Universal cumulants of the current in diffusive systems on a ring

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We calculate exactly the first cumulants of the integrated current and of the activity (which is the total number of changes of configurations) of the symmetric simple exclusion process on a ring with periodic boundary conditions. Our results indicate that for large system sizes the large deviation functions of the current and of the activity take a universal scaling form, with the same scaling function for both quantities. This scaling function can be understood either by an analysis of Bethe ansatz equations or in terms of a theory based on fluctuating hydrodynamics or on the macroscopic fluctuation theory of Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim.

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I. INTRODUCTION

The symmetric simple exclusion process (SSEP) [1–4] is one of the simplest lattice gas models studied in the theory of nonequilibrium systems. It consists of hard-core particles hopping with equal rates to either of their nearest neighbor sites, on a regular lattice. At equilibrium, when isolated, the system reaches in the long time limit an equilibrium where all accessible configurations are equally likely. Also, when equilibrium is achieved by contact with one or several reservoirs at a single density ρ, all sites are occupied with this density ρ and the occupation numbers of different sites are uncorrelated.

As soon as the system is maintained out of equilibrium, by contact with reservoirs at unequal densities, there is a current of particles and one observes long range correlations in the steady state [5]. In this out of equilibrium case several approaches have been developed to calculate steady state properties, such as the fluctuations or the large deviations of the density or of the current [6–18].

A lot of progress has been made in recent years on the study of the fluctuations and the large deviation functions of the current in equilibrium or nonequilibrium systems. The large deviation function of the current can be viewed as the dynamical analog of a free energy, as discussed by Ruelle in the early seventies [19]. The idea back then was to build up a thermodynamic formalism based upon probabilities over time realizations rather than over instantaneous configurations. Generic properties of these large deviation functions were later discovered such as the fluctuation theorem, which determines how the large deviation function of the current is changed under time reversal symmetry [20–28].

In the present work, we obtain exact expressions for the first cumulants of the integrated current and of the activity (which is the number of changes of configurations) during a long time t for the SSEP consisting of N particles on a ring of L sites. For large system sizes, these cumulants and the associated large deviation functions take universal scaling forms. We show how these scaling forms can be calculated for the SSEP by the Bethe ansatz or for more general diffusive systems on a ring by a theory based on fluctuating hydrodynamics or on the macroscopic fluctuation theory developed by Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim [9,10,16–18]. In the Bethe ansatz approach these scaling forms can be extracted from a detailed analysis of finite-size effects with results similar to those obtained recently for quantum spin chains in the context of string theory [29,30]. In the fluctuating hydrodynamics approach, it results from the discreteness of the wave vectors of the fluctuating modes on the ring.

Universal distributions of the current characteristic of the universality class of the KPZ (Kardar-Parisi-Zhang) equation [31–34], have been calculated in the past [35–38] for the asymmetric exclusion process (ASEP). The distributions obtained in the present paper are different and belong to the Edwards-Wilkinson universality class [39].

We begin by presenting in Sec. II exact expressions of the first cumulants of the current and of the activity for the SSEP on a ring. This is where we see that the cumulants of the integrated current and of the activity take scaling forms when the size of the ring becomes large and where emerges the idea that the large deviation function of the current and of the activity obey the same universal scaling function. This is confirmed in Sec. III by Bethe ansatz calculations. By resorting to fluctuating hydrodynamics in Sec. IV we are able to formulate the particular case of the SSEP within a more general framework using the Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim approach and to show that the same universal distribution of the current fluctuations are present in a larger family of diffusive systems.

II. EXACT EXPRESSIONS OF THE FIRST CUMULANTS

We consider a system of N particles on a one-dimensional lattice of L sites with periodic boundary conditions. Each site is either empty or occupied by a single particle. A microscopic configuration $\mathcal{C}=\{n_i\}_{i,1,\ldots,L}$ can be specified by L occupation numbers n_i (where $n_i=1$ if site i is occupied and

$n_i=0$ if site i is empty). In the simple symmetric exclusion process, SSEP, each particle hops to its right neighbor at rate 1 or to its left neighbor at rate 1, provided the target site is empty. In the present paper we try to determine the distribution of the total integrated current $Q(t)$ and of the total number $K(t)$ of changes of configuration (that we will call the activity [40]) during a time interval $(0,t)$. To do so we define the generating functions of the cumulants of Q and K as

$$\psi_Q(s) = \lim_{t\to\infty} \frac{\ln\langle e^{-sQ}\rangle}{t}, \quad \psi_K(s) = \lim_{t\to\infty} \frac{\ln\langle e^{-sK}\rangle}{t}, \tag{1}$$

where the brackets denote an average over the time evolutions during the time interval $(0,t)$. As the evolution is an irreducible Markov process with a finite number of states, the long time limits in Eq. (1) do not depend on the initial configuration and the generating functions defined in Eq. (1) can be calculated as the largest eigenvalue of a matrix [20,36,41].

Because the calculations are very similar for both observables K and Q, we shall first focus on the activity K and explain how to calculate the cumulant generating function $\psi_K(s)$ as a perturbation series in powers of s. We will then present only the results for $\psi_Q(s)$.

A. Cumulants of the activity $K(t)$

In order to determine ψ_K, as in [36], one can write a master equation for the probability $P(\mathcal{C},K,t)$ to find the system in configurations $\mathcal{C}$ at time t, given that the activity at time t is K [i.e., given that the system has changed K times of configurations during the time interval $(0,t)$].

$$\partial_t P(\mathcal{C},K,t) = -r(\mathcal{C})P(\mathcal{C},K,t) + \sum_{\mathcal{C}'} W(\mathcal{C}' \to \mathcal{C})P(\mathcal{C}',K-1,t), \tag{2}$$

where $W(\mathcal{C}\to\mathcal{C}')$ is the transition rate from configuration $\mathcal{C}$ to $\mathcal{C}'$, and $r(\mathcal{C})=\Sigma_{\mathcal{C}'}W(\mathcal{C}\to\mathcal{C}')$ is the escape rate from configuration $\mathcal{C}$.

If one introduces the generating function $\hat{P}(\mathcal{C},s,t) = \Sigma_K e^{-sK}P(\mathcal{C},K,t)$, its evolution satisfies

$$\partial_t \hat{P}(\mathcal{C},s,t) = \sum_{\mathcal{C}'} \mathbb{W}_K(\mathcal{C},\mathcal{C}')\hat{P}(\mathcal{C}',s,t), \tag{3}$$

where

$$\mathbb{W}_K(\mathcal{C},\mathcal{C}') = e^{-s}W(\mathcal{C}' \to \mathcal{C}) - r(\mathcal{C})\delta_{\mathcal{C},\mathcal{C}'}. \tag{4}$$

In the long time limit, $\hat{P}(\mathcal{C},s,t)$ grows (or decays) exponentially with time, with a rate given by the eigenvalue with largest real part [36] of the modified matrix $\mathbb{W}_K$. Thus $\psi_K(s)$ can be calculated as this largest eigenvalue of $\mathbb{W}_K$. For $s=0$, $\mathbb{W}_K$ reduces to the evolution operator of the master equation for the symmetric simple exclusion process, and this largest eigenvalue (which is 0) as well as the related eigenvector are known. We now present a way of obtaining the large deviation function ψ_K, by a perturbative expansion [41,42] in powers of s.

The idea is to start from the eigenvalue equation for ψ_K and its eigenvector $\tilde{P}$,

$$\psi_K(s)\tilde{P}(\mathcal{C},s) = \sum_{\mathcal{C}'} \mathbb{W}_K(\mathcal{C},\mathcal{C}')\tilde{P}(\mathcal{C}',s), \tag{5}$$

normalized such that $\Sigma_{\mathcal{C}}\tilde{P}(\mathcal{C},s)=1$. One can then define the average $\langle\mathcal{A}(\mathcal{C})\rangle_s$ of an observable $\mathcal{A}(\mathcal{C})$ in the corresponding eigenstate, [i.e., $\langle\mathcal{A}(\mathcal{C})\rangle_s=\Sigma_{\mathcal{C}}\mathcal{A}(\mathcal{C})\tilde{P}(\mathcal{C},s)$ and this is the same as averaging, in the limit of a long time interval $(0,t)$, over all trajectories weighted by a coefficient $e^{-sK(t)}$]. Note that, though the value of $K(t)$ is defined on trajectories running from 0 to t, the observable $\mathcal{A}(\mathcal{C})$ is evaluated at the final time t. From the eigenvalue Eq. (5), one gets

$$\psi_K(s)\langle\mathcal{A}(\mathcal{C})\rangle_s = e^{-s}\left\langle \sum_{\mathcal{C}'} W(\mathcal{C}\to\mathcal{C}')\mathcal{A}(\mathcal{C}')\right\rangle_s - \langle\mathcal{A}(\mathcal{C})r(\mathcal{C})\rangle_s, \tag{6}$$

where the escape rate $r(\mathcal{C})$ is twice the number of clusters of adjacent particles in the system

$$r(\mathcal{C}) = \sum_{\mathcal{C}'} W(\mathcal{C}\to\mathcal{C}') = 2\sum_{j=1}^{L} n_j(1-n_{j+1}). \tag{7}$$

Choosing $\mathcal{A}(\mathcal{C})=1$ in Eq. (6) leads to

$$\psi_K(s) = (e^{-s}-1)\langle r(\mathcal{C})\rangle_s = 2L(e^{-s}-1)[\rho - C_s(1)], \tag{8}$$

where $C_s(r)=\langle n_i n_{i+r}\rangle_s$ is the correlation function (which by translational invariance does not depend on i) computed within the eigenstate $\tilde{P}(\mathcal{C},s)$, and $\rho=N/L$ is the average density.

For the leading contribution as $s\to 0$, we can use the fact that at $s=0$ the eigenvector is known (this is the equilibrium distribution, for which all allowed microscopic configurations are equally likely), so that $\psi_K(s)=-2N(1-\frac{N-1}{L-1})s+\mathcal{O}(s^2)$. In order to compute the $\mathcal{O}(s^2)$ contribution from Eq. (8), we need to evaluate $C_s(1)$ at order s, which can be done by choosing $\mathcal{A}(\mathcal{C})=n_i n_j$ in Eq. (6). This requires the knowledge of the correlation function $C_s(r)=\langle n_i n_{i+r}\rangle_s$ at order $\mathcal{O}(s)$. For $\mathcal{A}(\mathcal{C})=n_i n_j$ in Eq. (6) one gets

$$C_s(1) - C_s(2) = sA_{N,L} + \mathcal{O}(s^2),$$

where

$$A_{N,L} = \frac{N(N-1)(L-N)(L-N-1)}{L(L-1)^2(L-2)},$$

$$C_s(r+1) + C_s(r-1) - 2C_s(r) = s\frac{2A_{N,L}}{L-3} + \mathcal{O}(s^2) \quad \text{for } 2 \leqslant r \leqslant L-2, \tag{9}$$

which have the following solution:

$$C_s(r) = \frac{N(N-1)}{L(L-1)} - sA_{N,L}\frac{6r(L-r)-L(L+1)}{6(L-3)} + \mathcal{O}(s^2). \tag{10}$$

We can therefore extract ψ_K up to $\mathcal{O}(s^2)$ and $\langle K^2\rangle_c/t$ follows.

