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Physica A 340 (2004) 714–724

PHYSICA A

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Universal properties of growing networks

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Received 12 February 2004; received in revised form 2 March 2004

This work is dedicated to the memory of Per Bak whose scientific work had an outstanding influence on the development of Statistical Physics and of the Theory of Complex Systems.

Abstract

Networks growing according to the rule that every new node has a probability p_k of being attached to k preexisting nodes, have a universal phase diagram and exhibit power-law decays of the distribution of cluster sizes in the non-percolating phase. The percolation transition is continuous but of infinite order and the size of the giant component is infinitely differentiable at the transition (though of course non-analytic). At the transition the average cluster size (of the finite components) is discontinuous.

PACS: 02.50. – r; 64.60.Ak; 02.10.Ox

Keywords: Percolation; Infinite cluster; Growing networks; Berezinskii–Kosterlitz–Thouless transition

1. Introduction

A number of recent works [1–7] have shown that simple models of growing networks exhibit an unexpected degree of universality with a percolation transition of infinite order. The goal of the present paper is to summarize and extend these recent results, and to show how the general case of a growing network can be understood via a simple theoretical approach based on the analysis of the differential equation (16) satisfied by the generating function of distribution of cluster sizes.

Perhaps the most studied [8,9] random network model is the random graph introduced by Erdös and Rényi [10] where each pair of vertices of a graph of N vertices are

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2004.05.020

directly connected with a probability c/N. The Erdös and Rényi random graph exhibits a percolation transition at $c=1$ and the number NG of vertices in the giant component vanishes linearly at the transition ($G \sim (c-1)$ as $c \to 1$). Therefore, it came as a surprise that the percolation transition in simple models of *growing* random networks [1] is infinitely gentle, namely every derivative of G vanishes at the transition. Subsequent recent works [2–7] confirmed that prediction for the original and more complicated models, and additionally demonstrated another surprising feature: the size distribution of the connected components has a power law decay in the non-percolating phase. This and other features are very reminiscent of the Berezinskii–Kosterlitz–Thouless phase transition [11,12]; the cause of this similarity remains not understood.

The growing networks which exhibit this infinite-order transition are constructed as follows: one starts with a single node and one adds new nodes one at each time step. When the network consists of N nodes and the $N+1$st node is added, this new node is connected to k randomly chosen existing nodes among the first N nodes. Thus, with probability p_k, there are k arrows emerging from the $N+1$st node, the targets of which are k nodes chosen at random among the N first nodes (whether we require the k target nodes to be distinct or not has no effect on the large N properties that we discuss here). The parameters which define the model are the probabilities p_k. Natural questions one can ask about such a network are whether it exhibits a percolation transition with the emergence of an infinite component and what are the critical behaviors associated with this transition. For a Poisson distribution $p_k=\beta^k/k!\mathrm{e}^{-\beta}$, the model has been investigated in Refs. [4,5], and the case of general p_k's was also recently studied by Coulomb and Bauer [6] who obtained a number of detailed results, in particular on the correlation between a cluster size and the times at which the vertices of a cluster were added.

Similar models have also been studied in the mathematical literature [13,14], and several properties of the percolation transition have been proved by Durrett [15] and Bollobás et al. [16]. The recent interest was partly driven by the desire to mimic biological networks [17,18]: the Poisson network arose as the limiting case of the protein interaction network [4] and as a toy model of a regulatory network [5].

2. The phase diagram and its critical behaviors

In this section, we summarize the properties of these growing networks, as derived in Sections 3–6. The first result which emerges from the analysis of the model is that the phase diagram (Fig. 1) as well as all the power laws depend only on two parameters, the first two moments of the distribution p_k

$$\beta = \langle k \rangle = \sum_k k p_k \quad \text{and} \quad \Delta = \langle k^2 \rangle - \langle k \rangle^2 \tag{1}$$

all the other parameters being irrelevant.

