PRAMANA
— journal of
physics

Vol. 64, No. 5
May 2005
pp. 695–707

Fluctuations and large deviations in non-equilibrium systems

B DERRIDA
Laboratoire de Physique Statistique, Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France
E-mail: derrida@lps.ens.fr

Abstract. For systems in contact with two reservoirs at different densities or with two thermostats at different temperatures, the large deviation function of the density gives a possible way of extending the notion of free energy to non-equilibrium systems. This large deviation function of the density can be calculated explicitly for exclusion models in one dimension with open boundary conditions. For these models, one can also obtain the distribution of the current of particles flowing through the system and the results lead to a simple conjecture for the large deviation function of the current of more general diffusive systems.

Keywords. Non-equilibrium systems; large deviations; current fluctuations.

PACS Nos 02.50.-r; 05.40.-a; 05.70.Ln; 82.20.-w

1. Introduction

The goal of this talk is to give a short review on results [1–8] obtained recently on the steady state of non-equilibrium systems such as systems in contact with two heat baths at different temperatures T_a and T_b.

When $T_a = T_b = T$, i.e. when the system is in equilibrium, one knows that each microscopic configuration $\mathcal{C}$ of energy $E(\mathcal{C})$ is occupied according to its Boltzmann weight

$$P(\mathcal{C}) = \frac{1}{Z} \exp\left[-\frac{E(\mathcal{C})}{T}\right]. \tag{1}$$

One can then define the free energy

$$F = -T \log Z = -T \log\left(\sum_{\mathcal{C}} \exp\left[-\frac{E(\mathcal{C})}{T}\right]\right) \tag{2}$$

from which macroscopic properties such as the average energy or the specific heat can be obtained.

When $T_a \neq T_b$, the system reaches in the long time limit a steady state, but there is no longer a general expression such as (1) for the steady state probabilities of microscopic configurations and there is no *a priori* definition of the free energy. Moreover new questions arise related to the flow of energy Q_t through the system during time t, i.e. the energy Q_t transferred from one reservoir to the other or equivalently the entropy production $Q_t\left(\frac{1}{T_b} - \frac{1}{T_a}\right)$. One can for example try to determine the probability distribution $P(Q_t)$ of this energy flow.

2. Large deviation function of the density

One way of extending the notion of free energy to non-equilibrium systems is to define the large deviation function [9] of the density. If one considers a box of volume V containing Vr particles, the probability $P_v(n)$ of finding n particles in a subvolume v has the following v dependence for large v,

$$P_v(n) \sim \exp\left[-va\left(\frac{n}{v}\right)\right], \tag{3}$$

where $a(\rho)$ is the large deviation function at density ρ. Figure 1 shows the typical shape of $a(\rho)$ for a homogeneous system (i.e. not at a coexistence between different phases) with a single minimum at $r = 0$ where $a(r) = 0$.

The large deviation function $a(\rho)$ can be defined for equilibrium systems as well as for non-equilibrium systems. For equilibrium systems, one can show that $a(\rho)$ is closely related to the free energy: if the volume v is sufficiently large and if the interactions are short ranged, the large deviation function $a(\rho)$ is given by

$$a(\rho) = \frac{f(\rho) - f(r) - (\rho - r)f'(r)}{kT}, \tag{4}$$

where $f(\rho)$ is the free energy per unit volume. This can be seen by noticing that for $v \ll V$ and when $v^{1/d}$ is much larger than the range of the interactions

$$P_v(n) = \frac{Z_v(n)\ Z_{V-v}(N-n)}{Z_V(N)} \exp[O(v^{\frac{d-1}{d}})], \tag{5}$$

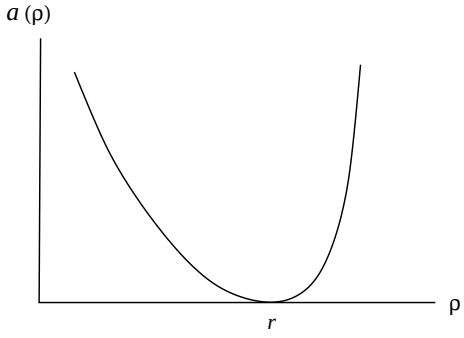

Figure 1. A typical shape of the large deviation function $a(\rho)$.

where the term $\exp[O(v^{\frac{d-1}{d}})]$ represents the interactions between pairs of particles, one of which is the volume v and the other one in $V - v$. Then taking the log of (5) and using the fact that

$$\lim_{v\to\infty} \frac{\log Z_v(v\rho)}{v} = -\frac{f(\rho)}{kT} \tag{6}$$

leads to (4).