To obtain higher cumulants, we have repeated the same procedure, with the observables $\mathcal{A}(\mathcal{C})=n_i n_j n_k$ and

$\mathcal{A}(\mathcal{C})=n_i n_j n_k n_l$. The calculations are longer but very similar. We found that the first cumulants of K, $\lim_{t\to\infty}\langle K^n\rangle_c/t = (-1)^n \frac{d^n\psi_K}{ds^n}|_{s=0}$, when expressed in terms of the system size L and of

$$\sigma(\rho) = 2\rho(1-\rho) = \frac{2N(L-N)}{L^2} \tag{11}$$

are given by (in the $t\to\infty$ limit)

$$\frac{\langle K\rangle}{t} = L^2\frac{\sigma}{L-1}, \quad \frac{\langle K^2\rangle_c}{t} = \frac{L^2\sigma(L^2\sigma+4L-4)}{6(L-1)^2},$$

$$\frac{\langle K^3\rangle_c}{t} = \frac{L^2\sigma[-L^5\sigma^2+L^4\sigma(2+3\sigma)-2L^3\sigma+48(L-1)^2]}{60(L-1)^3},$$

$$\begin{aligned}\frac{\langle K^4\rangle_c}{t} = L^2\sigma[&\sigma^3L^6(10L^3-70L^2+175L-153)\\ &-4\sigma^2L^4(L-1)(11L^3-69L^2+154L-126)\\ &+16\sigma L^2(L-1)^2(3L^3-17L^2+46L-63)\\ &+2112(L-1)^3(L-3)][2520(L-1)^4(L-3)]^{-1}.\end{aligned} \tag{12}$$

When L becomes large, while $\rho=N/L$ is kept fixed, the asymptotic behavior of the above cumulants reads

$$\frac{\langle K\rangle}{t} \simeq \sigma L, \quad \frac{\langle K^2\rangle_c}{t} \simeq \frac{\sigma^2}{6}L^2,$$

$$\frac{\langle K^3\rangle_c}{t} \simeq -\frac{\sigma^3}{60}L^4, \quad \frac{\langle K^4\rangle_c}{t} \simeq \frac{\sigma^4}{252}L^6. \tag{13}$$

One might have expected the derivatives at $s=0$ of the eigenvalue ψ_K to become extensive for a large system size L (after all, as we shall see it in Sec. III, it is always possible to view ψ_K as the ground state energy of a short range Hamiltonian). Yet this is not the case since the second and higher cumulants grow faster than linearly with L at fixed density ρ. This suggests that, in the large L limit, ψ_K/L becomes a singular function of s at $s=0$.

Also one can guess from Eq. (13) that for $n\geq 2$,

$$\frac{\langle K^n\rangle_c}{t} \sim \sigma^n L^{2n-2},$$

and that for $L\to\infty$ and $s\to 0$, the eigenvalue ψ_K takes a scaling form

$$\lim_{L\to\infty} L^2\left[\psi_K(s) + s\frac{\langle K\rangle}{t}\right] = \mathcal{F}_K\left(\frac{\sigma}{2}L^2 s\right), \tag{14}$$

where the scaling function $\mathcal{F}_K$ is given by

$$\mathcal{F}_K(u) = \frac{1}{3}u^2 + \frac{1}{45}u^3 + \frac{1}{378}u^4 + \mathcal{O}(u^5). \tag{15}$$

We shall see in Secs. III and IV that this scaling function can be fully determined and written as

$$\mathcal{F}_K(u) = -4\sum_{n\geq 1}[n\pi\sqrt{n^2\pi^2-2u} - n^2\pi^2 + u], \tag{16}$$

or equivalently (see Appendix A) as

$$\mathcal{F}_K(u) = \sum_{k\geq 2}\frac{B_{2k-2}}{(k-1)!k!}(-2u)^k, \tag{17}$$

where the Bernoulli numbers B_n are known to be simply the coefficients of the expansion $x(e^x-1)^{-1}=\Sigma_n B_n x^n/n!$. As a consequence, the generalization of Eq. (13) will be for $n\geq 2$,

$$\frac{\langle K^n\rangle_c}{t} \simeq \frac{B_{2n-2}}{(n-1)!}\sigma^n L^{2n-2}. \tag{18}$$

B. Cumulants of the current

The same procedure can be followed for the total integrated current Q [which can be defined by $Q=\Sigma_{j=1}^N x_j(t)$, where $x_j(t)$ is the total displacement of the jth particle during the time interval $(0,t)$]. Its cumulant generating function ψ_Q defined in Eq. (1) is the eigenvalue (with largest real part) of the matrix

$$\mathbb{W}_Q(\mathcal{C},\mathcal{C}') = W(\mathcal{C}'\to\mathcal{C})e^{-sj(\mathcal{C}',\mathcal{C})} - r(\mathcal{C})\delta_{\mathcal{C},\mathcal{C}'}, \tag{19}$$

where $j(\mathcal{C}',\mathcal{C})$ is +1 or −1 depending on whether a particle has moved to the right or to the left when the system jumps from configuration $\mathcal{C}'$ to configuration $\mathcal{C}$. Using an expansion in powers of s as in Sec. II A we have obtained (in the limit $t\to\infty$)

$$\frac{\langle Q^2\rangle}{t} = \frac{L^2\sigma}{L-1}, \quad \frac{\langle Q^4\rangle_c}{t} = \frac{1}{2}\frac{L^4\sigma^2}{(L-1)^2}, \tag{20}$$

$$\frac{\langle Q^6\rangle_c}{t} = -\frac{L^6\sigma^2[(L^2-L+2)\sigma-2(L-1)]}{4(L-1)^3(L-2)},$$

$$\frac{\langle Q^8\rangle_c}{t} = \frac{L^8\sigma^2[(10L^4-2L^3+27L^2-15L+18)\sigma^2-4(L-1)(11L^2-L+12)\sigma+48(L-1)^2]}{24(L-1)^4(L-2)(L-3)}, \tag{21}$$

with the corresponding large L behaviors (for $\rho=N/L$ fixed)

$$\frac{\langle Q^2\rangle_c}{t} \simeq \sigma L, \quad \frac{\langle Q^4\rangle_c}{t} \simeq \frac{\sigma^2}{2}L^2,$$

$$\frac{\langle Q^6\rangle_c}{t} \simeq -\frac{\sigma^3}{4}L^4, \quad \frac{\langle Q^8\rangle_c}{t} \simeq \frac{5\sigma^4}{12}L^6. \tag{22}$$

As for K, these results indicate that for $n \geqslant 2$,

$$\frac{\langle Q^{2n}\rangle_c}{t} \sim \sigma^n L^{2n-2},$$

and that ψ_Q takes a scaling form, in the limit $L\to\infty$ and $s\to 0$,

$$\lim_{L\to\infty} L^2\left[\psi_Q(s) - \frac{s^2}{2}\frac{\langle Q^2\rangle_c}{t}\right] = \mathcal{F}_Q\left(-\frac{\sigma}{4}L^2 s^2\right), \tag{23}$$

where, according to Eq. (22), the expansion of $\mathcal{F}_Q(u)$ in powers of u coincides with the expansion (15) of $\mathcal{F}_K(u)$, at least up to the fourth order in u.

We will see, in Sec. IV, that these two scaling functions [which appear in Eqs. (14) and in (23)] are in fact the same. Therefore the formula which generalizes Eq. (22) will be for $n \geqslant 2$,

$$\frac{\langle Q^{2n}\rangle_c}{t} \simeq \frac{(2n)!B_{2n-2}}{2^n(n-1)!n!}\sigma^n L^{2n-2}. \tag{24}$$

III. BETHE ANSATZ

It is well known that the Bethe ansatz allows one to calculate the eigenvalues of matrices such as $\mathbb{W}_K(\mathcal{C},\mathcal{C}')$ and $\mathbb{W}_Q(\mathcal{C},\mathcal{C}')$ defined in Eqs. (4) and (19) for exclusion processes [35–38,43–50]. In this section we show how to obtain the scaling forms (14) and (23) from the Bethe ansatz equations.

A. Relation to spin chains

It is possible to write the matrices $\mathbb{W}_K(\mathcal{C},\mathcal{C}')$ and $\mathbb{W}_Q(\mathcal{C},\mathcal{C}')$ as quantum spin-chain Hamiltonians [51]. We use the correspondence in which the z component of a two state spin operator is up when a particle is present at site i, and is down otherwise. In this basis one finds that

$$\hat{H}_K = \frac{L}{2} - \frac{1}{2}\sum_{i=1}^{L}\left[e^{-s}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + \sigma_i^z\sigma_{i+1}^z\right] = -\mathbb{W}_K,$$

$$\hat{H}_Q = \frac{L}{2} - \frac{1}{2}\sum_{i=1}^{L}\left[\cosh s(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + \sigma_i^z\sigma_{i+1}^z - i\sinh s(\sigma_i^x\sigma_{i+1}^y - \sigma_i^y\sigma_{i+1}^x)\right] = -\mathbb{W}_Q, \tag{25}$$

where we have resorted to the Pauli matrices $\sigma_i^{x,y,z}$. In this language, the quantities ψ_K and ψ_Q are the ground state energies of these operators. It also suggests that the methods of one-dimensional exactly solvable models apply in our case, such as the Bethe ansatz, as was exploited for similar systems in the past [36,52,53].

As the number of particles on the ring is fixed, we need to find the ground state with a fixed particle density ρ, that is, at fixed transverse magnetization $\Sigma_i\sigma_i^z$. The quantum operators appearing in Eq. (25) have of course been extensively studied [54], including within the framework of stochastic dynamics [55]. For instance, following the notations of Baxter [54] the operator $e^s\hat{H}_K$ is the ferromagnetic XXZ chain with anisotropy parameter $\Delta=e^s$. Similarly, $\hat{H}_Q$ corresponds to an XXZ chain with additional Dzyaloshinskii-Moriya interactions. A study of an operator closely related to $\hat{H}_Q$ was carried out by Kim [52] in 1995. His results will be recalled at the end of the present section.