Because the random variable k is an integer, the variance Δ is not only positive but it satisfies

$$\Delta \geqslant (\beta - [\beta]) - (\beta - [\beta])^2 \,, \tag{2}$$

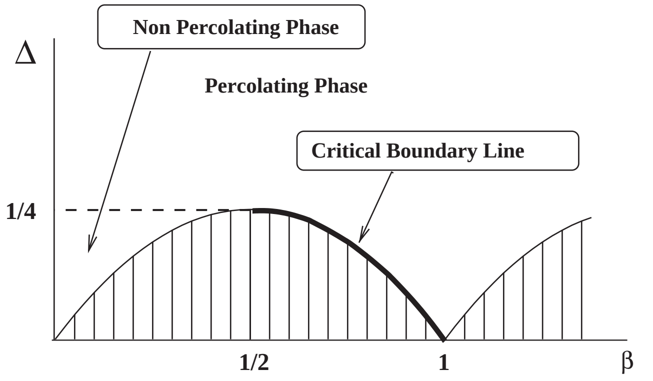

Fig. 1. The phase diagram.

where $[\beta]$ is the integer part of β: for a non-integer average β, a distribution concentrated on integers has a strictly positive variance Δ. The distribution which minimizes Δ, at fixed β, is concentrated on the two integers $[\beta]$ and $[\beta]+1$ with $p_{[\beta]}=1-\beta+[\beta]$ and $p_{[\beta]+1}=\beta-[\beta]$ leading to (2).

Let us summarize the main properties of the phase diagram:

(1) *The non-percolating phase* is the region

$$\beta-\beta^2 \leqslant \Delta < \tfrac{1}{4} \quad \text{for } \beta < \tfrac{1}{2}\,. \tag{3}$$

In this non-percolating phase, the density c_s of clusters of size s decays for large s like a power law with an exponent τ which varies continuously with Δ

$$c_s \sim s^{-\tau} \quad \text{with } \tau = 1+\frac{2}{1-\sqrt{1-4\Delta}}\,. \tag{4}$$

(2) *The percolation transition line* is given by

$$\Delta=\tfrac{1}{4} \quad \text{and} \quad \beta<\tfrac{1}{2} \tag{5}$$

along which for large s

$$c_s \simeq \frac{2}{(1-2\beta)^2}\,\frac{1}{s^3(\ln s)^2}\,. \tag{6}$$

(3) *The critical boundary line* is

$$\Delta=\beta-\beta^2 \quad \text{and} \quad \tfrac{1}{2}<\beta<1 \tag{7}$$

along which for large s,

$$c_s \sim s^{-1-(1/\beta)}\,. \tag{8}$$

(4) *The percolating phase* covers the rest of the phase diagram, namely

$$\begin{cases} \beta \leqslant \tfrac{1}{2} & \text{and } \Delta > \tfrac{1}{4}\,, \\ \tfrac{1}{2} \leqslant \beta < 1 & \text{and } \Delta > \beta-\beta^2\,, \\ 1 \leqslant \beta & \text{and } \Delta \text{ arbitrary}\,, \end{cases} \tag{9}$$

where c_s decays exponentially. As one approaches the boundaries of the percolating phase, the fraction G of sites in the giant component vanishes in the following ways

$$0<\beta<\tfrac{1}{2} \quad \text{and} \quad \Delta \to \tfrac{1}{4}\text{: } G \sim \exp\left(-\frac{\pi}{\sqrt{4\Delta-1}}\right) , \tag{10}$$

$$\beta=\tfrac{1}{2} \quad \text{and} \quad \Delta \to \tfrac{1}{4}\text{: } G \simeq \frac{2\mathrm{e}^{-1}}{\sqrt{4\Delta-1}} \exp\left(-\frac{\pi}{2\sqrt{4\Delta-1}}\right) , \tag{11}$$

$$\tfrac{1}{2}<\beta<1 \quad \text{and} \quad \Delta \to \beta-\beta^2\text{: } G \sim (\Delta-\beta+\beta^2)^{(1-\beta)/(2\beta-1)} . \tag{12}$$

These infinite-order transitions and continuously varying exponents are reminiscent of the Berezinskii–Kosterlitz–Thouless transition [11,12].