3. Large deviation of the current

One can define in a similar way the large deviation function of the current of particles for a system in contact with two reservoirs at densities ρ_a and ρ_b as in figure 2 (or of the heat current for a system in contact with two thermostats). The probability of observing during a long time t an average current j takes the form

$$\mathrm{Pro}\left(\frac{Q_t}{t} = j\right) \sim \mathrm{e}^{-tF(j)}, \tag{7}$$

where $F(j)$ is the large deviation function of the current j. This large deviation function $F(j)$ has a shape similar to $a(\rho)$, with a minimum at some value $\overline{j}$, the typical value where $F(\overline{j}) = 0$.

The knowledge of the large deviation function determines, by expanding around $\overline{j}$, all the cumulants of Q_t for large t.

4. The symmetric simple exclusion process (SSEP)

There are only few examples of non-equilibrium steady states for which the large deviation function $a(\rho)$ of the density or $F(j)$ of the current can be calculated [2,4–8]. One of the simplest cases for which the calculations can be done is the symmetric simple exclusion process [10–12] shown in figure 3.

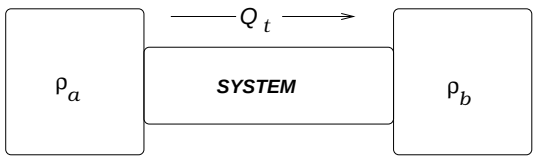

Figure 2. System in contact with two reservoirs of particles at densities ρ_a and ρ_b.

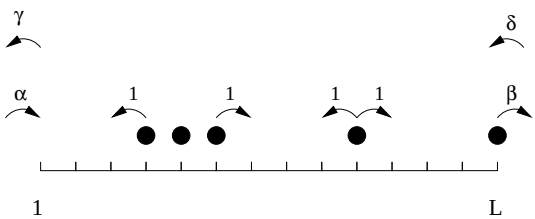

Figure 3. The symmetric simple exclusion process.

The model is defined as a one-dimensional lattice of L sites with open boundaries, each site being either empty or occupied by a single particle. During every infinitesimal time interval $\mathrm{d}t$, each particle has a probability $\mathrm{d}t$ of jumping to the left if the neighboring site on its left is empty, $\mathrm{d}t$ of jumping to the right if the neighboring site on its right is empty. At the two boundaries the dynamics is modified to mimic the coupling with reservoirs of particles: at the left boundary, during each time interval $\mathrm{d}t$, a particle is injected on site 1 with probability $\alpha\,\mathrm{d}t$ (if this site is empty) and a particle is removed from site 1 with probability $\gamma\,\mathrm{d}t$ (if this site is occupied). Similarly on site L, particles are injected at rate δ and removed at rate β.

One can show [4,5,8] that these choices of the rates $\alpha, \gamma, \beta, \delta$ correspond to the left boundary being connected to a reservoir at density ρ_a and the right boundary to a reservoir at density ρ_b with ρ_a and ρ_b given by

$$\rho_a = \frac{\alpha}{\alpha+\gamma}; \quad \rho_b = \frac{\delta}{\beta+\delta}. \tag{8}$$