The Bethe ansatz consists in looking for the ground state of $\hat{H}_{K \text{ or } Q}$ in the form of a linear combination of N-particle plane waves (see [43,52]). We denote by $\{x_j\}_{j=1,\ldots,N}$ the positions of the N particles and we postulate that the right eigenvector of $\mathbb{W}_K$ can be cast in the form

$$P(\{x_j\},s) = \sum_{\mathcal{P}}\mathcal{A}(\mathcal{P})\prod_{j=1}^{N}[\zeta_{p(j)}]^{x_j}, \tag{26}$$

where $\mathcal{P}=[p(1),\ldots,p(N)]$ is a permutation over the first N integers, and the ζ_j's are *a priori* complex numbers. This is an exact eigenstate provided these parameters satisfy the so-called Bethe equations. These take different forms for K and Q. We now discuss how to implement the Bethe ansatz to calculate $\psi_K(s)$ and $\psi_Q(s)$ defined in Eq. (1). Technical details have been gathered in the appendixes.

B. Bethe ansatz for *K*

For the expression (26) to be an eigenvector of $\hat{H}_K$ or $\hat{H}_Q$ the ζ_j's have to satisfy a number of constraints [56], the so-called Bethe (see, for example, [49]) equations

$$\zeta_i^L = \prod_{\substack{j=1\\ j\neq i}}^{N}\left[-\frac{1-2e^s\zeta_i+\zeta_i\zeta_j}{1-2e^s\zeta_j+\zeta_i\zeta_j}\right]. \tag{27}$$

The expression of $\psi_K(s)$ is given by

$$\psi_K(s) = e^{-s}\sum_{j=1}^{N}\left(\zeta_j + \frac{1}{\zeta_j}\right) - 2N. \tag{28}$$

Our goal is to obtain Eq. (14) from Eqs. (27) and (28) in the double limit $s\to 0$ and $L\to\infty$ keeping sL^2 and $N/L=\rho$ fixed. Because of the particle-hole symmetry the discussion below is limited to the case $\rho \leqslant \frac{1}{2}$.

For $s<0$ and L finite it is known [54,56] that the ζ_j's lie on the unit circle. In the large L limit, they become dense on a finite arc of the unit circle with a smooth density [54]. The calculation which follows is based on the assumption that, for large L, the ζ_j's become regularly spaced with distances between consecutive ζ_j's of order $1/L$. (The case $s>0$, where the ζ_j's are real can be approached by similar methods).

If one writes

$$e^s = \cos\delta, \tag{29}$$

and

$$\zeta_j = e^{ik_j\delta}, \tag{30}$$

Eq. (27) becomes

$$k_i = \frac{1}{L}\sum_{\substack{j=1\\ j\neq i}}^{N} U(k_i,k_j), \tag{31}$$

where

$$U(k_i,k_j) = \frac{1}{i\delta}\ln\left[-\frac{1-2e^{ik_i\delta}\cos\delta + e^{i(k_j+k_i)\delta}}{1-2e^{ik_j\delta}\cos\delta + e^{i(k_j+k_i)\delta}}\right]. \tag{32}$$

In the limit $\delta\to 0$, one can check that when $k_i-k_j=\mathcal{O}(1)$,

$$U(k_i,k_j) = 2\frac{1-k_ik_j}{k_i-k_j} + \mathcal{O}(\delta^2). \tag{33}$$

In the large L limit, however, the distance between consecutive k_i becomes of order $1/L\sim\delta$ and for $i-j$ of order 1, one should use instead

$$U(k_i,k_j) = \frac{1}{i\delta}\ln\left[\frac{k_i-k_j+i\delta(1-k_i^2)+i\delta k_i(k_i-k_j)-\delta^2k_i(1-k_i^2)}{k_i-k_j-i\delta(1-k_i^2)-i\delta k_i(k_i-k_j)+\delta^2k_i(1-k_i^2)}\right]. \tag{34}$$

Therefore one can rewrite Eq. (31) as

$$Lk_i \simeq \sum_{\substack{i-n_0\leq j\leq i+n_0\\ j\neq i}} \frac{1}{i\delta}\ln\left[\frac{k_i-k_j+i\delta(1-k_i^2)+i\delta k_i(k_i-k_j)-\delta^2k_i(1-k_i^2)}{k_i-k_j-i\delta(1-k_i^2)-i\delta k_i(k_i-k_j)+\delta^2k_i(1-k_i^2)}\right] + \sum_{j\notin[i-n_0,i+n_0]} 2\frac{1-k_ik_j}{k_i-k_j}, \tag{35}$$

where n_0 is a fixed large number $1\ll n_0\ll L$, so that one can use expression (33) for $|j-i|>n_0$ and Eq. (34) for $|j-i|\leq n_0$. As shown in Appendix B, the two sums (B4) and (B15) in Eq. (35) depend on the cutoff n_0 but this dependence disappears when the two terms in the right-hand side of Eq. (35) are added.

In the large L limit, the k_i become dense on an interval $(-\theta,\theta)$ of the real axis, with some density $g(k)$. In what follows we will assume that the k_i are regularly spaced according to this density, meaning that

$$L\int_{k_i}^{k_j} g(k)dk = j-i \quad\text{and}\quad L\int_{-\theta}^{\theta} g(k)dk = N. \tag{36}$$

Replacing the two sums in Eq. (35) by their expressions (B4) and (B15) obtained in Appendix B, one gets that for $k=k_i$ the density $g(k)$ should satisfy

$$k = 2\mathcal{P}\int_{-\theta}^{\theta} dk' g(k')\frac{1-k'^2}{k-k'} + \frac{1}{L}\left[\left(\frac{g'(k)(1-k^2)}{g(k)}-2k\right)\times\pi(1-k^2)g(k)L\delta\coth[\pi(1-k^2)g(k)L\delta]\right]. \tag{37}$$

If we make the change of variable $k'=\theta y$, $k=\theta x$, and

$$g(k)(1-k^2) = \phi(x), \tag{38}$$

Eq. (37) becomes

$$\mathcal{P}\int_{-1}^{1} dy\frac{\phi(y)}{y-x} = f(x), \tag{39}$$

where

$$f(x) = -\frac{\theta x}{2} + \frac{\pi(1-\theta^2x^2)\phi'(x)}{2\theta}\delta\coth[L\delta\pi\phi(x)] + \cdots . \tag{40}$$

As explained in Eqs. (C1) and (C2) of Appendix C one can invert Eq. (39) and express $\phi(x)$ in terms of $f(x)$,

$$\phi(x) = \frac{C}{\sqrt{1-x^2}} - \frac{1}{\pi^2\sqrt{1-x^2}}\mathcal{P}\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}f(y)dy, \tag{41}$$

where the constant C is so far an arbitrary constant.

For small δ, one can write Eq. (28), using Eqs. (30), (38), and (41), as

$$\psi_K(s) \simeq \sum_{j=1}^{N}\delta^2(1-k_j^2) \simeq L\delta^2\int_{-\theta}^{\theta} g(k)(1-k^2)dk$$
$$= L\delta^2\theta\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\left[C-\frac{1}{\pi^2}\mathcal{P}\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}f(y)dy\right],$$

which gives, using Eqs. (D5) and (D7)

$$\psi_K(s) \simeq L\delta^2\theta C\pi. \tag{42}$$

Also, as Eq. (36),

$$\int_{-\theta}^{\theta} g(k)dk = \rho,$$

one has Eqs. (38) and (41),

$$\rho = \theta\int_{-1}^{1} dx\left[\frac{C}{(1-\theta^2x^2)\sqrt{1-x^2}} - \frac{1}{\pi^2(1-\theta^2x^2)\sqrt{1-x^2}}\mathcal{P}\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}f(y)dy\right],$$

which can be simplified using Eqs. (8) and (12),

$$\rho = \frac{C\theta\pi}{\sqrt{1-\theta^2}} + \frac{\theta^3}{\pi\sqrt{1-\theta^2}}\int_{-1}^{1}\frac{f(y)y\sqrt{1-y^2}}{1-\theta^2y^2}dy. \tag{43}$$

C. Leading order in the large L limit

For large L (at fixed $L\delta$), Eq. (40) reduces to $f(x)=-\theta x/2$, so that Eq. (41) becomes to leading order using Eq. (D2),

$$\phi(x) = \frac{4\pi C-\theta}{4\pi\sqrt{1-x^2}} + \frac{\theta}{2\pi}\sqrt{1-x^2} + \mathcal{O}\left(\frac{1}{L}\right), \tag{44}$$

whereas Eq. (43) becomes using Eq. (D12),

$$\rho = \frac{C\theta\pi}{\sqrt{1-\theta^2}} + \frac{1}{2} + \frac{\theta^2-2}{4\sqrt{1-\theta^2}}. \tag{45}$$

Therefore for a fixed density ρ of particles, the constant C in Eqs. (41) and (44) and the eigenvalue (42) are given, to leading order in $\frac{1}{L}$, by

$$C = \frac{1}{\pi\theta}\left[\left(\rho-\frac{1}{2}\right)\sqrt{1-\theta^2} + \frac{2-\theta^2}{4}\right], \tag{46}$$

and

$$\psi_K(s) = L\delta^2\left[\left(\rho-\frac{1}{2}\right)\sqrt{1-\theta^2} + \frac{2-\theta^2}{4}\right]. \tag{47}$$

So far, the constant C remains undetermined.

The leading order corresponds to using expression (33) in Eq. (31) even when i and j differ by a few units. For the continuum description to be valid, we are now going to argue that $\phi(x)$ should remain finite as $x\to\pm 1$, or, in terms of the original density g, that $g(k)$ remains finite as $k\to\pm\theta$. This will impose [see Eq. (44)] that

$$C = \frac{\theta}{4\pi}.$$

Indeed if we order the N solutions k_i and focus on the ones closest to $\theta, \ldots < k_{N-1} < k_N \leq \theta$, then we may estimate using Eq. (36), the difference between k_N and θ, or between k_{N-1} and k_N. If $C \neq \frac{\theta}{4\pi}$, then $g(k)\sim(\theta-k)^{-1/2}$ as $k\to\theta$ implies that $k_N - k_{N-1}\sim L^{-2}$. This is not compatible with $k_N > \frac{2}{L}\frac{1-\theta^2}{k_N-k_{N-1}}$ [which follows from Eqs. (31) and (33)], where the right-hand side of this inequality would be $\mathcal{O}(L)$ in contradiction with the fact that $k_N\leq\theta$. Hence we must have $4\pi C=\theta$, in which case $k_N - k_{N-1}\sim L^{-2/3}$ and there is no contradiction. The fact that near the boundary the distance between consecutive k_j's is larger than $1/L$ is due to the vanishing of the density $g(k)$ at the boundary.

