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The Asymmetric Exclusion Process and Brownian Excursions*

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We consider the totally asymmetric exclusion process (TASEP) in one dimension in its maximal current phase. We show, by an exact calculation, that the non-Gaussian part of the fluctuations of density can be described in terms of the statistical properties of a Brownian excursion. Numerical simulations indicate that the description in terms of a Brownian excursion remains valid for more general one dimensional driven systems in their maximal current phase.

KEY WORDS: Density fluctuations; asymmetric simple exclusion process; open system; stationary nonequilibrium state; matrix method; Brownian excursion.

1. INTRODUCTION

Exclusion processes[(1–3)] with open boundaries have attracted much attention as simple models of an open non-equilibrium system in contact with two reservoirs having different chemical potentials.[(4–6)] Despite their simplicity, these models exhibit properties believed to be characteristic of realistic non-equilibrium systems, such as long range correlations[(7–11)] and phase transitions in one dimension.[(4–6)]

A number of exact results have been obtained for the one dimensional exclusion process with open boundaries, using the fact that the weights of the microscopic configurations in the stationary state can be calculated

* Dedicated to Gianni Jona-Lasinio, who was and continues to be a pioneer in this field, on the occasion of his 70th birthday.

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exactly.[5, 6, 12–17] The goal of the present paper is to show that the density fluctuations of the totally asymmetric exclusion process in its stationary maximal current phase can be expressed in terms of the statistical properties of a Brownian excursion.

A Brownian excursion[18] is a Brownian path $Y(x)$ conditioned to remain positive, i.e., such that $Y(x)>0$ for $0<x<1$, with the boundary condition $Y(0)=Y(1)=0$. In the following we will consider the Brownian excursion $Y(x)$ normalized in such a way that the probability density of being at heights $y_1, y_2,..., y_k$ at times $0<x_1<x_2<\cdots<x_k<1$ (i.e., that $Y(x_1)=y_1,..., Y(x_k)=y_k$) is given by

$$\text{Prob}(y_1,..., y_k; x_1,..., x_k)$$
$$=\frac{1}{\sqrt{\pi}}h(y_1; x_1)\, h(y_k; 1-x_k) \prod_{p=1}^{k-1} g(y_p, y_{p+1}; x_{p+1}-x_p), \tag{1.1}$$

where the functions h and g are defined by

$$h(y; x)=\frac{2y}{x^{3/2}}\, e^{-\frac{y^2}{x}} \tag{1.2}$$

and

$$g(y, y'; x)=\frac{1}{\sqrt{\pi x}}\,[e^{-\frac{(y-y')^2}{x}}-e^{-\frac{(y+y')^2}{x}}]. \tag{1.3}$$

One can check that (1.1) is normalized, i.e., that

$$\int_0^\infty dy_1 \cdots \int_0^\infty dy_k \,\text{Prob}(y_1,..., y_k; x_1,..., x_k)=1,$$

using the identities

$$\frac{1}{\sqrt{\pi}}\int_0^\infty dy\, h(y; x)\, h(y; 1-x)=1,$$

$$\int_0^\infty dy'\, h(y'; x)\, g(y', y; x')=h(y; x+x'),$$

$$\int_0^\infty dy''\, g(y, y''; x)\, g(y'', y'; x')=g(y, y'; x+x').$$

The one dimensional totally asymmetric simple exclusion process (TASEP) with open boundary conditions is defined as follows: each site i

(with $1 \leqslant i \leqslant L$) of a one dimensional lattice of L sites is either occupied by a single particle or empty, and the system evolves according to the following continuous time dynamics: if a particle is present on site i (for $1 \leqslant i \leqslant L-1$), it hops at rate 1 to site $i+1$ if this site is empty. At the left boundary, site 1 is filled at rate α by a particle if it is empty. At the right boundary, if a particle is present at site L, it is removed at rate β. Each microscopic configuration can be described by a set of L binary variables τ_i, the occupation numbers ($\tau_i = 1$ if site i is occupied and $\tau_i = 0$ if it is empty). When α and β lie in the interval (0, 1), the case we shall be concerned with here, the input and exit rates α and β can be thought as resulting from the system being in contact with a left and and a right reservoir at densities α and $1-\beta$ respectively.[11] In the large L limit, the system exhibits phase transitions[4, 12, 19] as α and β vary: a low density phase for $\alpha < \min(\frac{1}{2}, \beta)$ (where the bulk density is α), a high density phase for $\beta < \min(\frac{1}{2}, \alpha)$ (where the bulk density is $1-\beta$) and a maximal current phase for $\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$ (where the bulk density is $\frac{1}{2}$).