For the SSEP, it is easy to determine the steady state profile: if $\tau_i = 0$ or 1 is a binary variable indicating whether site i is occupied or empty, the time evolution of the average occupation $\langle\tau_i\rangle$ is given by

$$\frac{\mathrm{d}\langle\tau_1\rangle}{\mathrm{d}t} = \alpha - (\alpha+\gamma+1)\langle\tau_1\rangle + \langle\tau_2\rangle, \tag{9}$$

$$\frac{\mathrm{d}\langle\tau_i\rangle}{\mathrm{d}t} = \langle\tau_{i-1}\rangle - 2\langle\tau_i\rangle + \langle\tau_{i+1}\rangle, \quad \text{for} \;\; 2 \le i \le L-1 \tag{10}$$

$$\frac{\mathrm{d}\langle\tau_L\rangle}{\mathrm{d}t} = \langle\tau_{L-1}\rangle - (1+\beta+\delta)\langle\tau_L\rangle + \delta, \tag{11}$$

and the steady state density profile (obtained by writing that $\frac{\mathrm{d}\langle\tau_i\rangle}{\mathrm{d}t} = 0$) is [5]

$$\langle\tau_i\rangle = \frac{\rho_a(L + \frac{1}{\beta+\delta} - i) + \rho_b(i - 1 + \frac{1}{\alpha+\gamma})}{L + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta} - 1}. \tag{12}$$

The average current

$$\bar{j} = \langle\tau_i - \tau_{i+1}\rangle = \frac{\rho_a - \rho_b}{L + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta} - 1} \simeq \frac{\rho_a - \rho_b}{L} \tag{13}$$

is proportional to the gradient of the density (with a coefficient of proportionality which is simply 1 here) and therefore follows Fick's law.

5. Current fluctuations in the SSEP

In the long time limit the generating function of the total charge Q_t transferred from the left reservoir to the system during time t grows exponentially with time

$$\left\langle \mathrm{e}^{\lambda Q_t} \right\rangle \sim \mathrm{e}^{\mu(\lambda)t}, \tag{14}$$

where $\mu(\lambda)$ is related to the large deviation function $F(j)$ by a Legendré transformation

$$\mu(\lambda) = \max_j [\lambda j - F(j)]. \tag{15}$$

Because the evolution is Markovian, $\mu(\lambda)$ can be determined as the largest eigenvalue of a certain matrix [2]: if $P_t(\mathcal{C})$ is the probability of finding the system in a configuration $\mathcal{C}$ and $M(\mathcal{C},\mathcal{C}')\mathrm{d}t$ is the probability of transition from configuration $\mathcal{C}'$ to configuration $\mathcal{C}$ during time $\mathrm{d}t$, the evolution of $P_t(\mathcal{C})$ is given by

$$\frac{\mathrm{d}P_t(\mathcal{C})}{\mathrm{d}t} = \sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}')P_t(\mathcal{C}'). \tag{16}$$

Among the matrix elements $M(\mathcal{C},\mathcal{C}')$, some correspond to exchanges of particles with the left reservoir and some represent internal moves in the bulk or exchanges with the right reservoir. One can decompose the matrix $M(\mathcal{C},\mathcal{C}')$ into three matrices

$$M(\mathcal{C},\mathcal{C}') = M_1(\mathcal{C},\mathcal{C}') + M_0(\mathcal{C},\mathcal{C}') + M_{-1}(\mathcal{C},\mathcal{C}'), \tag{17}$$

where the index is the number of particles transferred from the left reservoir to the system during time $\mathrm{d}t$, when the system jumps from the configuration $\mathcal{C}'$ to the configuration $\mathcal{C}$. One can then show [2] that $\mu(\lambda)$ is simply the largest eigenvalue of the matrix

$$\mathrm{e}^{\lambda}M_1 + M_0 + \mathrm{e}^{-\lambda}M_{-1}. \tag{18}$$

The steady state weights $P(\mathcal{C})$ for the SSEP are known exactly [4,5,13]: they determine the eigenvector of the matrix $\mathrm{e}^{\lambda}M_1 + M_0 + \mathrm{e}^{-\lambda}M_{-1}$ associated to the eigenvalue $\mu(\lambda)$ when $\lambda = 0$.