It then follows that

$$\theta = 2\sqrt{\rho(1-\rho)}, \tag{48}$$

and therefore $\psi_K(s)=L\delta^2\rho(1-\rho)$ and Eq. (44),

$$\phi(x) = \frac{\theta\sqrt{1-x^2}}{2\pi} + \mathcal{O}\left(\frac{1}{L}\right). \tag{49}$$

which agrees with the density given on page 163 of [54]

D. Next order

Once ϕ is known to leading order (49), one can update the expression (40),

$$f(x) = -\frac{\theta x}{2} - \frac{(1-\theta^2x^2)x}{4\sqrt{1-x^2}}\delta\coth\left[\frac{L\delta\theta\sqrt{1-x^2}}{2}\right] + \cdots, \tag{50}$$

and one gets from Eq. (43),

$$\rho = \frac{C\theta\pi}{\sqrt{1-\theta^2}} + \frac{1}{2} - \frac{\theta^2-2}{4\sqrt{1-\theta^2}} - \frac{\theta^3\delta}{4\pi\sqrt{1-\theta^2}}\int_{-1}^{1}y^2\coth\left[\frac{L\delta\theta\sqrt{1-y^2}}{2}\right]dy. \tag{51}$$

Then using the fact that [see Eq. (A2) in Appendix A]

$$\int_{-1}^{1} y^2\coth(u\sqrt{1-y^2})dy = \frac{\pi}{2u} + \frac{\pi}{2u^3}\mathcal{F}\left(-\frac{u^2}{2}\right), \tag{52}$$

we get

$$\rho = \frac{C\theta\pi}{\sqrt{1-\theta^2}} + \frac{1}{2} + \frac{\theta^2-2}{4\sqrt{1-\theta^2}} - \frac{\theta^2}{4L\sqrt{1-\theta^2}} - \frac{1}{L^3\delta^2\sqrt{1-\theta^2}}\mathcal{F}\left(-\frac{L^2\delta^2\theta^2}{8}\right), \tag{53}$$

and this gives Eq. (42),

$$\psi_K(s) = L\delta^2 C\pi\theta = L\delta^2\left[\left(\rho-\frac{1}{2}\right)\sqrt{1-\theta^2} + \frac{2-\theta^2}{4} + \frac{\theta^2}{4L} + \frac{1}{L^3\delta^2}\mathcal{F}\left(-\frac{L^2\delta^2\theta^2}{8}\right)\right]. \tag{54}$$

The leading order [the first two terms of Eq. (54)] has a minimum for θ given by Eq. (48). Therefore to obtain $\psi_K(s)$ at first order in $\frac{1}{L}$ one can simply replace θ by Eq. (48) in Eq. (54) and one gets

$$\psi_K(s) = \frac{L\delta^2\theta^2}{4}\left(1+\frac{1}{L}\right) + \frac{1}{L^2}\mathcal{F}\left(-\frac{L^2\delta^2\theta^2}{8}\right), \tag{55}$$

which is equivalent [see Eqs. (29) and (48)] to Eq. (14).

It is shown in Eq. (A7) of Appendix A that for large negative u,

$$\mathcal{F}_K(u) \simeq \frac{2^{7/2}}{3\pi}(-u)^{3/2}, \quad u \to -\infty. \tag{56}$$

This implies that Eq. (14) becomes for small negative s (but large negative $L^2 s$),

$$\psi_K(s) \simeq L\left[-2s\rho(1-\rho) + \frac{2^{7/2}}{3\pi}[-s\rho(1-\rho)]^{3/2} + \cdots\right]. \tag{57}$$

So for s small, but $L^2 s$ large, the extensivity of $\psi_K(s)$ is recovered and Eq. (57) gives the beginning of the small s expansion in the large L limit.

One can also notice that the function $\mathcal{F}(u)$ [Eq. (16)] becomes singular as $u \to \frac{\pi^2}{2}$. This indicates the occurrence of a phase transition discussed at the end of Sec. IV: for $u > \frac{\pi^2}{2}$ the optimal profile to reduce K is no longer flat and the system adopts a deformed profile as in [16]. In fact in the limit $s \to +\infty$ the configurations which dominate are those formed of a single cluster of particles and the activity is limited to the two boundaries of this cluster.

The result (55) or equivalently, Eq. (14) with $\mathcal{F}$ given by Eq. (16),

$$\mathcal{F}_K(u) = -4\sum_{n\geq 1}[n\pi\sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u] \tag{58}$$

gives the leading finite-size correction to $\psi_K(s)$. These finite corrections have been calculated recently, starting from the Bethe ansatz equations, for several spin chains in the context of string theory, and expressions very similar to our $\mathcal{F}$ have been obtained [29]. Note also that a more systematic approach has been developed to calculate the next finite-size correction [30].

Bethe ansatz for Q

The eigenvector corresponding to the largest eigenvalue of $\mathbb{W}_Q$ can be written as in Eq. (26), with the Bethe equations (27) replaced by

$$\zeta_i^L = \prod_{\substack{j=1 \\ j\neq i}}^{N}\left[-\frac{e^s - 2\zeta_i + e^{-s}\zeta_i\zeta_j}{e^s - 2\zeta_j + e^{-s}\zeta_i\zeta_j}\right]. \tag{59}$$

Given the solutions ζ_j to Eq. (59), the expression of ψ_Q reads

$$\psi_Q(s) = -2N + e^{-s}[\zeta_1 + \cdots + \zeta_N] + e^s\left[\frac{1}{\zeta_1} + \cdots + \frac{1}{\zeta_N}\right]. \tag{60}$$

By a method following closely the steps of the Bethe ansatz for K, the basic ingredients of which are provided in Appendix E, we arrive at the following result for ψ_Q:

$$\psi_Q(s) = \frac{1}{2}\sigma(\rho)s^2(L+1) + L^{-2}\mathcal{F}\left(-\frac{L^2 s^2\sigma(\rho)}{4}\right), \tag{61}$$

which leads to the asymptotic behavior as $L \to \infty$,

$$\frac{\psi_Q(s)}{L} \simeq \frac{1}{2}\sigma(\rho)s^2 + \frac{2^{1/2}}{3\pi}\sigma^{3/2}|s|^3. \tag{62}$$

The Bethe equations (59) are very close to the ones considered by Kim [52] who worked out the asymmetric exclusion process case. As outlined in Appendix E, the SSEP is not in the range of validity of Kim's results and so there is no discrepancy between our expression (62) and Kim's results. There is, however, a discrepancy between Eq. (62) and expression (A.12) of [20] because we think that in [20] Kim's results were used outside their range of validity.

Before concluding this section devoted to the Bethe ansatz, let us mention that, both for the current or the activity, one can obtain $\psi_Q(s)$ or $\psi_K(s)$ in the $s \to \infty$ limit by directly solving Eq. (59) or Eq. (27). We do not give these expressions here because they are out of the universal regime, which is our main concern.

IV. FLUCTUATING HYDRODYNAMICS AND THE MACROSCOPIC FLUCTUATION THEORY

In this section we are going to show that the expressions (14) and (23) can be recovered by a macroscopic theory based on hydrodynamical large deviations [1,2,4].

A. Calculation of ψ_Q for a general diffusive system and derivation of Eq. (23)

The macroscopic fluctuation theory developed by Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim [6–10] is based on the fact that, for a large system of size L, the density and the current of a diffusive system take scaling forms. If one defines $\hat{\rho}_i(t)$, the density averaged in the neighborhood of site i at time t, and $\hat{Q}_i(t)$, the total flux between site i and $i+1$ during time t, these quantities take scaling forms [17,18]

$$\hat{\rho}_i(t) = \rho\left(\frac{i}{L}, \frac{t}{L^2}\right), \tag{63}$$

$$\hat{Q}_i(t) = LQ\left(\frac{i}{L}, \frac{t}{L^2}\right). \tag{64}$$

This allows one to define a rescaled current $j(x,\tau)$ as

$$j(x,\tau) = \frac{\partial Q(x,\tau)}{\partial\tau} = L\frac{d}{dt}\hat{Q}_{Lx}(L^2\tau).$$

The average microscopic current between site i and $i+1$ is related to the rescaled current j by

$$\frac{d\hat{Q}_i(t)}{dt} = \frac{1}{L}j\left(\frac{i}{L}, \frac{t}{L^2}\right).$$

The probability of observing a rescaled current $j(x,\tau)$ and a density profile $\rho(x,\tau)$ over a time $t = TL^2$ is given by [6–8,17,18]

$$\mathrm{Pro}(\{\rho(x,\tau),j(x,\tau)\}) \sim \exp\left[-L\int_0^T d\tau\int_0^1 dx\frac{[j(x,\tau)+D(\rho(x,\tau))\rho'(x,\tau)]^2}{2\sigma(\rho(x,\tau))}\right], \tag{65}$$

where the current $j(x,\tau)$ and the density profile $\rho(x,\tau)$ satisfy the conservation law

$$\frac{d\rho}{d\tau} = -\frac{dj}{dx}, \tag{66}$$

and the diffusive system under study is characterized by the two functions $D(\rho)$ and $\sigma(\rho)$. For the SSEP, these functions are known: $D(\rho)=1$ and $\sigma(\rho)=2\rho(1-\rho)$ (see [2]).

Note that Eq. (65) can be seen as the fact that the macroscopic density $\rho(x,\tau)$ and the macroscopic current $j(x,\tau)$ satisfy, in addition to the conservation law (66), a Langevin equation of the form [5]

$$j(x,\tau) = -\partial_x\rho(x,t) + \xi(x,\tau), \tag{67}$$

where $\xi(x,\tau)$ is a Gaussian white noise

$$\langle\xi(x,\tau)\xi(x',\tau')\rangle = L^{-1}\sigma(\rho(x,\tau))\delta(x-x')\delta(\tau-\tau'). \tag{68}$$

The contribution of a small time dependent perturbation to a constant profile ρ_0 and a constant rescaled current j_0,

$$\rho(x,\tau) = \rho_0 + \delta\rho(x,\tau),$$

$$j(x,\tau) = j_0 + \delta j(x,\tau),$$

to the quadratic form in Eq. (65) is

$$\frac{[j(x,t)+D(\rho(x,t))\rho'(x,t)]^2}{2\sigma(\rho(x,t))} = \frac{j_0^2}{2\sigma} + \frac{j_0}{\sigma}\delta j - \frac{j_0^2\sigma'}{2\sigma^2}\delta\rho + \frac{j_0D}{\sigma}\delta\rho' + \frac{\delta j^2 + 2D\,\delta j\,\delta\rho' + D^2\delta\rho'^2 + 2j_0D'\,\delta\rho\delta\rho'}{2\sigma} - \frac{j_0\sigma'(\delta j\,\delta\rho + D\,\delta\rho\delta\rho')}{\sigma^2} + j_0^2\left(\frac{\sigma'^2}{2\sigma^3} - \frac{\sigma''}{4\sigma^2}\right)\delta\rho^2, \tag{69}$$

where the functions D,σ,σ,σ'' are evaluated at the density ρ_0.