In the steady state, which is unique for such a Markov process, one can try to determine correlation functions, which we will denote $\langle \tau_{i_1} \tau_{i_2} \cdots \tau_{i_k} \rangle$. One can also divide the L sites into k boxes of $L_1, L_2,..., L_k$ sites and try to determine the probability that in the steady state N_1 particles are present in the first box, N_2 in the second box,... N_k in the kth box.

In the present paper we are going to show that in the maximal current phase of the stationary TASEP, corresponding to

$$\alpha > \tfrac{1}{2} \qquad \text{and} \qquad \beta > \tfrac{1}{2}, \tag{1.4}$$

(where the bulk density is equal to $\frac{1}{2}$[4, 11, 12, 19]), the correlation functions of the occupation numbers τ_i in the steady state are given for large L by

$$\left\langle \left(\tau_{i_1} - \frac{1}{2}\right)\left(\tau_{i_2} - \frac{1}{2}\right) \cdots \left(\tau_{i_k} - \frac{1}{2}\right) \right\rangle \simeq \frac{1}{2^k L^{\frac{k}{2}}} \frac{d^k}{dx_1\, dx_2 \cdots dx_k} \overline{Y(x_1) \cdots Y(x_k)}, \tag{1.5}$$

where $i_1 < i_2 < \cdots < i_k$ and the right hand side is to be evaluated at $x_p = i_p / L$. The averages are taken with respect to (1.1).

Our second result concerns the fluctuations of the numbers $N_1,..., N_k$ of particles in boxes of length $L_1 = L(x_1 - x_0)$, $L_2 = L(x_2 - x_1),...,$ $L_k = L(x_k - x_{k-1})$ with $x_0 = 0 < x_1 < \cdots < x_{k-1} < x_k = 1$. Define μ_p to be the rescaled fluctuations of the number of particles, in box p,

$$\mu_p = \frac{N_p - L(x_p - x_{p-1})/2}{\sqrt{L}}. \tag{1.6}$$

We are going to show that their probability density $Q(\mu_1,..., \mu_k; x_1,..., x_{k-1})$ is given for large L by

$$Q(\mu_1,..., \mu_k; x_1,..., x_{k-1}) = \int_0^\infty dy_1 \cdots \int_0^\infty dy_{k-1} \,\mathrm{Prob}(y_1,..., y_{k-1}; x_1,..., x_{k-1}) \times \prod_{i=1}^{k} \frac{2}{\sqrt{\pi(x_i - x_{i-1})}} \exp\left[\frac{-(2\mu_i + y_{i-1} - y_i)^2}{x_i - x_{i-1}}\right] \tag{1.7}$$

where $y_0 = y_k = 0$, $x_0 = 0$, $x_k = 1$.

As we shall see in Section 2 the product in the integrand of (1.7) is just the conditional probability of $\mu_1,..., \mu_k$ given the values of the Y process, $Y(x_1) = y_1,..., Y(x_{k-1}) = y_{k-1}$. This conditional probability is just a product of Gaussians with means $(y_i - y_{i-1})/2$ and variances $(x_i - x_{i-1})/8$. Since (1.7) is valid for arbitrary number and sizes of boxes it is equivalent to the statement that the "fluctuation field" of the particle density $\rho(x)$ in the maximum current phase can be written as a sum of two independent processes,

$$\sqrt{L}\,[\rho(x) - \tfrac{1}{2}] \simeq \tfrac{1}{2}\,[\dot{B}(x) + \dot{Y}(x)]. \tag{1.8}$$

Here $\rho(x)$ is the empirical density at x, defined by

$$\sqrt{L} \int_{x_{p-1}}^{x_p} \left(\rho(x) - \frac{1}{2}\right) dx \simeq \frac{1}{\sqrt{L}} \sum_{i = x_{p-1}L}^{x_p L} \left(\tau_i - \frac{1}{2}\right) = \mu_p,$$

while $\dot{Y}(x)$ is the (generalized) derivative of the Brownian excursion described by (1.1), $\dot{B}(x)$ is a white noise, the derivative of a Brownian path, normalized so that

$$\overline{[B(x) - B(x')]^2} = \tfrac{1}{2}\,|x - x'|, \tag{1.9}$$

and B and Y are independent.

The fact that the fluctuations in $\rho(x)$ given in (1.8) are a superposition of the $\dot{B}$ and $\dot{Y}$ processes has its origin in the fact, see Section 2.5, that the probability of a microscopic configuration $\{\tau_i\}$ can be obtained by first considering an excursion type random walk giving rise to Y and then using a product measure for the $\{\tau_i\}$ conditioned on the walk.