In [2] a perturbation theory was developed to calculate $\mu(\lambda)$ in powers of λ. The main outcome of this perturbation theory is that $\mu(\lambda)$, which in principle depends on L, λ and on the four parameters $\alpha, \beta, \gamma, \delta$, takes for large L a simple form

$$\mu(\lambda) = \frac{1}{L}R(\omega) + O\left(\frac{1}{L^2}\right), \tag{19}$$

where ω is defined as

$$\omega = (\mathrm{e}^{\lambda} - 1)\rho_a + (\mathrm{e}^{-\lambda} - 1)\rho_b - (\mathrm{e}^{\lambda} - 1)(1 - \mathrm{e}^{-\lambda})\rho_a\rho_b. \tag{20}$$

The perturbation theory gives up to fourth order in ω,

$$R(\omega) = \omega - \frac{\omega^2}{3} + \frac{8\omega^3}{45} - \frac{4\omega^4}{35} + O(\omega^5). \tag{21}$$

The fact that $\mu(\lambda)$ depends only on ρ_a, ρ_b and λ through the single parameter ω is the outcome of the calculation, but so far there is no physical explanation why it is so. However, ω remains unchanged under a number of symmetries [2] (left–right, particle–hole, Gallavotti–Cohen [14,15] symmetry) implying that $\mu(\lambda)$ remains unchanged as it should under these symmetries.

From the knowledge of $R(\omega)$ up to fourth order in ω, one can determine the first four cumulants of the integrated current Q_t for arbitrary ρ_a and ρ_b:

- For $\rho_a = 1$ and $\rho_b = 0$, one finds

$$\frac{\langle Q_t \rangle}{t} = \frac{1}{L} + O\left(\frac{1}{L^2}\right), \tag{22}$$

$$\frac{\langle Q_t^2 \rangle_c}{t} = \frac{1}{3L} + O\left(\frac{1}{L^2}\right), \tag{23}$$

$$\frac{\langle Q_t^3 \rangle_c}{t} = \frac{1}{15L} + O\left(\frac{1}{L^2}\right), \tag{24}$$

$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{-1}{105L} + O\left(\frac{1}{L^2}\right). \tag{25}$$

  Surprisingly these cumulants are the same as the ones known for a different problem of current flow: the case of non-interacting fermions through a mesoscopic disordered conductor [16–18].
- For $\rho_a = \rho_b = \frac{1}{2}$ which corresponds to an equilibrium case with the same density $1/2$ in the two reservoirs, one finds that all odd cumulants vanish as they should and that

$$\frac{\langle Q_t^2 \rangle_c}{t} = \frac{1}{2L} + O\left(\frac{1}{L^2}\right), \tag{26}$$

$$\frac{\langle Q_t^4 \rangle_c}{t} = O\left(\frac{1}{L^2}\right). \tag{27}$$

6. A conjecture for the distribution of current

Because $\mu(\lambda)$ depends on the parameters ρ_a, ρ_b and λ through the single parameter ω, if one knows $\mu(\lambda)$ for one single choice of ρ_a and ρ_b, then (20),(21) determine $\mu(\lambda)$ for all other choices of ρ_a, ρ_b.

In [2], it was conjectured that for the particular case $\rho_a = \rho_b = \frac{1}{2}$, not only the fourth cumulant vanishes as in (27), but also all the higher cumulants vanish, so that the distribution of Q_t is Gaussian (to leading order in $1/L$). This fully determines the function $R(\omega)$ to be

$$R(\omega) = \left[\log\left(\sqrt{1+\omega} + \sqrt{\omega}\right)\right]^2. \tag{28}$$

One can then check that, with this expression of $R(\omega)$, all the higher cumulants of Q_t in the case $\rho_a = 1$ and $\rho_b = 0$ coincide with those of fermions through mesoscopic conductors [2,17].

7. The additivity principle

One can formulate another conjecture, the additivity principle [1], based on a simpler physical interpretation, which leads to the same expressions (20),(28) as predicted by the previous conjecture and can be generalized to other diffusive systems.

We have seen that, for a system of length $L + L'$ in contact with two reservoirs of particles at densities ρ_a and ρ_b, the probability of observing, during a long time t, an integrated current $Q_t = jt$ has the following form

$$\mathrm{Pro}_{L+L'}\left(j, \rho_a, \rho_b\right) \sim \mathrm{e}^{-tF_{L+L'}(j,\rho_a,\rho_b)}. \tag{29}$$