3. The generating function of cluster sizes

When N becomes large one can show [5] that the total number of clusters A_s of clusters of size s becomes extensive ($A_s \sim Nc_s$) with fluctuations of order of order $N^{1/2}$ (see Appendix A). Therefore for large N, the random variable A_s is well characterized by its average $\langle A_s \rangle$ which evolves according to

$$\frac{\mathrm{d}\langle A_s\rangle}{\mathrm{d}N} = -\beta\,\frac{s\langle A_s\rangle}{N} + \sum_{k=0}^{\infty} p_k \sum_{s_1\dots s_k} \prod_{j=1}^{k} \frac{s_j\langle A_{s_j}\rangle}{N} , \tag{13}$$

where the sum is taken over all $s_1 \geqslant 1,\dots,s_k \geqslant 1$ such that $s_1+\cdots+s_k+1=s$. On the right-hand side of (13) the negative term corresponds to the decrease of the number of clusters of size s, when a new node is introduced which connects a cluster of size s to other sites, creating that way a new cluster of size large than s. The positive terms corresponds to all the ways the new node can make up a cluster of size s by connecting preexisting clusters.

Writing $\langle A_s\rangle = Nc_s$ we reduce Eqs. (13) to

$$(1+\beta s)c_s = \sum_{k=0}^{\infty} p_k \sum_{s_1\dots s_k} \prod_{j=1}^{k} s_j c_{s_j} . \tag{14}$$

This allows one to determine all the c_s recursively: $c_1=p_0/(1+\beta)$, $c_2=p_1c_1/(1+2\beta)$, $c_3=(p_1c_2+p_2c_1^2)/(1+3\beta)$, etc. The infinite set of Eq. (14) can be converted into a single differential equation for the generating function

$$g(z) = \sum_{s=1}^{\infty} sc_s\mathrm{e}^{sz} . \tag{15}$$

Multiplying both sides of Eq. (14) by $s\mathrm{e}^{sz}$ and summing over s gives

$$g+\beta g' = \mathrm{e}^z(\mathscr{P}[g]+g'\mathscr{Q}[g]) , \tag{16}$$

where $g' = \mathrm{d}g/\mathrm{d}z$, the function $\mathscr{P}[g]$ is the generating function of the p_k's and $\mathscr{Q}[g]$ its derivative:

$$\mathscr{P}[g] = \sum_{k\geqslant 0} p_k g^k, \quad \mathscr{Q}[g] = \frac{\mathrm{d}\mathscr{P}[g]}{\mathrm{d}g} . \tag{17}$$

Up to a change of notation, Eq. (16) is the same as Eq. (17) of Ref. [6]. We will see below that all the properties summarized in Section 2 follow from (16).

4. The average cluster size

If a node is chosen at random (outside the giant component) the average size $\langle s\rangle$ of the cluster it belongs to is given by

$$\langle s\rangle = \frac{\sum s^2 c_s}{\sum s c_s} = \frac{g'(0)}{g(0)} . \tag{18}$$

The region without the giant component is characterized mathematically by $g(0)=1$. If we replace $g(z)$ by its expansion $g(z) = 1 + g'(0)z + o(z)$ in Eq. (16), and use that

$$\mathscr{P}[g] = 1 + \beta(g-1) + \frac{1}{2}\gamma(g-1)^2 + \cdots \tag{19}$$

with

$$\gamma = \varDelta + \beta^2 - \beta \tag{20}$$

we obtain a quadratic equation satisfied by $g'(0)$

$$(\varDelta + \beta^2 - \beta)[g'(0)]^2 + (2\beta - 1)g'(0) + 1 = 0 . \tag{21}$$

Eq. (21) has no positive solution when $\beta \geqslant \frac{1}{2}$ [see (2)]. For $\beta \leqslant \frac{1}{2}$ and $\varDelta < \frac{1}{4}$, the solution to Eq. (21) reads

$$\langle s\rangle = g'(0) = \frac{1 - 2\beta - \sqrt{1-4\varDelta}}{2(\varDelta + \beta^2 - \beta)} = \frac{2}{1 - 2\beta + \sqrt{1-4\varDelta}} . \tag{22}$$