For the integrated fluctuation $r(x, x')$ of the density in the macroscopic segment (x, x'),

$$r(x, x') = \int_x^{x'} dy \left(\rho(y) - \frac{1}{2}\right) \simeq \frac{B(x') + Y(x') - B(x) - Y(x)}{2\sqrt{L}}, \tag{1.10}$$

one has

$$\overline{r(x, x')^2} \simeq \frac{\overline{[B(x)-B(x')]^2}+\overline{[Y(x)-Y(x')]^2}}{4L}. \tag{1.11}$$

If one considers now the fluctuation in a very small segment away from the end points, then $Y(x')-Y(x)$ behaves like $B(x')-B(x)$ and so

$$\overline{r(x, x')^2} \simeq \frac{|x'-x|}{4L} \qquad \text{for} \quad |x-x'| \to 0,$$

indicating that locally the measure is Bernoulli. On the other hand if one considers the particle number fluctuation in the whole system

$$\overline{r(0, 1)^2} \simeq \frac{1}{8L},$$

which means that the fluctuation of the total number of particles is one half of what it would be for a Bernoulli measure at density 1/2.[8]

One can check that (1.5) extends to arbitrary correlations what was already known for the one-point and two-point functions (Eqs. (47) and (52) of ref. 8) when $\alpha = \beta = 1$. Also (1.7) extends to an arbitrary number of boxes the result (6.15) of ref. 11 valid for a single box.

Our derivation presented in Section 2 is a generalization of the method used in ref. 11. In particular, in Section 2.5, we describe the microscopic (or discrete) version of (1.8) which allows to generate random steady state configurations of the TASEP by generating random walks constrained to remain positive. Numerical simulations reported in Section 3 indicate that the description in terms of a Brownian excursion remains valid in the maximal current phase of other models for which the steady state is not known exactly.

2. DERIVATION OF (1.5) AND (1.7)

2.1. The Matrix Method

For the steady state of the open TASEP described in Section 1, the probability $P(\{\tau_i\})$ of any microscopic configuration $\{\tau_i\}$ can be written as[12]

$$P(\{\tau_i\}) = \frac{\langle W| \prod_{i=1}^{N} [D\tau_i + E(1-\tau_i)] |V\rangle}{Z_L}, \tag{2.1}$$

where the normalization factor Z_L is given by

$$Z_L = \langle W| \, (D+E)^L \, |V\rangle, \tag{2.2}$$

and the matrices D, E and the vectors $\langle W|$, $|V\rangle$ satisfy the relations,

$$DE = D+E, \tag{2.3}$$

$$\beta D \, |V\rangle = |V\rangle, \tag{2.4}$$

$$\langle W| \, \alpha E = \langle W|. \tag{2.5}$$

From the algebra (2.1)–(2.5) all equal time steady state properties can be calculated. For example the average occupation $\langle \tau_i \rangle$ of site i is given by

$$\langle \tau_i \rangle = \frac{\langle W| \, (D+E)^{i-1} \, D(D+E)^{L-i} \, |V\rangle}{Z_L}, \tag{2.6}$$

and the two point function is, for $i < j$,

$$\langle \tau_i \tau_j \rangle = \frac{\langle W| \, (D+E)^{i-1} \, D(D+E)^{j-i-1} \, D(D+E)^{L-j} \, |V\rangle}{Z_L}. \tag{2.7}$$

The probability of finding $N_1,..., N_k$ particles in subsystems of lengths $L_1,..., L_k$ can be written as

$$q_{L_1, L_2 \cdots L_k}(N_1,..., N_k) = \frac{\langle W| \, X_{L_1}(N_1) \, X_{L_2}(N_2) \cdots X_{L_k}(N_k) \, |V\rangle}{Z_L} \tag{2.8}$$

where $X_L(N)$ is the sum over all the products of L matrices containing exactly N matrices D and $L-N$ matrices E. This can be written as

$$X_L(N) = \int_0^1 d\theta (De^{2i\pi\theta} + E)^L \, e^{-2i\pi N\theta}. \tag{2.9}$$

The algebraic rules (2.3)–(2.5) allow one to calculate all the matrix elements appearing in (2.1), (2.2), (2.6)–(2.8) without using any explicit representation of the matrices D and E or of the vectors $\langle W|$ and $|V\rangle$. Working with a particular representation might be convenient but of course the steady state properties, such as correlation functions and current, do not depend on the particular representation used.