If one considers a fluctuation of the form

$$\delta\rho = k[a_{k,\omega}e^{i\omega\tau+ikx} + a^*_{k,\omega}e^{-i\omega t-ikx}], \tag{70}$$

one has

$$\delta\rho' = ik^2[a_{k,\omega}e^{i\omega\tau+ikx} - a^*_{k,\omega}e^{-i\omega t-ikx}],$$

and due to Eq. (66),

$$\delta j = -\omega[a_{k,\omega}e^{i\omega\tau+ikx} + a^*_{k,\omega}e^{-i\omega t-ikx}].$$

The ring geometry ($x\equiv x+1$) imposes that the wave numbers k are discrete

$$k = 2\pi n \quad \text{with} \quad n \geq 1.$$

Also, because one considers a finite time interval T, the frequencies ω are also discrete and

$$\omega = \frac{2\pi m}{T} \quad \text{with} \quad m \in \mathbb{Z}.$$

Integrating over the time interval $0<\tau<T$ and over space, one gets

$$\langle\delta\rho^2\rangle = 2k^2|a_{k,\omega}|^2T,$$

$$\langle\delta\rho'^2\rangle = 2k^4|a_{k,\omega}|^2T,$$

$$\langle\delta j^2\rangle = 2\omega^2|a_{k,\omega}|^2T,$$

$$\langle\delta j\,\delta\rho\rangle = -2k\omega|a_{k,\omega}|^2T,$$

$$\langle\delta\rho\delta\rho'\rangle = \langle\delta j\,\delta\rho'\rangle = 0.$$

Therefore the superposition of all the fluctuations (70) leads to

$$\mathrm{Pro}(j_0,\{a_{k,\omega}\}) \sim \exp\left[-\frac{j_0^2}{2\sigma}\frac{t}{L} - \frac{t}{L}\sum_{\omega,k}|a_{k,\omega}|^2 \times\left(\frac{(\sigma\omega+j_0\sigma'k)^2}{\sigma^3} + \frac{D^2k^4}{\sigma} - \frac{j_0^2\sigma''k^2}{2\sigma^2}\right)\right],$$

where some terms independent of j_0 have been forgotten (they will be fixed later by normalization). After integrating over the Gaussian fluctuations and if one replaces the sum over ω by an integral one gets

$$\mathrm{Pro}(j_0) \sim \exp\left[-\frac{j_0^2}{2\sigma}\frac{t}{L} - \frac{t}{2\pi L^2}\sum_{1\leq k\leq k_{\max}}\int_{-\omega_{\max}}^{\omega_{\max}}d\omega \times\ln\left(\frac{(\omega\sigma+j_0\sigma'k)^2}{\sigma^3} + \frac{D^2k^4}{\sigma} - \frac{j_0^2\sigma''k^2}{2\sigma^2}\right)\right], \tag{71}$$

where we have introduced cutoffs $k_{\max}$ and $\omega_{\max}$. The reason for these cutoffs is that the macroscopic fluctuation theory (65) is valid only on hydrodynamic space and time scales. For $x=O(L^{-1})$ or $\tau=O(L^{-2})$ it has no validity at all, meaning that the cutoffs should satisfy $k_{\max}<L$ and $\omega_{\max}<L^2$.

For large L, i.e., for large $k_{\max}$ and $\omega_{\max}$, one can see by integrating over ω that only the constant term and the term proportional to j_0^2 depend on the cutoffs so that

$$\frac{1}{2\pi}\sum_{1\leqslant k\leqslant k_{\max}}\int_{-\omega_{\max}}^{\omega_{\max}} d\omega \ln\left(\frac{(\omega\sigma+j_0\sigma' k)^2}{\sigma^3}+\frac{D^2k^4}{\sigma}-\frac{j_0^2\sigma'' k^2}{2\sigma^2}\right)$$
$$\simeq A(k_{\max},\omega_{\max})+B(k_{\max},\omega_{\max})j_0^2$$
$$+\sum_{n=1}^{\infty}\left\{\sqrt{D^2(2\pi n)^4-\frac{j_0^2\sigma''}{2\sigma}(2\pi n)^2}-4\pi^2n^2D+\frac{j_0^2\sigma''}{4D\sigma}\right\}$$
$$=A(k_{\max},\omega_{\max})+B(k_{\max},\omega_{\max})j_0^2-D\mathcal{F}\left(\frac{j_0^2\sigma''}{16D^2\sigma}\right), \qquad (72)$$

where we have used the definition (16) of $\mathcal{F}$.

If the averaged rescaled current is j_0 over a macroscopic time T, the sum of the microscopic flux over all the bonds is $Q=TL^2j_0=tj_0$. Thus as $\lim_{t\to\infty}\frac{\langle Q^2\rangle}{t}=\frac{L^2}{L-1}\sigma$ [see Eq. (21)] one can determine the cutoff-dependent constants and get

$$\text{Pro}(j_0)\sim\exp\left[-\frac{j_0^2(L-1)t}{2\sigma L^2}+\frac{t}{L^2}D\mathcal{F}\left(\frac{j_0^2\sigma''}{16D^2\sigma}\right)\right], \qquad (73)$$

where $\mathcal{F}$ is defined in Eq. (16). This becomes, at order $1/L^2$, using the fact that $\psi_Q(s)=\max_{j_0}[-j_0s+t^{-1}\ln \text{Pro}(j_0)]$,

$$\psi_Q(s)-\frac{s^2\langle Q^2\rangle}{2t}=\frac{1}{L^2}D\mathcal{F}\left(\frac{\sigma\sigma''}{16D^2}L^2s^2\right). \qquad (74)$$

This formula is, in principle, valid for arbitrary diffusive systems, i.e., for arbitrary functions $\sigma(\rho)$ and $D(\rho)$. As $\sigma=2\rho(1-\rho)$, $D=1$, $\sigma''=-4$ for the SSEP this leads to the announced results (23) and (16).

For a general diffusive system the expressions of the cumulants (22) would therefore become

$$\lim_{t\to\infty}\frac{\langle Q^{2n}\rangle_c}{t}=B_{2n-2}\frac{(2n)!}{n!(n-1)!}D\left(\frac{-\sigma\sigma''}{8D^2}\right)^nL^{2n-2}, \qquad (75)$$

where $\sigma(\rho)$ and $D(\rho)$ are the two functions which appear in Eq. (65) and the B_n's are the Bernoulli numbers.

B. Calculation of ψ_K for the SSEP and derivation of Eq. (14)

To obtain Eq. (14), one can first write the activity K as

$$K=2L^3\int_0^T d\tau\int_0^1 dx\rho(x,\tau)[1-\rho(x,\tau)].$$

Then one has

$$K-\langle K\rangle\simeq 2L^3\int_0^T d\tau\int_0^1 dx[\langle\delta\rho^2\rangle-\delta\rho(x,\tau)^2].$$

Then one can proceed as above [Eqs. (69)–(74)] and get, up to terms constant or proportional to s, in the exponential

$$\langle e^{-s(K-\langle K\rangle)}\rangle\sim\int dj_0\int da_{k,\omega}\exp\left[-\frac{j_0^2}{2\sigma}\frac{t}{L}-\frac{t}{L}\sum_{\omega,k}|a_{k,\omega}|^2\right.$$
$$\left.\times\left(\frac{(\sigma\omega+j_0\sigma' k)^2}{\sigma^3}+\frac{D^2k^4}{\sigma}-\frac{j_0^2\sigma''k^2}{2\sigma^2}+4k^2sL^2\right)\right].$$

The rest of the calculation is the same as Eqs. (72)–(74), with a maximum over j_0 achieved at $j_0=0$, and one finally gets

$$\psi_K(s)=-s\frac{\langle K\rangle}{t}+L^{-2}\mathcal{F}_K\left(\frac{\sigma}{2}L^2s\right), \qquad (76)$$

which is exactly the same as Eq. (14).

C. Calculation of ψ_Q in the case of a weak asymmetry

One can also repeat the above calculation in the case of weakly driven systems, i.e., for systems where there is an additional driving force of strength $1/L$. In particular, this would be the case for the weakly asymmetric exclusion process (WASEP) [16] for which the hopping rates to the right and to the left are, respectively, $\exp\frac{\nu}{L}$ and $\exp(-\frac{\nu}{L})$.

For such systems, Eq. (65) becomes

$$\text{Pro}(\{\rho(x,\tau),j(x,\tau)\})\sim\exp\left[-L\int_0^T d\tau\int_0^1 dx\right.$$
$$\left.\times\frac{[j(x,\tau)+D(\rho(x,\tau))\rho'(x,\tau)-\nu\sigma(\rho(x,\tau))]^2}{2\sigma(\rho(x,\tau))}\right]. \qquad (77)$$

Following exactly the same steps as before, one gets an additional term $\frac{\nu^2\sigma''}{4}\delta\rho^2$ in Eq. (69), with everything else remaining the same. Then Eq. (73) becomes in this case

$$\text{Pro}(j_0)\sim\exp\left[-\frac{(j_0-\nu\sigma)^2(L-1)t}{2\sigma L^2}+\frac{t}{L^2}D\mathcal{F}\left(\frac{(j_0^2-\nu^2\sigma^2)\sigma''}{16D^2\sigma}\right)\right], \qquad (78)$$

where we have adjusted as in Eq. (73) the terms linear and quadratic in j_0 which are cutoff dependent.

D. Phase transitions

The function $\mathcal{F}(u)$ becomes singular as $u\to\frac{\pi^2}{2}$ [see Eq. (16)]. For systems for which $\sigma''<0$, this implies the occurrence of a phase transition in the expression (76) of $\psi_K(s)$ or in the large deviation function (78) of the current in the case of a weak asymmetry. These phase transitions are exactly the same as the one discussed in [9,10,16,17]: beyond the transition the system does not fluctuate anymore about a flat density profile, but the profile becomes deformed on a macroscopic scale. For systems such as the Kipnis Marchioro Presutti model [57,58] which have $\sigma''>0$, a similar phase transition occurs in ψ_Q even in the absence of a weak asymmetry.