This suggests that different behaviors occur below, at, and above the critical line (5). We shall see in Section 6 that the giant component is born when the critical line (5) is crossed. When line (5) is approached from below, i.e., $\varDelta \to \frac{1}{4} - 0$, one sees that

$$\langle s\rangle \to \frac{2}{1-2\beta} . \tag{23}$$

Turn now to the phase with the giant component. Setting $z=0$, Eq. (16) gives $g'(0)$ in terms of the size $G = 1 - g(0)$ of the giant component

$$g'(0) = \frac{\mathscr{P}[1-G] + G - 1}{\beta - \mathscr{Q}[1-G]} . \tag{24}$$

There is no explicit formula for the average size in the percolating phase due to the lack of explicit expression for G. However, near the critical line (5) one can use the fact that $G \to 0$ to simplify Eq. (24) and get as $\varDelta \to \frac{1}{4} + 0$

$$\langle s\rangle \to \frac{1-\beta}{\varDelta + \beta^2 - \beta} = \frac{2}{1-2\beta}\,\frac{2-2\beta}{1-2\beta} . \tag{25}$$

Comparing (23) and (25) shows that the average size $\langle s\rangle$ of finite components jumps discontinuously as one crosses the percolation transition line (5).

5. Component size distribution

We now examine the component size distribution c_s.

5.1. Sub-critical regime $\beta < \frac{1}{2}$ and $\Delta < \frac{1}{4}$

The large s behavior of c_s can be read off from the behavior of the generating function (15) in the $z \to 0$ limit. A power-law decay (see Appendix B)

$$c_s \sim B s^{-\tau} \quad \text{as } s \to \infty \tag{26}$$

implies the following small z expansion of the generating function (15):

$$g(z) = 1 + g'(0)z + B\Gamma(2-\tau)(-z)^{\tau-2} + \cdots \, . \tag{27}$$

Substituting this expansion into Eq. (16) or rather into

$$g - e^z\mathscr{P}[g] = g'(e^z\mathscr{Q}[g] - \beta) \tag{28}$$

we find by balancing the contributions of the order of $(-z)^{\tau-2}$

$$\tau - 2 = -\frac{\beta - 1 + (\Delta + \beta^2 - \beta)g'(0)}{\beta + (\Delta + \beta^2 - \beta)g'(0)} \tag{29}$$

which using (22) gives (4).

Thus in contrast to ordinary critical phenomena, one gets a power-law (26) in the whole non-percolating phase and the exponent τ given in (4) depends continuously on the single parameter Δ.

The amplitude B in (26) is, as usual for critical amplitudes, more difficult to determine. It can nevertheless be calculated [6] along the line

$$\Delta = \beta - \beta^2 \quad \text{and} \quad \beta < \tfrac{1}{2} \, , \tag{30}$$

where only p_0 and p_1 are non-zero. Then recursion (14) reduces to $(1+\beta s)c_s = \beta(s-1)c_{s-1}$ which is easily solved

$$c_s = \frac{1-\beta}{\beta}\,\frac{\Gamma(1+\frac{1}{\beta})\Gamma(s)}{\Gamma(s+1+\frac{1}{\beta})} \simeq \frac{1-\beta}{\beta}\,\frac{\Gamma(1+\frac{1}{\beta})}{s^{1+(1/\beta)}} \quad \text{for large } s \, . \tag{31}$$

As claimed in (8), expressions (31) remain valid along the critical boundary line (7), where only two probabilities p_0 and p_1 are non-zero. Note (see Appendix B) that for small z and $\beta > \frac{1}{2}$, one has

$$g(z) = 1 + \frac{1-\beta}{\beta}\,\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1-\frac{1}{\beta}\right)(-z)^{\frac{1}{\beta}-1} + \cdots \tag{32}$$

At $\beta = \frac{1}{2}$, Eq. (31) becomes $c_s = 2/[s(s+1)(s+2)]$ leading to $g = 2e^{-z} - 1 - 2e^{-z}(1-e^{-z})\log(e^{-z}-1)$ which gives for small negative z

$$g = 1 - 2z\log(-z) - 2z + \cdots \tag{33}$$

5.2. Critical line $0 < \beta < \frac{1}{2}$ and $\Delta = \frac{1}{4}$

We write as in Appendix A

$$g(z) = 1 + zv(z) \tag{34}$$

and we obtain (A.6) an implicit form of $v(z)$

$$\ln\left(1 - \left(\tfrac{1}{2} - \beta\right) v\right) + \ln(-z) + \frac{1}{(1-2\beta)[1-(1/2-\beta)v]} = A(\beta)\ . \tag{35}$$

The integration constant $A(\beta)$ cannot be determined without integrating the full equation (16) with appropriate boundary condition: $g \to c_1 e^z$ with $c_1 = p_0/(1+\beta)$ as $z \to -\infty$.

