit of a very large population, the weights W_α and their moments Y_p defined by (1) fluctuate, if one chooses the right time scale for T (i.e. if T is of order M). If one defines τ as

$$\tau = \frac{T}{M}$$

and if Y_2 is the probability that 2 individuals belong to the same family, one finds that for the first two moments of $Y_2 \equiv Y$

$$\langle Y \rangle = 1 - e^{-\tau} \tag{15}$$

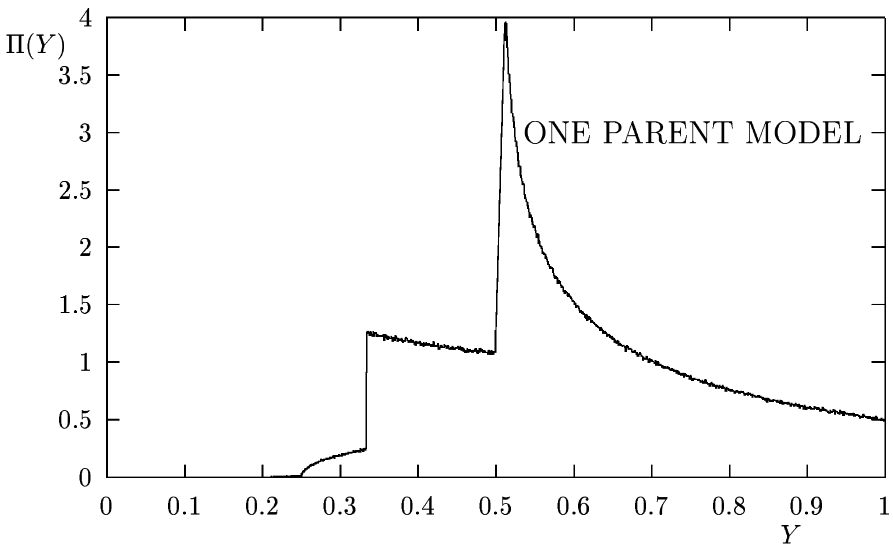

Figure 3: Distribution $\Pi(Y)$ for the evolution model when $\langle Y \rangle = \frac{2}{3}$.

$$\langle Y^2 \rangle = \frac{15\langle Y \rangle^2 + 10\langle Y \rangle^3 - 15\langle Y \rangle^4 + 6\langle Y \rangle^5 - \langle Y \rangle^6}{15} \tag{16}$$

One can also determine the τ-dependence of all the moments of the Y_p (Y_p is the probability that p individuals belong to the same family). For example one finds that

$$\langle Y_3 \rangle = \frac{3\langle Y \rangle^2 - \langle Y \rangle^3}{2} \tag{17}$$

$$\langle Y_4 \rangle = \frac{15\langle Y \rangle^3 - 15\langle Y \rangle^4 + 6\langle Y \rangle^5 - \langle Y \rangle^6}{5} \tag{18}$$

where $\langle . \rangle$ denotes an average over the history (i.e. over time t). These relations differ from (8-11) of spin glasses and from (12-13) of the random map model. All the moments of the Y_p can be calculated [16,15] and the distribution $\Pi(Y)$ for $\langle Y \rangle = 2/3$ is shown in figure 3. The difference of shape with spin glasses (figure 1) or the random map model (figure 2) is clearly visible.

As for spin glasses (5), one can define the overlap q_{ab} between two individuals a and b at time t as

$$q_{ab} = \frac{1}{N} \sum_{i=1}^{N} S_i^a S_i^b \tag{19}$$

and consider at any time t the distribution $P(q)$ of overlaps

$$P(q) = \frac{1}{M^2} \sum_{a=1}^{M} \sum_{b=1}^{M} \delta(q - q_{ab}) \tag{20}$$

As in spin glasses, the distribution $P(q)$ remains broad and non-self-averaging [15] ($\langle P(q) \rangle \neq \langle P(q) \rangle^2$) even in the limit of a very large population ($M \to \infty$).

The calculation of the statistical properties of $P(q)$ is simpler in the limit of a very long genome ($N \to \infty$). In this limit, the matrix of overlaps evolves according to [15]

$$q_{ab}(t+1) = e^{-4\mu} \; q_{G(a)G(b)}(t) \qquad \text{if} \quad a \neq b \tag{21}$$

with

$$q_{aa}(t+1) = 1 \tag{22}$$

All the fluctuations in the matrix q_{ab} are due to the random choice (21,22) of the ancestors $G(a)$ at each generation.

One finds [15] that for the evolution rule (21,22), the average of $P(q)$ is given by

$$\langle P(q) \rangle = \lambda q^{\lambda - 1} \tag{23}$$

where $\lambda = (4\mu M)^{-1}$.

It is quite remarkable that a rule as simple for the evolution of the matrix q_{ab} as (21,22) is sufficient to produce non-self-averaging effects similar to those predicted for spin glasses by the replica theory, with a non-trivial $\Pi(Y)$ and a broad and fluctuating $P(q)$.

5 Conclusion

This lecture was an attempt to show that very similar non-self-averaging effects occur in a variety of systems (spin glasses, random networks of automata, simple models of evolution). Similar non-self-averaging effects are present in a large class of systems (sums of random variables, random walks [8], randomly broken objects [6,17], asymmetric spin models [18,19,20], genetics [21]). The simplest quantities which exhibit non-self-averaging effects seem to be the Y_p defined by (1). It has been shown in a number of examples [6,8,17] that the probability distributions of the Y_p have singularities at some simple rational values. For example $\Pi(Y)$ shown in figures 1,2 and 3 is singular [6,17] at all the values $1/n$ with weaker and weaker singularities with increasing n. Higgs in [21] shows that a distribution similar to those of figures 1,2 and 3 is consistent with observed histograms of homozygosity in Drosophila.

From the theoretical point of view, it would be interesting to develop theoretical tools to calculate the non-self-averaging properties of random boolean networks (see section 2) or of more sophisticated models of evolution [22].

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