To derive the expressions (1.5), (1.7) we find it convenient to use a particular representation of (2.3)–(2.5) (which was already used in Section 6.3 of ref. 11):

$$D = \sum_{n=1}^{\infty} |n\rangle\langle n| + |n\rangle\langle n+1|, \tag{2.10}$$

$$E = \sum_{n=1}^{\infty} |n\rangle\langle n| + |n+1\rangle\langle n|, \tag{2.11}$$

where the vectors $|1\rangle, |2\rangle,..., |n\rangle,...$ form an orthonormal basis of an infinite dimensional space (with $\langle n \mid m\rangle = \delta_{n,m}$). In this basis, the vectors $|V\rangle$ and $\langle W|$ satisfying (2.4), (2.5) are given by

$$|V\rangle = \sum_{n=1}^{\infty} \left(\frac{1-\beta}{\beta}\right)^n |n\rangle, \tag{2.12}$$

$$\langle W| = \sum_{n=1}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^n \langle n|. \tag{2.13}$$

In the following we will assume that

$$1 < \alpha + \beta, \qquad \alpha < 1, \qquad \text{and} \qquad \beta < 1. \tag{2.14}$$

so that $\langle W \mid V\rangle$ and all the matrix elements $\langle W| X_1 X_2 \cdots |V\rangle$ are finite and positive when the matrices $X_1, X_2,...$ are polynomials of matrices D and E with positive coefficients. This condition is not the same as the condition (1.4) of being in the maximal current phase. We will see later (2.26) how condition (1.4) appears.

Note that as long as L is finite, all the matrix elements are rational functions of α and β and so all the expressions derived assuming (2.14) could be analytically continued to the whole range of values of α and β.

2.2. The Sum over Walks

Let us introduce the set of discrete walks w of L steps which, at each step, either increase by one unit, decrease by one unit, or stay constant, with the constraint that they remain positive. Each such walk w can be described by a sequence of $L+1$ integers $\{n_i(w)\}$ satisfying for all $i = 0, 1,..., L$

$$n_i > 0 \qquad \text{and} \qquad |n_i - n_{i-1}| \leqslant 1.$$

To each such walk w, one associates a weight $\Omega(w)$ defined by

$$\Omega(w) = \left(\frac{1-\alpha}{\alpha}\right)^{n_0} \left(\frac{1-\beta}{\beta}\right)^{n_L} \prod_{i=1}^{L} v(n_{i-1}, n_i) \tag{2.15}$$

where $v(n, n')$ is given by

$$v(n, n') = \begin{cases} 2 & \text{if} \quad |n-n'| = 0 \\ 1 & \text{if} \quad |n-n'| = 1 \\ 0 & \text{if} \quad |n-n'| > 1. \end{cases}$$

It is easy to check from (2.10), (2.11) that for $n \geqslant 1$ and $n' \geqslant 1$, one has

$$v(n, n') = \langle n| \; D+E \; |n'\rangle$$

and it follows that

$$\langle W| \; (D+E)^L \; |V\rangle = \sum_w \Omega(w). \tag{2.16}$$

These weights define a measure $\nu(w)$ on the walks w

$$\nu(w) = \frac{\Omega(w)}{\sum_{w'} \Omega(w')}. \tag{2.17}$$

It follows from (2.10) and (2.11) that

$$\langle n| \; D \; |n'\rangle = \frac{(1+n'-n) \; \langle n| \; D+E \; |n'\rangle}{2},$$

which combined with (2.7) yields, for $i < j$,

$$\langle \tau_i \tau_j \rangle = \tfrac{1}{4} \sum_w \nu(w)[1+n_i(w)-n_{i-1}(w)][1+n_j(w)-n_{j-1}(w)]. \tag{2.18}$$

More generally, for $i_1 < i_2 < \cdots < i_k$,

$$\langle \tau_{i_1} \cdots \tau_{i_k} \rangle = \frac{1}{2^k} \sum_w \nu(w)[1+n_{i_1} - n_{i_1-1}] \cdots [1+n_{i_k} - n_{i_k-1}], \tag{2.19}$$

(where to avoid heavy notation we have not repeated the w dependence of all the n_i's). This can be rewritten as

$$\left\langle \left(\tau_{i_1}-\frac{1}{2}\right)\cdots\left(\tau_{i_k}-\frac{1}{2}\right)\right\rangle=\frac{1}{2^k}\sum_w v(w)[n_{i_1}-n_{i_1-1}]\cdots[n_{i_k}-n_{i_k-1}], \quad (2.20)$$

which is the exact finite L version of (1.5).

The expressions of $\langle n|\,(D+E)^L\,|n'\rangle$ and of $\langle n|\,X_L(M)\,|n'\rangle$ defined in (2.9) are known (see Eqs. (6.24) and (6.65) of ref. 11 which had been derived by recursions over L in refs. 20 and 21).

$$\langle n|\,(D+E)^L\,|n'\rangle=\frac{(2L)!}{(L+n-n')!\,(L+n'-n)!}-\frac{(2L)!}{(L+n+n')!\,(L-n'-n)!}, \quad (2.21)$$

$$\langle n|\,X_L(N)\,|n'\rangle=\frac{(L!)^2}{(N)!\,(L-N)!\,(N+n-n')!\,(L-N-n+n')!}-\frac{(L!)^2}{(N+n)!\,(L-N-n)!\,(N-n')!\,(L-N+n')!}, \quad (2.22)$$

where any negative factorial is defined to be infinite (i.e., the matrix elements are non-zero only when $N-L\leqslant n'-n\leqslant N\leqslant L$).