V. CONCLUSION

In the present paper we have obtained exact expressions (12) and (21) of the first cumulants of the activity K and of the integrated current Q for the SSEP. In the large L limit, these cumulants take scaling forms (13) and (22).

We have shown in Sec. III that these scaling forms can be understood starting from the Bethe ansatz equations (27) and

(59), by calculating the leading finite-size corrections. These finite-size corrections are similar to the ones calculated recently for spin chains in the context of quantum strings [29,30].

We have also shown in Sec. IV that they can also be understood starting from the macroscopic fluctuation theory (65) of Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim. This enabled us to extend Eqs. (73)–(75) our results for the SSEP to arbitrary diffusive systems and to see that the occurrence of phase transitions can be predicted from the scaling form of the cumulants of the current. In order to better understand these phase transitions it might be interesting to characterize the eigenstate of the s-dependent evolution operator by, e.g., determining correlation functions in those states.

We have discussed here systems governed by diffusive dynamics with a single conserved field. How the universal scaling forms would be modified for systems with several conserved fields is an interesting open question.

Furthermore, there has recently been some progress made in the study of exclusion processes by Bethe ansatz methods, both with periodic and open boundary conditions [49–60]. It would be interesting to see whether our approach of Sec. III, which tries to extract the finite-size contribution due to the discreteness of the roots, could be extended to other cases, such as driven diffusive systems or systems with open boundaries.

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APPENDIX A: SEVERAL REPRESENTATIONS OF THE FUNCTION $\mathcal{F}$

In this appendix we show the equivalence between several representations [Eqs. (16), (17), and (52)] of the function $\mathcal{F}$ defined in Eq. (16),

$$\mathcal{F}(u) = -4\sum_{n\geq 1}\left[n\pi\sqrt{n^2\pi^2-2u} - n^2\pi^2 + u\right]. \tag{A1}$$

To do so consider the integral I

$$I = \frac{2u^3}{\pi}\int_{-1}^{1} y^2 dy \coth(u\sqrt{1-y^2}).$$

Then by using the fact that

$$\coth z = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2+n^2\pi^2},$$

and by integrating over y, one gets

$$\begin{aligned} I &= \frac{2u^3}{\pi}\int_{-1}^{1} y^2 dy \coth(u\sqrt{1-y^2}) \\ &= u^2 + \sum_{n\geq 1}\left[2u^2 + 4n^2\pi^2 - 4n\pi\sqrt{n^2\pi^2+u^2}\right] \\ &= u^2 + \mathcal{F}\left(-\frac{u^2}{2}\right). \end{aligned} \tag{A2}$$

This establishes Eq. (52). Now as

$$\frac{x}{e^x-1} = \sum_{n\geq 0}\frac{B_n}{n!}x^n = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \cdots, \tag{A3}$$

which is simply the definition of the Bernoulli numbers B_n (so that $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$ $B_6=\frac{1}{42},\ldots$), one can show that

$$\coth x = \frac{1}{x} + \sum_{k\geq 2} 2^{2k-2}x^{2k-3}\frac{B_{2k-2}}{(2k-2)!}\,.. \tag{A4}$$

Therefore

$$\begin{aligned} I &= \frac{2u^3}{\pi}\int_{-1}^{1} y^2 dy \coth(u\sqrt{1-y^2}) \\ &= \frac{2u^2}{\pi}\int_{-1}^{1}\frac{y^2}{\sqrt{1-y^2}}dy \\ &\quad + \sum_{k\geq 2}\frac{2^{2k-1}}{\pi}\frac{B_{2k-2}}{(2k-2)!}u^{2k}\int_{-1}^{1} y^2(1-y^2)^{(2k-3)/2}dy, \end{aligned}$$

i.e.,

$$I = u^2 + \sum_{k\geq 2}\frac{B_{2k-2}}{\Gamma(k)\Gamma(k+1)}u^{2k}. \tag{A5}$$

Comparing Eqs. (A2) and (A5), one gets

$$\begin{aligned} \mathcal{F}(u) &= \sum_{k\geq 2}\frac{B_{2k-2}}{\Gamma(k)\Gamma(k+1)}(-2u)^k \\ &= \frac{u^2}{3} + \frac{u^3}{45} + \frac{u^4}{378} + \frac{u^5}{2700} + \cdots, \end{aligned} \tag{A6}$$

so that Eqs. (17) and (15) are consistent with Eq. (16).

For large negative u, one gets, by replacing in Eq. (A1) the sum over n by an integral,

$$\mathcal{F}(u) \simeq \frac{2^{7/2}(-u)^{3/2}}{3\pi}. \tag{A7}$$

APPENDIX B: CALCULATION OF THE TWO SUMS APPEARING IN EQ. (35)

In this appendix we calculate the two sums which appear in Eq. (35) when $\delta\to 0$ and $L\to\infty$ keeping $L\delta$ fixed.

1. First sum in Eq. (35)

If the k_i are distributed according to a density $g(k)$ on the real axis and regularly spaced one can write that

$$L\int_{k_i}^{k_{i+n}} g(k')dk' = n. \tag{B1}$$

Therefore for fixed n and large L, one has

$$L(k_{i+n}-k_i)g(k_i) + L(k_{i+n}-k_i)^2\frac{g'(k_i)}{2} + \cdots = n,$$

so that

$$k_{i+n}-k_i = \frac{n}{g(k_i)L} - \frac{n^2 g'(k_i)}{2g(k_i)^3L^2} + \cdots. \tag{B2}$$

Replacing k_j by expression (B2) into the first sum in Eq. (35) one gets

$$\sum_{j=i-n_0}^{i-1} + \sum_{j=i+1}^{i+n_0} U(k_i,k_j) \simeq \sum_{n=1}^{n_0}\left(4k_i - \frac{2g'(k_i)(1-k_i^2)}{g(k_i)}\right)\frac{n^2}{n^2+(1-k_i^2)^2 g(k_i)^2 L^2\delta^2}.$$

Using the fact that for $n_0 \gg 1$ [and $b < \mathcal{O}(1)$],

$$\sum_{n=1}^{n_0}\frac{1}{n^2+b^2} = -\frac{1}{2b^2} + \frac{\pi}{2b}\coth \pi b, \tag{B3}$$

the first sum in Eq. (35) can be replaced by

$$\sum_{j=i-n_0}^{i-1} + \sum_{j=i+1}^{i+n_0} U(k_i,k_j) \simeq \left(4k_i - \frac{2g'(k_i)(1-k_i^2)}{g(k_i)}\right)n_0 - \left(2k_i - \frac{g'(k_i)(1-k_i^2)}{g(k_i)}\right) \times[-1+\pi(1-k_i^2)g(k_i)L\delta\coth[\pi(1-k_i^2)g(k_i)L\delta]]. \tag{B4}$$

2. Second sum in Eq. (35)

Let us consider the following integral:

$$I = \mathcal{P}\int_{-\theta}^{\theta} g(k')dk'\frac{1-k_ik'}{k_i-k'}. \tag{B5}$$

We are now going to compare this integral with the sum

$$S = \sum_{j\notin[i-n_0,i+n_0]}\frac{1-k_ik_j}{k_i-k_j}.$$

We assume [Eq. (B1)] that the k_j are given by

$$L\int_{-\theta}^{k_j} g(q)dq = j-\alpha, \tag{B6}$$

and for the moment α is arbitrary. Therefore

$$k_{j+1}-k_j \simeq \frac{1}{g(k_i)L}. \tag{B7}$$

One can decompose the integral I as

$$I = \mathcal{P}\int_{k_{i-n_0}}^{k_{i+n_0}} g(q)dq\frac{1-k_iq}{k_i-q} + \sum_{j=1}^{i-n_0-1}\int_{k_j}^{k_{j+1}} g(q)dq\frac{1-k_iq}{k_i-q} + \sum_{j=i+n_0}^{N-1}\int_{k_j}^{k_{j+1}} g(q)dq\frac{1-k_iq}{k_i-q} + \int_{-\theta}^{k_1} g(q)dq\frac{1-k_iq}{k_i-q} + \int_{k_N}^{\theta} g(q)dq\frac{1-k_iq}{k_i-q}. \tag{B8}$$

As $k_{j+1}-k_j$ is small and of order $1/L$ and because of Eqs. (B6) and (B7),

$$\int_{k_j}^{k_{j+1}} g(q)dq\frac{1-k_iq}{k_i-q} \simeq \frac{1}{L}\frac{1-k_ik_j}{k_i-k_j} + \frac{g(k_j)(k_{j+1}-k_j)^2}{2}\frac{d}{dk_j}\left(\frac{1-k_ik_j}{k_i-k_j}\right) \simeq \frac{1}{L}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{2L^2g(k_j)}\frac{d}{dk_j}\left(\frac{1-k_ik_j}{k_i-k_j}\right) \simeq \frac{1}{L}\frac{1-k_ik_{j+1}}{k_i-k_{j+1}} - \frac{1}{2L^2g(k_{j+1})}\frac{d}{dk_{j+1}}\left(\frac{1-k_ik_{j+1}}{k_i-k_{j+1}}\right). \tag{B9}$$

Therefore, using Eq. (B9) in the sum $1 \le j \le i-n_0-1$ and in the sum $i+n_0 \le j \le N-1$, one can rewrite Eq. (B8) as

$$I \simeq \mathcal{P}\int_{k_{i-n_0}}^{k_{i+n_0}} g(q)dq\frac{1-k_iq}{k_i-q} + \frac{1}{L}\sum_{j=1}^{i-n_0-1}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{L}\sum_{j=i+n_0+1}^{N}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{2L^2}\sum_{j=1}^{i-n_0-1}\frac{1}{g(k_j)}\frac{d}{dk_j}\left(\frac{1-k_ik_j}{k_i-k_j}\right) - \frac{1}{2L^2}\sum_{j=i+n_0+1}^{N}\frac{1}{g(k_j)}\frac{d}{dk_j}\left(\frac{1-k_ik_j}{k_i-k_j}\right) + \int_{-\theta}^{k_1} g(q)dq\frac{1-k_iq}{k_i-q} + \int_{k_N}^{\theta} g(q)dq\frac{1-k_iq}{k_i-q}.$$