From (34) and (35), we get the small z of $g(z)$:

$$g(z) = 1 + \frac{2}{1-2\beta} z + \frac{2}{(1-2\beta)^2} \frac{z}{\ln(-z)} + \cdots\ . \tag{36}$$

Inverting this expansion (see Appendix B) yields (6).

Thus, the component size distribution acquires a logarithmic correction in the critical regime with a remarkable degree of universality: it depends only on the average connectivity β of the network.

5.3. Super-critical regime $0 < \beta < \frac{1}{2}$ and $\Delta > \frac{1}{4}$

Above the phase transition point, both $g(0) = 1 - G$ and $g'(0)$ are finite. Repeatedly differentiating Eq. (16) and setting $z = 0$ we find that all following derivatives are finite as well. This implies that for $\Delta > 1/4$, the component size distribution decays faster than any power law. We now argue that

$$c_s \sim s^{-5/2} e^{-s/s_*} \quad \text{as } s \to \infty\ . \tag{37}$$

An exponential factor in the component size distribution, $c_s \propto e^{-s/s_*}$, shifts a singularity of the generating function $g(z)$ to $z_* = 1/s_*$. From Eq. (16) which is useful to re-write in the form

$$g' = \frac{e^z \mathscr{P}[g] - g}{\beta - e^z \mathscr{Q}[g]} \tag{38}$$

we see that the singularity arises when the denominator on the right-hand side of (38) vanishes: $\beta = e^{z_*} \mathscr{Q}[g_*]$ where $g_* = g(z_*)$. The derivative on the left-hand side of (38) should be therefore singular suggesting an algebraic asymptotic $g(z) - g_* \propto (z_* - z)^\alpha$ with $\alpha < 1$. Plugging this into (38) gives $\alpha = \frac{1}{2}$. This type of singularity corresponds to the $s^{-3/2}$ decay of the sequence $s c_s e^{s z_*}$ (Appendix B) and hence we finally obtain (37). A more detailed analysis would allow to see that $s_* \sim 1/G$ diverges as the percolation line is approached [6].

6. Giant component

To determine the size of the giant component G we need to understand the behavior of solutions to Eq. (16) near $z = 0$ and this can be done analytically near the transition line or near the critical boundary line (see Appendix A).

6.1. For $0 < \beta < \frac{1}{2}$ and $\Delta > \frac{1}{4}$

If we write $g(z)$ as (34), set $\varepsilon = \Delta - \frac{1}{4} \ll 1$ and we use expression (A.5) for $v(z)$, we get

$$\ln(1-(1/2-\beta)v) + \ln(-z) - \frac{\pi}{2\sqrt{\varepsilon}} + \frac{1}{(1-2\beta)[1-(1/2-\beta)v]} = \ln\left[G\sqrt{\frac{1}{2}-\beta}\right] . \tag{39}$$

For this solution to be consistent with (35) in the limit $\varepsilon \to 0$ the size G of the giant component should satisfy

$$G \sim \frac{2e^{A(\beta)}}{1-2\beta} \exp\left\{-\frac{\pi}{2\sqrt{\varepsilon}}\right\} = \frac{2e^{A(\beta)}}{1-2\beta} \exp\left\{-\frac{\pi}{\sqrt{4\Delta-1}}\right\} \tag{40}$$

as claimed in (10). Therefore the transition is of infinite order since all derivatives of G vanish as $\Delta \to 1/4$. Behavior (40) appears quite universal as it was observed numerically [1] and confirmed analytically [2–6] for other growing networks.

6.2. $\beta = \frac{1}{2}$ and $\Delta > \frac{1}{4}$

In this region we may use either (A.4) or (A.5) to obtain for small γ

$$\ln(-z) - \frac{\pi}{4\sqrt{\gamma}} + \frac{v}{2} = \ln(\sqrt{\gamma}G) \tag{41}$$

To be consistent with (33), i.e., with $v = -2 - 2\ln(-z)$ in the limit $\gamma = \Delta - 1/4 \to 0$, the size of the giant component should satisfy

$$G(\gamma) \simeq \frac{e^{-1}}{\sqrt{\gamma}} \exp\left[-\frac{\pi}{4\sqrt{\gamma}}\right] \tag{42}$$

as claimed in (11). In comparison with (40) there is a factor $\frac{1}{2}$ in the exponential and the prefactor is determined here analytically. The transition still is of infinite order.