Let

$$F_{L,N}(n,n')=\frac{\langle n|\,X_L(N)\,|n'\rangle}{\langle n|\,(D+E)^L\,|n'\rangle}. \quad (2.23)$$

The probability $q_{L_1,L_2\cdots L_k}(N_1,\ldots,N_k)$ that there are N_1 particles in the first L_1 sites, N_2 in the next L_2 sites, etc., is then given by

$$q_{L_1,L_2\cdots L_k}(N_1,\ldots,N_k)=\sum_w v(w)\prod_{i=1}^{k} F_{L_i,N_i}(n_{M_{i-1}}(w),n_{M_i}(w)) \quad (2.24)$$

where $M_0=0$ and $M_i=M_{i-1}+L_i$. This is the exact finite L version of (1.7). It shows clearly that given w the $\{N_i\}$ are independent random variables.

2.3. Derivation of (1.5)

Let us first evaluate for large L the normalization factor (2.16)

$$\sum_w \Omega(w) = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^n \left(\frac{1-\beta}{\beta}\right)^{n'} \langle n| (D+E)^L |n'\rangle. \tag{2.25}$$

Using (2.21) (or its large L behavior easily obtained by the Stirling formula) one can show that if

$$\alpha > \tfrac{1}{2} \qquad \text{and} \qquad \beta > \tfrac{1}{2}, \tag{2.26}$$

then the walks which dominate the sum for large L are those which have both $n_0(w)$ and $n_L(w)$ of order 1 and which remain at distances of order $L^{1/2}$ from the origin. Condition (2.26) corresponds to the maximal current phase.[12, 19] It is more restrictive than (2.14) which assures only that $\langle W | V\rangle$ is finite. In the range where (2.14) is satisfied but (2.26) is not, the walks which dominate (2.25) are walks such that either $n_0(w)$ or $n_L(w)$ is of order L, and where the Brownian excursion picture does not apply.

The large L expressions of matrix elements of the form (2.21) can be easily obtained using Stirling formula and one gets, if and n and n' are of order $\sqrt{L}$,

$$\langle n| (D+E)^L |n'\rangle \simeq \frac{4^L}{\sqrt{\pi L}} \left[\exp\left(\frac{-(n-n')^2}{L}\right) - \exp\left(\frac{-(n+n')^2}{L}\right)\right]. \tag{2.27}$$

Hence for

$$i_p = Lx_p, \qquad n_{i_p} = L^{1/2} y_p, \tag{2.28}$$

one gets

$$\langle n_{i_p}| (D+E)^{i_{p+1}-i_p} |n_{i_{p+1}}\rangle \simeq \frac{4^{L(x_{p+1}-x_p)}}{\sqrt{L}} g(y_p, y_{p+1}; x_{p+1}-x_p)$$

where g is defined as in (1.3). This formula remains valid even if n_{i_p} and/or $n_{i_{p+1}}$ are of order 1. For example one obtains that way that if n_{i_p} is of order 1 and $n_{i_{p+1}} \simeq \sqrt{L}\, y_{p+1}$,

$$\langle n_{i_p}| (D+E)^{i_{p+1}-i_p} |n_{i_{p+1}}\rangle \simeq \frac{4^{L(x_{p+1}-x_p)}}{\sqrt{\pi\, L}} 2n_{i_p} h(y_{p+1}; x_{p+1}-x_p)$$

so that (2.25) and (2.27) give

$$\langle W| (D+E)^L |V\rangle \simeq \frac{4^{L+1}}{\sqrt{\pi}\, L^{3/2}} \frac{(1-\alpha)\,\alpha}{(2\alpha-1)^2} \frac{(1-\beta)\,\beta}{(2\beta-1)^2}. \tag{2.29}$$

The correlation function $\overline{n_{i_1} \cdots n_{i_k}}$ of the heights of a walk w at positions i_1,..., i_k is then given by

$$\overline{n_{i_1} \cdots n_{i_k}} = \sum_{n_0, n_{i_1},..., n_{i_k}, n_L} n_{i_1} \cdots n_{i_k} \times \frac{\langle n_0| (D+E)^{i_1} |n_{i_1}\rangle \cdots \langle n_{i_k}| (D+E)^{L-i_k} |n_L\rangle (\frac{1-\alpha}{\alpha})^{n_0} (\frac{1-\beta}{\beta})^{n_L}}{\sum_w \Omega_w},$$

and for $i_1 = Lx_1$,..., $i_k = Lx_k$ one gets

$$\overline{n_{i_1} \cdots n_{i_k}} \simeq L^{k/2}\, \overline{Y(x_1) \cdots Y(x_k)}$$

which using (2.20) leads to (1.5).