This becomes

$$I \simeq \mathcal{P}\int_{k_{i-n_0}}^{k_{i+n_0}} g(q)dq\frac{1-k_iq}{k_i-q} + \frac{1}{L}\sum_{j=1}^{i-n_0-1}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{L}\sum_{j=i+n_0+1}^{N}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{2L}\left[\frac{1-k_ik_{i-n_0-1}}{k_i-k_{i-n_0-1}} + \frac{1-k_ik_{i+n_0+1}}{k_i-k_{i+n_0+1}} - \frac{1-k_ik_1}{k_i-k_1} - \frac{1-k_ik_N}{k_i-k_N}\right] + \int_{-\theta}^{k_1} g(q)dq\frac{1-k_iq}{k_i-q} + \int_{k_N}^{\theta} g(q)dq\frac{1-k_iq}{k_i-q},$$

which can be rewritten as

$$I \simeq \mathcal{P}\int_{k_{i-n_0}}^{k_{i+n_0}} g(q)dq\frac{1-k_iq}{k_i-q} + \frac{1}{L}\sum_{j=1}^{i-n_0-1}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{L}\sum_{j=i+n_0+1}^{N}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{2L}\left[\frac{1-k_ik_{i+n_0+1}}{k_i-k_{i+n_0+1}} + \frac{1-k_ik_{i-n_0-1}}{k_i-k_{i-n_0-1}}\right] + \frac{1-k_ik_1}{k_i-k_1}\left[-\frac{1}{2L}+\int_{-\theta}^{k_1}g(q)dq\right] + \frac{1-k_ik_N}{k_i-k_N}\left[-\frac{1}{2L}+\int_{k_N}^{\theta}g(q)dq\right]. \quad \text{(B10)}$$

From Eq. (B2) one can show that

$$\mathcal{P}\int_{k_{i-n_0}}^{k_{i+n_0}} g(q)dq\frac{1-k_iq}{k_i-q} \simeq \frac{2k_in_0}{L} - \frac{k_in_0(1-k_i^2)g'(k_i)}{Lg(k_i)}, \quad \text{(B11)}$$

and that

$$\frac{1-k_ik_{i+n_0}}{k_i-k_{i+n_0}} + \frac{1-k_ik_{i-n_0-1}}{k_i-k_{i-n_0-1}} \simeq 2k_i - (1-k_i^2)\frac{g'(k_i)}{g(k_i)} + \mathcal{O}\left(\frac{1}{L}\right). \quad \text{(B12)}$$

Lastly, because one expects the symmetry $k_j=-k_{N+1-j}$ and because $L\int_{-\theta}^{\theta}g(q)dq=N$, one gets that $\alpha=1/2$ in Eq. (B6) and therefore the last two terms of Eq. (B10) vanish.

Then, using Eqs. (B11) and (B12) in and (B10), one gets that

$$\frac{1}{L}\sum_{j=1}^{i-n_0-1}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{L}\sum_{j=i+n_0+1}^{N}\frac{1-k_ik_j}{k_i-k_j} \simeq I - \frac{1}{L}\left(2k_i-(1-k_i^2)\frac{g'(k_i)}{g(k_i)}\right)\left(n_0+\frac{1}{2}\right), \quad \text{(B13)}$$

where the integral I is defined in Eq. (B5). Lastly using the fact that $g(k)=g(-k)$, one can rewrite the integral I in Eq. (B5) as

$$I = \mathcal{P}\int_{-\theta}^{\theta} g(k')dk'\frac{1-k'^2}{k_i-k'}, \quad \text{(B14)}$$

so that Eq. (B13) becomes

$$\frac{1}{L}\sum_{j=1}^{i-n_0-1}\frac{1-k_ik_j}{k_i-k_j} + \frac{1}{L}\sum_{j=i+n_0+1}^{N}\frac{1-k_ik_j}{k_i-k_j} \simeq \mathcal{P}\int_{-\theta}^{\theta} g(k')dk'\frac{1-k'^2}{k_i-k'} - \frac{1}{L}\left(2k_i-(1-k_i^2)\frac{g'(k_i)}{g(k_i)}\right) \times\left(n_0+\frac{1}{2}\right). \quad \text{(B15)}$$

Note that Eq. (B6) is not accurate for i close to 1 or N, i.e., near the singularities of $g(k)$. A more detailed analysis of these two neighborhoods would only contribute to higher orders in the $1/L$ expansion [30].

APPENDIX C: SOLUTION OF THE AIRFOIL EQUATION (39)

In this appendix we show, in the spirit of [61], that the solution $\phi(x)$ of

$$f(x) = \mathcal{P}\int_{-1}^{1} dy\frac{\phi(y)}{y-x} \quad \text{(C1)}$$

is

$$\phi(x) = \frac{C}{\sqrt{1-x^2}} - \frac{1}{\pi^2}\mathcal{P}\int_{-1}^{1} dy\sqrt{\frac{1-y^2}{1-x^2}}\frac{f(y)}{y-x}. \quad \text{(C2)}$$

This solution is used to obtain Eq. (41) as the solution of Eq. (39).

Let us choose

$$\phi(x) = \frac{\sqrt{1-x^2}}{x-\alpha}. \quad \text{(C3)}$$

Then for $x\notin[-1,1]$ and $\alpha\notin[-1,1]$ one can see, using Eq. (D3), that

$$\int_{-1}^{1} dy\frac{\phi(y)}{y-x} = \pi\left[\frac{\sqrt{\alpha^2-1}}{\alpha-x} - \frac{\sqrt{x^2-1}}{\alpha-x} - 1\right], \quad \text{(C4)}$$

and therefore

$$f(x) = \mathcal{P}\int_{-1}^{1} dy\frac{\phi(y)}{y-x} = \pi\left[\frac{\sqrt{\alpha^2-1}}{\alpha-x} - 1\right]. \quad \text{(C5)}$$

Now the following integral of this function $f(x)$ can be computed [using Eqs. (D1) and (D3)] for $x\notin[-1,1]$:

$$-\frac{1}{\pi^2}\int_{-1}^{1} dy\frac{\sqrt{1-y^2}}{y-x}f(y) = \sqrt{\alpha^2-1}\left(\frac{\sqrt{\alpha^2-1}}{\alpha-x} - \frac{\sqrt{x^2-1}}{\alpha-x} - 1\right) + \sqrt{x^2-1} - x, \quad \text{(C6)}$$

so that

$$-\frac{1}{\pi^2}\mathcal{P}\int_{-1}^{1} dy\frac{\sqrt{1-y^2}}{y-x}f(y) = \frac{\alpha^2-1}{\alpha-x} - \sqrt{\alpha^2-1} - x = \alpha - \sqrt{\alpha^2-1} - \frac{1-x^2}{\alpha-x}. \quad \text{(C7)}$$

Comparing with Eq. (C3) we see that

$$-\frac{1}{\pi^2}\mathcal{P}\int_{-1}^{1} dy\sqrt{\frac{1-y^2}{1-x^2}}\frac{f(y)}{y-x} = \frac{\alpha-\sqrt{\alpha^2-1}}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{x-\alpha} = \frac{\alpha-\sqrt{\alpha^2-1}}{\sqrt{1-x^2}} + \phi(x). \quad \text{(C8)}$$

Therefore Eq. (C2) is the solution of Eq. (C1) with a constant C which depends through α on $\phi(x)$ when one chooses Eq. (C2) for $\phi(x)$.

As the inversion formula (C2) is valid for arbitrary α, it would also be valid when $f(x)$ is any polynomial in x, and as the polynomials are dense in the set of continuous functions on $(-1,1)$, one can consider that Eqs. (C1) and (C2) are valid for "arbitrary functions" $f(x)$.

APPENDIX D: USEFUL INTEGRALS

In this appendix we list a few integrals which are used in various places of the paper.

First, for $x \notin [-1,1]$ one has

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}dy = \sqrt{x^2-1}-x, \tag{D1}$$

so that

$$\frac{1}{\pi}\mathcal{P}\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}dy = -x. \tag{D2}$$

As a consequence of Eq. (D1) one has for $x \notin [-1,1]$ and $\alpha \notin [-1,1]$,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}\frac{dy}{y-\alpha} = \frac{\sqrt{\alpha^2-1}}{\alpha-x} - \frac{\sqrt{x^2-1}}{\alpha-x} - 1, \tag{D3}$$

and thus for $x \in [-1,1]$ and $\alpha \notin [-1,1]$,

$$\frac{1}{\pi}\mathcal{P}\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}\frac{dy}{y-\alpha} = \frac{\sqrt{\alpha^2-1}}{\alpha-x} - 1. \tag{D4}$$

One can also show that

$$\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \pi, \tag{D5}$$

and that for $y \notin [-1,1]$,

$$\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\frac{1}{y-x} = \frac{\pi}{\sqrt{y^2-1}}. \tag{D6}$$

As a consequence of Eqs. (D4) and (D6), one has

$$\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\mathcal{P}\int_{-1}^{1}\frac{dy}{y-x}F(y) = 0 \tag{D7}$$

for an arbitrary function $F(y)$ as it is valid for any polynomial.

For $\theta<1$ one can show using Eq. (D6) that

$$\int_{-1}^{1}\frac{dx}{(1-\theta^2x^2)\sqrt{1-x^2}} = \frac{\pi}{\sqrt{1-\theta^2}}. \tag{D8}$$

One can also show

$$\int_{-1}^{1}\frac{\sqrt{1-x^2}}{(1-\theta^2x^2)}dx = \pi\frac{1-\sqrt{1-\theta^2}}{\theta^2}, \tag{D9}$$

and that

$$\int_{-1}^{1}\frac{y^2\sqrt{1-y^2}}{1-\theta^2y^2}dy = \pi\left(\frac{1}{\theta^4} - \frac{1}{2\theta^2} - \frac{\sqrt{1-\theta^2}}{\theta^4}\right), \tag{D10}$$

and for $y \notin [-1,1]$,

$$\int_{-1}^{1}\frac{1}{(1-\theta^2x^2)\sqrt{1-x^2}}\frac{dx}{y-x} = \frac{\pi}{(1-\theta^2y^2)\sqrt{y^2-1}} - \frac{\pi\theta^2 y}{(1-\theta^2y^2)\sqrt{1-\theta^2}}, \tag{D11}$$

and therefore for any function $F(y)$,

$$\int_{-1}^{1}\frac{dx}{(1-\theta^2x^2)\sqrt{1-x^2}}\mathcal{P}\int_{-1}^{1}\frac{dy}{y-x}F(y) = -\frac{\pi\theta^2}{\sqrt{1-\theta^2}}\int_{-1}^{1}\frac{yF(y)}{1-\theta^2y^2}dy. \tag{D12}$$

APPENDIX E: BETHE ANSATZ CALCULATION FOR THE CURRENT LARGE DEVIATION FUNCTION $\psi_Q(s)$

This appendix describes how a Bethe ansatz calculation of $\psi_Q(s)$, similar to the one conducted for ψ_K, can be implemented. The operator $\mathbb{W}_Q$ whose largest eigenvalue is ψ_Q reads, in the spin language already used in Eq. (25),

$$\mathbb{W}_Q(s) = \sum_{i=1}^{L}\left[\frac{\sigma_i^z\sigma_{i+1}^z-1}{2} + e^{-s}\sigma_i^+\sigma_{i+1}^- + e^{s}\sigma_i^-\sigma_{i+1}^+\right]. \tag{E1}$$