The idea of the additivity principle is to relate the large deviation function $F_{L+L'}(j, \rho_a, \rho_b)$ of the current to the large deviation functions of subsystems by writing that for large t,

$$\mathrm{Pro}_{L+L'}\left(j, \rho_a, \rho_b\right) \sim \max_{\rho} \left[\mathrm{Pro}_{L}\left(j, \rho_a, \rho\right) \times \mathrm{Pro}_{L'}\left(j, \rho, \rho_b\right)\right]. \tag{30}$$

This means that the probability of transporting a current j over a distance $L + L'$ between two reservoirs at densities ρ_a and ρ_b is the same (up to boundary effects which give for large L subleading contributions) as the probability of transporting the same current j over a distance L between two reservoirs at densities ρ_a and ρ times the probability of transporting the current j over a distance L' between two reservoirs at densities ρ and ρ_b. One can then argue that choosing the optimal ρ makes this probability maximum. From (30) one gets the following additivity property of the large deviation function

$$F_{L+L'}\left(j, \rho_a, \rho_b\right) = \max_{\rho} \left[F_{L}\left(j, \rho_a, \rho\right) + F_{L'}\left(j, \rho, \rho_b\right)\right]. \tag{31}$$

Suppose we consider a diffusive system for which we know the two functions $D(\rho)$ and $\sigma(\rho)$ defined as follows:

1. For $\rho_a - \rho_b$ small,

$$\lim_{t\to\infty} \frac{\langle Q_t \rangle}{t} = D(\rho_a) \frac{\rho_a - \rho_b}{L}. \tag{32}$$

2. For $\rho_a = \rho_b$,

$$\lim_{t\to\infty} \frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho_a)}{L}. \tag{33}$$

If one accepts the additivity property (31) of the large deviation function, one can cut the system into more and more pieces so that [1] when these pieces have length $L\,\mathrm{d}x$, one gets

$$F_{L}\left(j, \rho_a, \rho_b\right) = \frac{1}{L} \max_{\rho(x)} \left[\int_0^1 \frac{[jL + D(\rho)\rho']^2}{2\sigma(\rho)} \mathrm{d}x\right], \tag{34}$$

where the optimal $\rho(x)$ should satisfy $\rho(0) = \rho_a$ and $\rho(1) = \rho_b$.

Using the fact that for the SSEP $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$, one can recover expression (28) from (34) [1]. For more general diffusive systems, one can also determine all the cumulants of the integrated current for arbitrary ρ_a and ρ_b. For example

$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{1}{L} \frac{3(5I_4 I_1^2 - 14 I_1 I_2 I_3 + 9 I_2^3)}{I_1^5}, \tag{35}$$

where

$$I_n = \int_{\rho_b}^{\rho_a} D(\rho) \sigma(\rho)^{n-1} \, \mathrm{d}\rho. \tag{36}$$

Recently, (34) was also derived in the context of fermions in mesoscopic conductors [18]. It has been checked for weakly interacting lattice gases [19]. It was also shown [20] that there are restrictions on $\sigma(\rho)$ and $D(\rho)$ for (34) to be valid and that, as $F_L(j, \rho_a, \rho_b)$ defined in (29) should be a convex function of j, for some $\sigma(\rho)$ and $D(\rho)$ one has to replace the expression (34) of $F_L(j, \rho_a, \rho_b)$ by its convex envelope.

8. Free energy functional

If one divides a system of linear size L into n boxes of linear size l (in dimension d, one has $n = L^d/l^d$ such boxes), one can try to determine the probability of finding a certain density profile $\{\rho_1, \rho_2, \ldots, \rho_n\}$, i.e. the probability of seeing $l^d \rho_1$ particles in the first box, $l^d \rho_2$ particles in the second box,..., $l^d \rho_n$ in the nth box. For large L one expects the following L dependence of this probability

$$\mathrm{Pro}(\rho_1, \ldots, \rho_n) \sim \exp[-L^d \mathcal{F}(\rho_1, \rho_2, \ldots, \rho_n)], \tag{37}$$

where $\mathcal{F}$ is a large deviation function which generalizes $a(\rho)$ introduced in (3). If one defines a reduced coordinate $\vec{x}$ by

$$\vec{r} = L\vec{x} \tag{38}$$

and if one takes the limit $l \to \infty$ with $l \ll L$ so that the number of boxes becomes infinite, one can define a functional $\mathcal{F}(\rho(\vec{x}))$ for an arbitrary density profile $\rho(\vec{x})$,

$$\mathrm{Pro}(\rho(\vec{x})) \sim \exp[-L^d \mathcal{F}(\rho(\vec{x}))]. \tag{39}$$