2.4. Derivation of (1.7)

For large L, with n, n' and $N-L/2$ of order $\sqrt{L}$, one can easily see from (2.22) that

$$\langle n| X_L(N) |n'\rangle \simeq 2 \frac{4^L}{\pi L} [e^{-2\frac{(\Delta N)^2}{L}-2\frac{(\Delta N+n-n')^2}{L}} - e^{-2\frac{(\Delta N+n)^2}{L}-2\frac{(\Delta N-n')^2}{L}}],$$

where $\Delta N = N-L/2$, which can be rewritten as

$$\langle n| X_L(N) |n'\rangle \simeq 2 \frac{4^L}{\pi L} e^{\frac{-(2\Delta N+n-n')^2}{L}} [e^{-\frac{(n-n')^2}{L}} - e^{-\frac{(n+n')^2}{L}}].$$

Thus $F_{L,N}(n, n')$ defined in (2.23) becomes for n, n' and ΔN of order $\sqrt{L}$

$$F_{L,N}(n, n') \simeq \frac{2}{\sqrt{\pi L}} \exp\left[\frac{-(2\,\Delta N+n-n')^2}{L}\right],$$

and this shows that in the large L limit (2.24) reduces to (1.7).

2.5. Origin of (1.8)

As already noted in Section 1, expression (1.8) is essentially equivalent to (1.7) so the derivation of (1.7) above also gives (1.8). It is interesting, however, to understand the origin of (1.8) directly from the microscopic picture involving the walks w. To do that let us define the joint distribution $\tilde{v}(w, \tau)$ of w and $\tau = \{\tau_i\}$. It follows directly from (2.19) that

$$\tilde{v}(w, \tau) = v(w)\, P(\tau \mid w), \tag{2.30}$$

where $v(w)$ is defined in (2.17) and the conditional probability of the $\{\tau_i\}$ given w is a product measure

$$P(\tau \mid w) = \prod_{i=1}^{L} \frac{1 + (n_i - n_{i-1})(2\tau_i - 1)}{2}. \tag{2.31}$$

This leads to a simple way of generating steady state configurations of the occupation numbers $\{\tau_i\}$. First one generates a random walk w of L steps according to the measure $v(w)$. Then according to (2.31) a steady state configuration $\{\tau_i\}$ is obtained by taking $\tau_i = 1$ whenever $n_i(w) - n_{i-1}(w) = 1$, $\tau_i = 0$ whenever $n_i(w) - n_{i-1}(w) = -1$ and by choosing $\tau_i = 0$ or 1 with equal probabilities for each i such that $n_i(w) - n_{i-1}(w) = 0$. Therefore the fluctuations of the density have two contributions: the random choice of the walk w which is at the origin of $\dot{Y}(x)$ in (1.8) and once w is chosen, the random choices of the τ_i for the flat parts of the walk, which correspond to $\dot{B}(x)$ in (1.8). The normalization of $B(x)$ in (1.9) arises from the fact that, for large L, $n_i(w) = n_{i-1}(w)$ for approximately half the steps of the walk w.

3. SIMULATIONS

An interesting question is to know whether the fluctuations of density of one dimensional driven diffusive systems in their maximal current phase take always the form (1.8), once properly normalized.

First, we believe that our results also hold for the general ASEP where particles can also jump to the left with rate $q < 1$. In this case also, one can choose a representation of the matrices, used recently to calculate the large deviation function in the weak asymmetry limit,[22] such that matrix elements can be thought as sums over weighted walks which do not cross the origin. We will not discuss this further here.

In order to test whether the results obtained here for the TASEP remain valid for a broader class of models, we consider in this section a generalisation of the TASEP[23, 24] in which particles are extended. In this model, each particle occupies d consecutive sites of a lattice of $L+d-1$ sites and the exclusion rule forbids that any site of the lattice is occupied by more than one particle. There are thus L possible positions for a single particle of size d on the lattice of $L+d-1$ sites. By convention, we say that a particle is at site i when it covers sites $i, i+1,..., i+d-1$. The system evolves according to the following rule: during each infinitesimal time interval dt, each particle jumps one step to its right with probability dt provided that this is allowed by the exclusion rule. Moreover if the first d sites are empty, a new particle is injected at site 1 with probability $\alpha\, dt$ and if a particle covers sites $L, L+1,..., L+d-1$, it is removed with probability $\beta\, dt$. For $d=1$ the problem reduces to the TASEP discussed in Sections 1 and 2.