6.3. $\frac{1}{2} < \beta < 1$ and $\Delta > \beta - \beta^2$

In this range of parameters one can use (A.3). When one approaches the critical boundary line (7), i.e., for small γ, expression (A.3) becomes

$$\frac{\beta}{2\beta-1}\ln[1+(2\beta-1)v] + \ln(-z) = \ln\{\gamma^{-(1-\beta)/(2\beta-1)}G\} + \frac{\ln(2\beta-1)}{2\beta-1} . \tag{43}$$

To be consistent with (32)

$$v \sim C(-z)^{(1/\beta)-2},\ C = -\frac{(1-\beta)\Gamma(1+\frac{1}{\beta})\Gamma(1-\frac{1}{\beta})}{\beta} = -\frac{1-\beta}{\beta^2}\frac{\pi}{\sin(\frac{\pi}{\beta})} \tag{44}$$

for small z, one needs the size G of the giant component to scale as

$$G(\beta,\gamma) \sim D\gamma^{\nu}, \qquad \nu = \frac{1-\beta}{2\beta-1} , \tag{45}$$

where $D = C^{\beta/(2\beta-1)}(2\beta-1)^{-(1-\beta)/(2\beta-1)}$ and C is given by (44). The exponent ν is again universal as it depends only on the average connectivity β but not on any other parameter of the distribution p_k. Yet it varies continuously with β along the critical boundary line (7) thereby showing a richer behavior than in random graph models where $\nu = 1$.

Appendix A. Analysis of (16) for z small and $1-g$ small

In this appendix, we analyze the solution of (16) when both z and $1-g$ are small. If we write

$$g(z) = 1 + zv(z)\ , \tag{A.1}$$

Eq. (16) gives to leading order

$$(1 - v + 2\beta v + \gamma v^2)z + (\beta + \gamma v)v' z^2 = 0$$

so that $v(z)$ satisfies

$$\frac{\gamma v + \beta}{\gamma v^2 + (2\beta - 1)v + 1}\,\mathrm{d}v + \frac{\mathrm{d}z}{z} = 0\ . \tag{A.2}$$

If the size G of the giant component is non-zero, $\lim_{z\to 0} zv = g(0) - 1 = -G$ and this fixes the constant of integration and leads to the following expression valid for $\beta > 1/2$, γ small and $\gamma < (\beta - 1/2)^2$:

$$\frac{1}{2}\ln[\gamma v^2 + (2\beta-1)v + 1] + \ln(-z) - \frac{1}{2\sqrt{(2\beta-1)^2 - 4\gamma}} \ln \frac{2\beta - 1 + 2\gamma v + \sqrt{(2\beta-1)^2 - 4\gamma}}{2\beta - 1 + 2\gamma v - \sqrt{(2\beta-1)^2 - 4\gamma}} = \ln(\sqrt{\gamma}G)\ . \tag{A.3}$$

If one tries to analytic continue (A.3) to the region $\beta > 1/2$, γ small and $\gamma > (\beta - 1/2)^2$ one gets

$$\frac{1}{2}\ln[\gamma v^2 + (2\beta-1)v + 1] + \ln(-z) + \frac{i}{2\sqrt{4\gamma - (2\beta-1)^2}} \ln \frac{2\beta - 1 + 2\gamma v + i\sqrt{4\gamma - (2\beta-1)^2}}{2\beta - 1 + 2\gamma v - i\sqrt{4\gamma - (2\beta-1)^2}} = \ln(\sqrt{\gamma}G)\ . \tag{A.4}$$

and to the region $\beta < 1/2$, γ small and $\gamma > (\beta - 1/2)^2$ one gets (the extra term coming from the continuation of the logarithm)

$$\frac{1}{2}\ln[\gamma v^2 - (1-2\beta)v + 1] + \ln(-z) - \frac{\pi}{\sqrt{4\gamma - (1-2\beta)^2}} + \frac{i}{2\sqrt{4\gamma - (1-2\beta)^2}} \ln \frac{1 - 2\beta - 2\gamma v - i\sqrt{4\gamma - (1-2\beta)^2}}{1 - 2\beta - 2\gamma v + i\sqrt{4\gamma - (1-2\beta)^2}} = \ln(\sqrt{\gamma}G)\ . \tag{A.5}$$