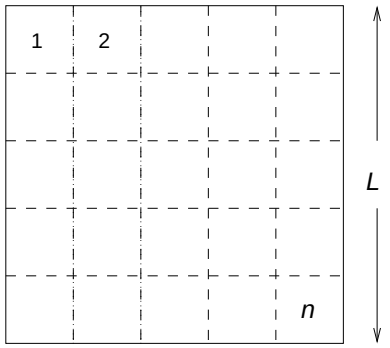

Figure 4. One specifies the density ρ_i in each box i.

At equilibrium the functional $\mathcal{F}$ can be expressed in terms of the free energy $f(\rho)$ per unit volume (6) at density ρ: if one considers Vr particles in a volume $V = L^d$, one can show that for systems with short range pair interactions

$$\mathrm{Pro}(\rho_1, \ldots, \rho_n) = \frac{Z_v(v\rho_1)\ldots Z_v(v\rho_n)}{Z_V(Vr)} \exp\left[O\left(\frac{L^d}{l}\right)\right], \tag{40}$$

where $v = l^d$. Comparing with (37), in the limit $L \to \infty$, $l \to \infty$, keeping n fixed gives

$$\mathcal{F}(\rho_1, \rho_2, \ldots, \rho_n) = \frac{1}{kT}\frac{1}{n}\sum_{i=1}^{n}[f(\rho_i) - f(r)]. \tag{41}$$

In the limit of an infinite number of boxes, this becomes

$$\mathcal{F}(\rho(\vec{x})) = \frac{1}{kT}\int \mathrm{d}\vec{x}\ [f(\rho(\vec{x})) - f(r)]. \tag{42}$$

Thus the large deviation functional $\mathcal{F}$ is fully determined by the knowledge of the free energy $f(\rho)$ per unit volume. From (42), we see that this functional is a *local* functional of $\rho(\vec{x})$. It is also *convex* (as the free energy is a convex function of ρ). By expanding (42) around the most likely profile $\overline{\rho}(\vec{x}) = r$, one obtains from (42) that the fluctuations of the density profile are Gaussian with a variance related to the second derivative of the free energy, i.e. related to the compressibility as predicted by Smoluchowski and Einstein. Lastly one knows (by the Landau argument) that, with short range interactions, there is no phase transition in one dimension.

For non-equilibrium systems, the large deviation functional of the density has been calculated so far in very few cases [4–8,19,21]. For the asymmetric simple exclusion process (ASEP) with open boundaries, which is an extension of the SSEP discussed in §4, where the hopping rates in the bulk become asymmetric (a rate 1 to jump to the right and a rate q to jump to the left), the exact expression of $\mathcal{F}(\rho(\vec{x}))$ obtained in [6,7] shows that, depending on q, ρ_a, ρ_b, the functional may be *non-local*, *non-convex*. It was also shown that for the ASEP, the fluctuations of the density are non-Gaussian [3,6,7] and are not related to the expansion of the functional around the most likely profile. It is also well-known that non-equilibrium systems may exhibit phase transitions in one dimensions [13,22–25].

For the SSEP (in one dimension), the functional $\mathcal{F}(\rho(x))$ is given by the following exact expression:

- at equilibrium, i.e. for $\rho_a = \rho_b = r$,

$$\mathcal{F}(\rho(x)) = \int_0^1 B(\rho(x), r)\mathrm{d}x, \tag{43}$$

where

$$B(\rho, r) = (1-\rho)\log\frac{1-\rho}{1-r} \ + \ \rho\log\frac{\rho}{r}. \tag{44}$$

This can be derived easily. When $\rho_a = \rho_b = r$, the steady state is a Bernoulli measure where all the sites are occupied independently with probability r. Therefore if one divides a chain of length L into L/l intervals of length l, one has

$$\mathrm{Pro}(\rho_1, \ldots, \rho_n) = \prod_i^{L/l} \frac{l!}{[l\rho_i]![l(1-\rho_i)]!} r^{l\rho_i}(1-r)^{l(1-\rho_i)} \tag{45}$$

and using Stirling's formula one gets (43), (44).