For general d, the current J_d and the density ρ_d in the maximal current phase are given by[23]

$$J_d = \frac{1}{(\sqrt{d}+1)^2} \tag{3.1}$$

$$\rho_d = \frac{1}{\sqrt{d}(\sqrt{d}+1)}. \tag{3.2}$$

These expressions can be understood by considering a system of M such particles on a ring of L sites, and by using the fact that all allowed configurations are equally likely in the steady state (see Appendix A).

In this ring geometry, the number m of particles on l consecutive sites fluctuates in the steady state and it is possible to show (see Appendix A) that for $d \ll l \ll L$, one has

$$\frac{\langle m^2\rangle - \langle m\rangle^2}{l} \simeq \Delta_d(\rho) \tag{3.3}$$

with

$$\Delta_d(\rho) = \rho(1-\rho) - \rho^2(d-1)(2-d\rho).$$

This expression can also be recovered from the pressure[25] of this system $p = \log(\frac{1-\rho(d-1)}{1-\rho d})$ using $\Delta_d(\rho) = \rho\,\frac{d\rho}{dp}$. In our simulations we tried to see whether, for large system sizes with open boundary conditions, the fluctuations of density in the maximal current phase would still be given by the statistics of a Brownian excursion with (1.8) and (1.10) being replaced by

$$\rho(x) - \rho_d = \sqrt{\frac{\Delta_d(\rho_d)}{L}}\,[\dot{B}(x) + \dot{Y}(x)] \tag{3.4}$$

and

$$r(x, x') = \int_x^{x'} dy(\rho(y) - \rho_d) = \sqrt{\frac{\Delta_d(\rho_d)}{L}}\,[B(x') + Y(x') - B(x) - Y(x)]. \tag{3.5}$$

If we look at the statistical properties of the normalized number $s(x)$ of particles in a box of size Lx centered at the middle of the system

$$s(x) = \sqrt{\frac{L}{\Delta_d(\rho_d)}}\, r\left(\frac{1-x}{2}, \frac{1+x}{2}\right) \tag{3.6}$$

we get, using (3.5) and the properties (1.1) of $Y(x)$

$$\overline{s(x)^2} = \frac{2x - 3x^2}{2} + \frac{3 - 2x + 3x^2}{2\pi} \cos^{-1}\left(\frac{1-x}{1+x}\right) - \frac{3(1-x)\sqrt{x}}{\pi} \tag{3.7}$$

and

$$\overline{s(x)^4} = \frac{3x^2(x-2)(5x-2)}{4} + \frac{15 - 3x^2(x-2)(5x-2)}{4\pi} \cos^{-1}\left(\frac{1-x}{1+x}\right) + \frac{1-x}{2\pi(1+x)} \sqrt{x}\,(15x^3 - 11x^2 - 25x - 15). \tag{3.8}$$

We have simulated this model for three sizes of particles $d = 1$, 3, and 5 for an open system of $L = 601$ sites when $\alpha = \beta = 1$. Let N_x be the number of particles in the interval $L\frac{1-x}{2} \leqslant i \leqslant L\frac{1+x}{2}$ (we count as N_x the number of sites in the interval covered by a particle, divided by the length d of one particle). The normalized number of particles $s(x)$ is related to N_x by

$$s(x) = \frac{N_x - Lx\rho_d}{\sqrt{L\Delta_d(\rho_d)}} \tag{3.9}$$

We have measured the second and fourth moments of $s(x)$ in the steady state, averaged over typically 10^8 updates per site. In Fig. 1, we compare the results of our simulations with the theoretical predictions (3.7) and (3.8). The fact that the curves for the different choices of d coincide indicates that the fluctuations, once properly normalised, are universal (i.e., do not depend on d).

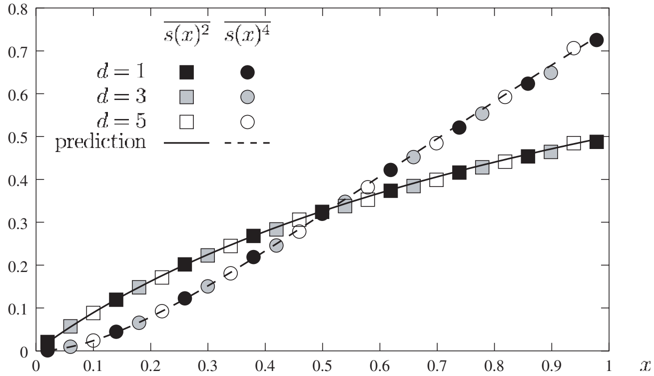

Fig. 1. The moments $\overline{s(x)^2}$ and $\overline{s(x)^4}$ of the fluctuations of the number of particles $s(x)$ (normalized as in (3.9)) between sites $L\frac{1-x}{2}$ and $L\frac{1+x}{2}$ for a system of $L = 601$ sites and for 3 sizes of particles ($d = 1, 3, 5$). The curves are the analytic predictions (3.7) and (3.8).