The Bethe ansatz equation analogous to Eq. (27) takes the form Eq. (59),

$$\zeta_i^L = \prod_{\substack{j=1\\ j\neq i}}^{N}\left[-\frac{1-2e^{-s}\zeta_i + e^{-2s}\zeta_i\zeta_j}{1-2e^{-s}\zeta_j + e^{-2s}\zeta_i\zeta_j}\right]. \tag{E2}$$

In terms of the ζ_j's, we have that

$$\psi_Q(s) = -2N + e^{-s}[\zeta_1 + \cdots + \zeta_N] + e^{s}\left[\frac{1}{\zeta_1} + \cdots + \frac{1}{\zeta_N}\right]. \tag{E3}$$

Kim [52] has studied the spectrum of $\mathcal{H}=-\mathbb{W}_Q/(\cosh s/2)$ by means of a Bethe ansatz calculation: in the notations of his Eq. (1), the parameters $\tilde{\Delta}$ and S are given by

$$\tilde{\Delta} = \frac{1}{\cosh s}, \quad S = \tanh s, \tag{E4}$$

but unfortunately his results do not apply to our particular case, which turns out to correspond to a critical point of the related six-vertex model. The defining parameters of the latter, denoted by Δ, H, and ν, are related to Kim's by $\tilde{\Delta}=\Delta/\cosh(2H)$, $S=\tanh(2H)$, $\Delta=\cosh\nu$. Thus, in terms of our original parameters, we get that

$$\Delta = 1, \quad 2H = s, \quad \nu = 0, \tag{E5}$$

a limiting case specifically excluded in Eq. (7) of Kim [52], which lies at the critical point of the six-vertex model.

We choose to write that $\zeta_j = e^{-is(k_j+2i\rho)}$. The two main differences with the calculation of ψ_K is that the ζ_j's dependence in s is different. We have also shifted them by $2i\rho$ for convenience. Just as was the case previously, the k_j's will be densely distributed on a connected curve $\mathcal{C}$ of the complex plane that is invariant upon complex conjugation. Given that the equations for the ζ_j's are invariant under complex conjugation, we expect the contour $\mathcal{C}$ to be symmetric with respect to the vertical axis in the complex k plane. We shall denote the end points of $\mathcal{C}$ by $-\theta^*$ and θ.

Given that Eq. (E2) becomes

$$-i(k_i+i2\rho)=\frac{1}{L}\sum_{j=1,j\neq i}^{N}U(k_i,k_j),$$

where

$$U(k_i,k_j)=\frac{1}{s}\ln\left[-\frac{1-2e^{-s}\zeta_i+e^{-2s}\zeta_i\zeta_j}{1-2e^{-s}\zeta_j+e^{-2s}\zeta_i\zeta_j}\right]. \tag{E6}$$

For $|i-j|\gg 1$ we have that

$$U(k_i,k_j)=\frac{2i(k_i+i\alpha)(k_j+i\alpha)}{k_i-k_j},\quad \alpha=2\rho-1, \tag{E7}$$

while for $i-j$ of order 1, s will be of the order $1/L$ and k_i-k_j as well. We define $g(k)$ as the root density along contour $\mathcal{C}$, so that

$$L\int_{k_i}^{k_j}g(k)dk=j-i$$

[note that $g(k)$ is in general complex but along the contour $g(k)dk$, is real]. If k_j and k_i are n roots apart, we have that $k_j-k_i=\frac{n}{g(k_i)L}-\frac{n^2g'(k_i)}{2g(k_i)^3L^2}+\cdots$. Expanding U at fixed sL in powers of L^{-1} leads to

$$\begin{aligned}U(k_i,k_j)=&\frac{1}{s}\ln\frac{n-ig(k_i)(k_i+i\alpha)^2sL}{n+ig(k_i)(k_i+i\alpha)^2sL}\\&-i(k_i+i\alpha)\left(2+\frac{g'(k_i)(k_i+i\alpha)}{g(k_i)}\right)\\&\times\frac{n^2}{n^2+[g(k_i)(k_i+i\alpha)^2(sL)]^2}.\end{aligned} \tag{E8}$$

Equations (E7) and (E8) play a role analogous to Eqs. (33) and (32) in the study of K. After using the methods of Appendixes B and C we arrive at the following equation for g which we express in terms of $\phi(x)=(\theta x+i\alpha)^2g(\theta x)$ and $r=\theta^*/\theta$:

$$\begin{aligned}\theta\left(x+i\frac{\alpha+1}{\theta}\right)=&2\mathcal{P}\int_{-r}^{1}dy\frac{\phi(y)-(y-x)\left(y+i\frac{\alpha}{\theta}\right)^{-1}\phi(y)}{y-x}\\&-\frac{\theta}{L}\left(x+i\frac{\alpha}{\theta}\right)^2\frac{\phi'(x)}{\phi(x)}[\pi\phi(x)(sL)]\\&\times\coth[\pi\phi(x)(sL)].\end{aligned} \tag{E9}$$

Let us denote $\phi_0(x)$ the solution of the above equation, in the $L\to\infty$ limit

$$\theta x/2+h=\mathcal{P}\int_{-r}^{1}dy\frac{\phi_0(y)}{y-x}, \tag{E10}$$

where $h=i(\alpha+1)/2+\int dy\,\phi_0(y)(y+i\alpha/\theta)^{-1}$ is a density-dependent constant to be determined. The general solution of Eq. (E10) can be written [see (C1) and (C2)] as

$$\begin{aligned}\phi_0(x)=&-\frac{C}{\sqrt{(1-x)(r+x)}}-\frac{\theta(r+1)^2}{16\pi\sqrt{(1-x)(r+x)}}\\&+\frac{\theta x^2}{2\pi\sqrt{(1-x)(r+x)}}+\frac{2h(r-1+x)}{2\pi\sqrt{(1-x)(r+x)}}\\&+\frac{\theta x(r-1)}{4\pi\sqrt{(1-x)(r+x)}}.\end{aligned} \tag{E11}$$

The four unknowns C, θ, r, and h are determined by requiring that ϕ_0 remains finite as $x\to 1$ and as $x\to -r$, and by noting that by definition

$$\int_{-r}^{+1}dx\frac{\phi_0(x)}{\left(x+i\frac{\alpha}{\theta}\right)^2}=\theta\rho, \tag{E12}$$

while ϕ_0 must verify the self-consistency equation $h=i(\alpha+1)/2+\int dy\,\phi_0(y)(y+i\alpha/\theta)^{-1}$. After explicitly evaluating the latter integral and that appearing in Eq. (E12), we arrive at $r=1$, $h=0$, and $4\pi C=\theta=2\sqrt{\rho(1-\rho)}$, which leads to $\phi_0(x)=-\theta\frac{\sqrt{1-x^2}}{2\pi}$. Up to a sign, this is exactly the same function as that found in the study of K, and this is the same end point $\theta=2\sqrt{\rho(1-\rho)}$ for the contour on which the k_j's lie.

We may now simplify Eq. (E9) into

$$\begin{aligned}\theta\left(x+i\frac{\alpha+1}{\theta}\right)=&2\mathcal{P}\int dy\frac{\phi(y)-(y-x)(y+i\alpha/\theta)^{-1}\phi(y)}{y-x}\\&+\frac{1}{\theta L}\frac{x(x+i\alpha/\theta)^2}{1-x^2}([\theta\sqrt{1-x^2}(sL)/2]\\&\times\coth[\theta\sqrt{1-x^2}(sL)/2]),\end{aligned} \tag{E13}$$

whose solution reads $\phi(x)=\phi_0(x)+\delta\phi(x)$,

$$\begin{aligned}\delta\phi(x)=&-\frac{\delta C}{\sqrt{1-x^2}}+\frac{2\,\delta hx}{2\pi\sqrt{1-x^2}}\\&-\frac{1}{\pi^2}\frac{1}{\sqrt{1-x^2}}\mathcal{P}\int dy\frac{\sqrt{1-y^2}}{y-x}\delta F(y).\end{aligned} \tag{E14}$$

We have denoted by $\delta F(x)$ the function

$$\begin{aligned}\delta F(x)=&-\frac{\theta}{2L}\frac{x(x+i\alpha/\theta)^2}{1-x^2}\\&\times[\theta\sqrt{1-x^2}(sL)/2]\coth[\theta\sqrt{1-x^2}(sL)/2]\\=&-\frac{\theta}{2L}\frac{x(x+i\alpha/\theta)^2}{1-x^2}\left[\sum_{p\geq 2}\frac{B_p}{p!}(\theta sL)^p(1-x^2)^{p/2}+1\right].\end{aligned} \tag{E15}$$

The new constants δC and δh are determined by $\int\frac{\delta\phi}{(x+i\alpha/\theta)^2}=0$ and $\delta h=\int\frac{\delta\phi}{(x+i\alpha/\theta)}$. After performing explicit integrations along the lines of Appendix D, we obtain the final result through the following equality:

$$\psi_Q(s)/L = - s^2\theta \int dx \phi(x) = \frac{\theta^2}{4}s^2 + s^2 \delta C \theta\pi + s^2\theta \frac{1}{\pi^2}\int dx \frac{1}{\sqrt{1-x^2}}\mathcal{P}\int dy \frac{\sqrt{1-y^2}}{y-x}\delta F(y), \tag{E16}$$

where

$$\delta C\theta\pi = \frac{\theta^2}{2\pi L}\int_{-1}^{1} dx x^2\left[\sum_{p\geq 2}\frac{B_p}{p!}(\theta s L)^p (1-x^2)^{(p-1)/2} + \frac{1}{\sqrt{1-x^2}}\right] = \frac{1}{L^3 s^2}\mathcal{F}(-L^2 s^2\theta^2/8) + \frac{\theta^2}{4L}. \tag{E17}$$

After noting that, as before, we have

$$\frac{1}{\pi^2}\int dx \frac{1}{\sqrt{1-x^2}}\mathcal{P}\int dy \frac{\sqrt{1-y^2}}{y-x}\delta F(y) = 0, \tag{E18}$$

it only remains to substitute the value of δC into Eq. (E16). This allows us to conclude that

$$\psi_Q(s) = \frac{\theta^2}{4}s^2(L+1) + L^{-2}\mathcal{F}(-L^2 s^2\theta^2/8), \tag{E19}$$

which is the announced result of Eq. (61).

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