- In the non-equilibrium case, i.e. for $\rho_a \neq \rho_b$, it was shown in [4,5,21] that

$$\mathcal{F}(\rho(x)) = \int_0^1 \mathrm{d}x \ \left[B(\rho(x), F(x)) + \log \frac{F'(x)}{\rho_b - \rho_a} \right], \tag{46}$$

where the function $F(x)$ is the monotone solution of the differential equation

$$\rho(x) = F + \frac{F(1-F)F''}{F'^2} \tag{47}$$

satisfying the boundary conditions $F(0) = \rho_a$ and $F(1) = \rho_b$.

This expression shows that $\mathcal{F}$ is a *non-local* functional of the density profile $\rho(x)$ as $F(x)$ depends (in a non-linear way) on the profile $\rho(y)$ at all points y. For example if the difference $\rho_a - \rho_b$ is small, one can expand $\mathcal{F}$ and obtain an expression where the non-local character of the functional is clearly visible

$$\mathcal{F} = \int_0^1 \mathrm{d}x \ B(\rho(x), \rho(x)) + \frac{(\rho_a - \rho_b)^2}{[\rho_a(1-\rho_a)]^2}$$
$$\times \int_0^1 \mathrm{d}x \int_x^1 \mathrm{d}y \ x(1-y)(\rho(x) - \overline{\rho}(x))(\rho(y) - \overline{\rho}(y)) + O(\rho_a - \rho_b)^3.$$

Here $\overline{\rho}(x)$ is the most likely profile given by

$$\overline{\rho}(x) = (1-x)\rho_a + x\rho_b. \tag{48}$$

9. Derivation of the functional for the SSEP

The expressions (46), (47) have been derived by two different approaches:

- *The matrix approach:* For exclusion processes in one dimension with open boundary conditions, the weights of all microscopic configurations are known exactly [13,26,27]. In [13] it was shown that the probability of a microscopic configuration $\{\tau_1, \tau_2, \ldots, \tau_L\}$ can be written as the matrix element of a product of L matrices

$$\mathrm{Pro}(\tau_1, \tau_2, \ldots, \tau_L) = \frac{\langle W | X_1 X_2 \ldots X_L | V \rangle}{\langle W | (D+E)^L | V \rangle}, \tag{49}$$

where the matrix X_i depends on the occupation τ_i of site i,

$$X_i = \tau_i D + (1 - \tau_i) E \tag{50}$$

and the matrices D and E satisfy the following algebraic rules:

$$\begin{aligned} &DE - ED = D + E, \\ &\langle W|(\alpha E - \gamma D) = \langle W|, \\ &(\beta D - \delta E)|V\rangle = |V\rangle. \end{aligned} \tag{51}$$

Let us introduce the following left and right eigenvectors of the operators $\rho_a E - (1 - \rho_a)D$ and $(1 - \rho_b)D - \rho_b E$,

$$\begin{aligned} &\langle \rho_a, a| \ [\rho_a E - (1 - \rho_a)D] = a\langle \rho_a, a|, \\ &[(1 - \rho_b)D - \rho_b E] \ |\rho_b, b\rangle = b|\rho_b, b\rangle. \end{aligned} \tag{52}$$

It is easy to see, using the definition (8), that the vectors $\langle W|$ and $|V\rangle$ are given by

$$\begin{aligned} &\langle W| = \langle \rho_a, (\alpha + \gamma)^{-1}|, \\ &|V\rangle = |\rho_b, (\beta + \delta)^{-1}\rangle. \end{aligned} \tag{53}$$