4. CONCLUSION

The main result of the present paper is that the steady state fluctuations of the density profile in the TASEP can be written in terms of a Brownian excursion as in (1.5), (1.7), (1.8).

Our simulations of a more general model in which particles are extended indicates that the fluctuations of the density profile may be universal. It would of course be nice to make other numerical tests of this universality and to see whether it could be understood by a more macroscopic approach, such as in refs. 26 and 27.

Another interesting question would be to see whether the time dependent fluctuations of the density profile would arise from some simple stochastic dynamics of the Brownian excursion.

APPENDIX A

Let us consider the TASEP on a ring of L sites with N particles of size d as in Section 3. This hard rod problem has a long history starting with the works of Lee and Yang.[25] In the steady state, all allowed configurations (i.e., configurations which satisfy the exclusion rule) are equally likely. Let us call $Z_L(N)$ their number.

The number $z_L(N)$ of configurations of N particles on a lattice of L sites with open boundary conditions is given by, $z_0(N) = z_1(N) = z_2(N) = \cdots = z_{d-1}(N) = \delta_{N,0}$, and for $L > Nd$

$$z_L(N) = \frac{(L - Nd + N)!}{N!\,(L - Nd)!}$$

(one can easily check that $z_L(N)$ satisfies the recursion $z_L(N) = z_{L-1}(N) + z_{L-d}(N-1)$).

By considering that site 1 on a ring is either empty or covered by one particle, one can express $Z_L(N)$ in terms of the partition functions of the open systems

$$Z_L(N) = z_{L-1}(N) + d z_{L-d}(N-1) = \frac{L(L-Nd+N-1)!}{N!\,(L-Nd)!}.$$

The current J on the ring is given by

$$J = \frac{z_{L-d-1}(N-1)}{Z_L(N)} = \frac{N(L-Nd)}{L(L-Nd+N-1)}.$$

In the limit L and N going to infinity, at fixed density $\rho = N/L$, J becomes

$$J = \frac{\rho(1-\rho d)}{1-\rho(d-1)}$$

which gives the expressions (3.2) and (3.1) when the current is maximal.

As the steady state weights of all configurations are equal, all the correlation functions of the occupation numbers τ_i of the sites can be computed exactly. The two point function $\langle \tau_i \tau_{i+k} \rangle$ between the occupations on the ring is given by

$$\langle \tau_i \tau_{i+k} \rangle = \frac{1}{Z_L(N)} \sum_{n=0}^{N} z_{k-d}(n)\, z_{L-k-d}(N-n-2).$$

Using the above expressions of $Z_L(N)$ and $z_L(N)$, one gets for large L and N, keeping $\rho = N/L$, i and k fixed

$$\langle \tau_i \tau_{i+k} \rangle = \sum_{n \geqslant 0} z_{k-d}(n)\, \rho^{n+2} (1-\rho d)^{k-nd-d} \,[1-\rho(d-1)]^{(n+1)(d-1)-k}.$$

Then noting that

$$\sum_{L \geqslant Nd} z_L(N)\, x^L = \frac{x^{Nd}}{(1-x)^{N+1}}$$

one gets that (for arbitrary ϵ)

$$\sum_{k \geqslant 1} \langle \tau_i \tau_{i+k} \rangle\, e^{-k\epsilon} = \frac{\rho^2 e^{-d\epsilon}}{(1-\rho d)(1-e^{-\epsilon}) + \rho(1-e^{-d\epsilon})}.$$

On the other hand

$$\sum_{k \geqslant 1} \langle \tau_i \rangle \langle \tau_{i+k} \rangle \, e^{-k\epsilon} = \frac{\rho^2 e^{-\epsilon}}{1-e^{-\epsilon}}.$$

Expanding the above expressions in powers of ϵ one finds that

$$\sum_{k \geqslant 1} \langle \tau_i \tau_{i+k} \rangle - \langle \tau_i \rangle \langle \tau_{i+k} \rangle = -\frac{(d-1)\, \rho^2 (2-\rho d)}{2}$$

so that the fluctuations of the number m of particles on l consecutive sites, with $d \ll l \ll L$ is

$$\frac{\langle m^2 \rangle - \langle m \rangle^2}{l} = (\rho - \rho^2) - (d-1)\, \rho^2 (2-\rho d)$$

as given in (3.3).

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