Lastly when $G=0$ and $\gamma=(\beta-1/2)^2$ one gets by integrating (A.2)

$$\ln(1-(1/2-\beta)v)+\ln(-z)+\frac{1}{(1-2\beta)[1-(1/2-\beta)v]}=A(\beta)\,, \tag{A.6}$$

where $A(\beta)$ is an integration constant.

Appendix B. Extracting the asymptotics

Consider the generating function

$$A(z)=\sum_{s=1}^{\infty}a_s e^{sz}\,. \tag{B.1}$$

The dominant singularity of the generating function allows one to extract the large s asymptotic of a_s. This can be done by a variety of techniques [19,20]. Here, we give an elementary exposition that is sufficient to extract the asymptotics used in this paper.

Let us first see what kind of the singular behavior is associated with the power-law asymptotic

$$a_s \simeq As^{\alpha} \qquad \text{as } s\to\infty\,. \tag{B.2}$$

If $\alpha>-1$, then $A(z)$ diverges as $z\uparrow 0$. The dominant contribution is found by replacing summation by integration:

$$A(z)\to\int_0^{\infty}\mathrm{d}s\,As^{\alpha}\,e^{sz}=A\Gamma(1+\alpha)(-z)^{-1-\alpha}\,. \tag{B.3}$$

If $-2<\alpha<-1$, the sum $\sum_{s\geqslant 1}a_s$ converges and instead of (B.3) we get

$$A(z)=A(0)+A\Gamma(1+\alpha)(-z)^{-1-\alpha}+\cdots\,. \tag{B.4}$$

Similarly for $-3<\alpha<-2$,

$$A(z)=A(0)+A'(0)z+A\Gamma(1+\alpha)(-z)^{-1-\alpha}+\cdots\,, \tag{B.5}$$

where of course $A(0)=\sum_{s\geqslant 1}a_s$ and $A'(0)=\sum_{s\geqslant 1}sa_s$. Thus if the dominant *singular* term has the power-law form $(-z)^{-1-\alpha}A\Gamma(1+\alpha)$, the asymptotic must have the form (B.2).

Above we assumed that α is not a negative integer. Otherwise logarithms can arise. For instance, if $\alpha=-1$ we get

$$A(z)\to\int_1^{\infty}\mathrm{d}s\,As^{-1}e^{sz}\to -A\ln(-z)\,. \tag{B.6}$$

Imagine now that the dominant singular term is a power of $\ln(-z)$. It is tempting to test the asymptotic

$$a_s\simeq As^{-1}(\ln s)^{-a} \quad \text{as } s\to\infty\,. \tag{B.7}$$

For $a\leqslant 1$, it leads to

$$A(z)\to A\int^{1/(-z)}\frac{\mathrm{d}s}{s(\ln s)^a}\to A\times\begin{cases}(1-a)^{-1}[-\ln(-z)]^{1-a}, & a<1\,;\\ \ln[-\ln(-z)], & a=1\,.\end{cases} \tag{B.8}$$

For $a > 1$, the sum $\sum_{s\geqslant 1} a_s$ converges and we get

$$A(z) = A(0) - \frac{A}{a-1}[-\ln(-z)]^{1-a} + \cdots . \tag{B.9}$$

As an example (36), imagine that we have found the small z expansion of the generating function $g(z) = \sum sc_s e^{sz}$ that reads

$$g(z) = A_0 + A_1 z + A\frac{z}{\ln(-z)} + \cdots . \tag{B.10}$$

Differentiating gives

$$g'(z) = A_1 + A\frac{1}{\ln(-z)} + \cdots , \tag{B.11}$$

which, in conjunction with (B.9), shows that $a=2$. Using asymptotic (B.7) and $g'(z) = \sum s^2 c_s e^{sz}$ we get as in (6)

$$c_s \simeq As^{-3}(\ln s)^{-2} \quad \text{as } s \to \infty . \tag{B.12}$$

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