It is then possible to show, using (51) that $DE - ED = D + E$ and the definition of the eigenvectors (52) that (for $\rho_b < \rho_a$)

$$\begin{aligned} \frac{\langle \rho_a, a|Y_1 Y_2|\rho_b, b\rangle}{\langle \rho_a, a|\rho_b, b\rangle} = &\oint\limits_{\rho_b < |\rho| < \rho_a} \frac{\mathrm{d}\rho}{2i\pi} \frac{(\rho_a - \rho_b)^{a+b}}{(\rho_a - \rho)^{a+b}(\rho - \rho_b)} \\ &\times \frac{\langle \rho_a, a|Y_1|\rho, b\rangle}{\langle \rho_a, a|\rho, b\rangle} \frac{\langle \rho, 1 - b|Y_2|\rho_b, b\rangle}{\langle \rho, 1 - b|\rho_b, b\rangle}, \end{aligned} \tag{54}$$

where Y_1 and Y_2 are arbitrary polynomials of matrices D and E. (To prove (54) it is sufficient to establish it when both Y_1 and Y_2 are of the form $E^n D^{n'}$ as any polynomial can be reduced to a sum of such terms by the relation $DE - ED = D + E$. One can also, and this is easier, prove (54) for Y_1 of the form $[\rho_a E - (1 - \rho_a)D]^n [D + E]^{n'}$ and Y_2 of the form $[D + E]^{n''}[(1 - \rho_b)D - \rho_b E]^{n'''}$ and show using $DE - ED = D + E$ that any polynomial Y_1 or Y_2 can be reduced to a finite sum of such terms.)

To calculate the large deviation functional for a given profile $\rho(x)$, one needs to sum the weights of all microscopic configurations consistent with this profile. One way of doing it [4,5] consists in breaking the system into subsystems and in using (54) to relate the density functional of a large system to the functionals of its subsystems (when the subsystems are themselves large enough, one can evaluate the integral in (54) by a saddle point method).

- *The macroscopic fluctuation theory:* Bertini *et al* [21,28,29] have developed a different and more general approach to calculate this large deviation functional. Their idea is that to observe a certain density profile $\rho(x)$ at time t, one has to find out how this fluctuation is produced. For large L, one has to find the optimal path $\rho(x,s)$ for $-\infty < s < t$ in the space of profiles which goes from the typical profile $\overline{\rho}(x)$ to the desired profile $\rho(x)$. In other words,

$$\mathrm{Pro}(\rho(x)) \sim \max_{\rho(x,s)} \exp\left\{ -L \int_{-\infty}^{t} \mathrm{d}s \int_0^1 \mathrm{d}x \frac{[j + D(\rho)\frac{\mathrm{d}\rho}{\mathrm{d}x}]^2}{2\sigma(\rho)} \right\}, \tag{55}$$

  where the current $j(x,s)$ is related to the density profile $\rho(x,s)$ by

$$\frac{\mathrm{d}\rho(x,s)}{\mathrm{d}s} = -\frac{\mathrm{d}j(x,s)}{\mathrm{d}x} \tag{56}$$

  and the optimal path $\rho(x,s)$ satisfies

$$\rho(x,-\infty) = \overline{\rho}(x),$$
$$\rho(x,t) = \rho(x).$$

So far it has not been possible to find the explicit expression of the functional $\mathcal{F}$ for general $D(\rho)$ and $\sigma(\rho)$. For the SSEP, however, this approach gives (46), (47). It also leads to the same expression of $\mathcal{F}$ as found by the matrix approach [8] in the weak asymmetric ASEP.

10. Conclusion

The results on the distribution of current flowing through a system in its non-equilibrium steady state are so far rather limited. It would be nice to calculate $\mu(\lambda)$ exactly for a SSEP of finite length L and to justify that way the conjecture (28). It would also be interesting to establish under what conditions on $\sigma(\rho)$ and $D(\rho)$, the additivity principle [1] is valid.

Extending the results on the current fluctuations to the case of a strong asymmetry, the case of several species of particles or situations where more than one quantity is conserved (numbers of particles, energy, momentum..) would also be very interesting.

Lastly it would be nice to obtain an explicit expression of the functional $\mathcal{F}(\rho(x))$ for general $\sigma(\rho)$ and $D(\rho)$ or for models with several species of particles [25].

Acknowledgements

The author would like to thank T Bodineau, B Douçot, C Enaud, J L Lebowitz, P E Roche and E R Speer. Most results reported in this talk were obtained in a series of works done in collaboration with them. The author would also like to thank E Brunet, M R Evans, V Hakim, D Mukamel and V Pasquier for useful discussions.

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