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The distance between the two BBM leaders

Julien Berestycki[1], **Éric Brunet**[2], **Cole Graham**[3,*], **Leonid Mytnik**[4], **Jean-Michel Roquejoffre**[5] **and Lenya Ryzhik**[3]

[1] Department of Statistics and Magdalen College, University of Oxford, United Kingdom
[2] Laboratoire de Physique de l'École normale supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université de Paris, F-75005 Paris, France
[3] Department of Mathematics, Stanford University, Stanford, CA 94350, United States of America
[4] Faculty of Industrial Engineering and Management, Technion, Technion City, Haifa 3200003, Israel
[5] Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France

E-mail: julien.berestycki@stats.ox.ac.uk, eric.brunet@ens.fr, grahamca@stanford.edu, leonid@ie.technion.ac.il, Jean-Michel.Roquejoffre@math.univ-toulouse.fr and ryzhik@stanford.edu

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Abstract

We study the distance between the two rightmost particles in branching Brownian motion. Derrida and the second author have shown that the long-time limit d_{12} of this random variable can be expressed in terms of PDEs related to the Fisher–KPP equation. We use such a representation to determine the sharp asymptotics of $\mathbb{P}(d_{12} > a)$ as $a \to +\infty$. These tail asymptotics were previously known to 'exponential order;' we discover an algebraic correction to this behavior.

Keywords: branching Brownian motion, Fisher–KPP equation, branching processes

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(Some figures may appear in colour only in the online journal)

*Author to whom any correspondence should be addressed.
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1. Introduction

Branching Brownian motion (BBM) is a point process consisting of particles moving and reproducing stochastically. In this work, we study the tail behavior of the distance between the two rightmost particles in BBM. To do so, we rely on the close relationship between BBM and the Fisher–KPP equation, a reaction–diffusion PDE.

We consider a BBM beginning from a single particle at the position $x = 0$ at time $t = 0$, with branching occurring at rate 1. At each branching event, the particle splits into a random number of particles. Let p_k denote the probability of k offspring in a given branching event. We assume that $p_0 = p_1 = 0$, so that

$$\sum_{k=2}^{\infty} p_k = 1.$$

We let N denote the expected number of offspring:

$$N = \sum_{k=2}^{\infty} k p_k < \infty,$$

and we assume that the offspring distribution has a higher moment:

$$\sum_{k=2}^{\infty} k^{1+\beta} p_k < \infty \quad \text{for some } \beta \in (0, 1). \tag{1.1}$$

To simplify the diffusion constant in the Fisher–KPP equation below, we assume that the variance of each individual Brownian motion is $2t$ (rather than t). Let $x_1(t) \geqslant x_2(t) \geqslant \ldots \geqslant x_{n(t)}$ denote the positions of the particles that exist at time $t \geqslant 0$. Here, $n(t)$ is the total number of particles present at time t. Note that $n(t)$ is itself random. It is well known from works of Bramson in [4, 5] and Lalley and Sellke [15] that the maximal particle $x_1(t)$ is typically at distance $O(1)$ from the position

$$m(t) = c_* t - \frac{3}{2\lambda_*} \log(t + 1) \tag{1.2}$$

as $t \to \infty$, with

$$c_* = 2\sqrt{N - 1}, \qquad \lambda_* = \sqrt{N - 1}. \tag{1.3}$$

See also Roberts [21] for a more recent shorter proof. This result does not depend on the precise nature of branching, so both the spreading speed c_* and the logarithmic correction in (1.2) depend only on the expected number of offspring N.

These probabilistic results transfer to PDEs via an identity of Ikeda, Nagasawa, and Watanabe [11–13] and McKean [17]: the cumulative distribution function of the maximal particle

$$H(t, x) = \mathbb{P}[x_1(t) \geqslant x] \tag{1.4}$$

satisfies the Fisher–KPP equation [8, 14]

$$\partial_t H = \partial_x^2 H + f(H), \tag{1.5}$$

with the initial condition

$$H(0, x) = 1_{(-\infty,0]}(x). \tag{1.6}$$

The nonlinear reaction in (1.5) has the form

$$f(u) = 1 - u - \sum_{k=2}^{\infty} p_k(1-u)^k. \tag{1.7}$$

Bramson's results on BBM imply that there is a constant $\overline{x}_0 \in \mathbb{R}$ such that

$$H(t, x + m(t)) \to U(x - \overline{x}_0) \quad \text{as } t \to +\infty \tag{1.8}$$

uniformly on $\mathbb{R}$. Here, $U(x)$ is a Fisher–KPP traveling wave, so that $U(x - c_* t)$ is a solution to (1.5) moving with speed c_*. The function $U(x)$ satisfies

$$-c_* U' = U'' + f(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0 \tag{1.9}$$

and has the asymptotics

$$\begin{aligned} U(x) &\sim x\mathrm{e}^{-\lambda_* x} && \text{as } x \to +\infty, \\ U(x) &= 1 - A\mathrm{e}^{\gamma_* x} + O\left(\mathrm{e}^{(\gamma_* + c)x}\right) && \text{as } x \to -\infty \end{aligned} \tag{1.10}$$

for

$$\gamma_* = \sqrt{N} - \sqrt{N-1}$$

and some constants $A > 0$ and $c > 0$. Any translate of a solution to (1.9) is also a solution, and Uchiyama [22] showed that each resembles $Cx\mathrm{e}^{-\lambda_* x}$ as $x \to \infty$ for some $C > 0$ depending on the translation. We select the unique shift of the wave such that the pre-factor C is one, as in the first line of (1.10). The second line in (1.10) follows from elementary phase-plane analysis.

More generally, let $u(t, x)$ solve (1.5)

$$\partial_t u = \partial_x^2 u + f(u) \tag{1.11}$$

with non-negative bounded initial data

$$u(0, x) = \phi(x)$$

such that $\phi \not\equiv 0$ and $\phi|_{(L_0,\infty)} = 0$ for some $L_0 \in \mathbb{R}$. Assume that the nonlinearity $f : [0, 1] \to [0, \infty)$ is $\mathcal{C}^{1,\beta}$, so that f' exists and is β-Hölder continuous for some $\beta \in (0, 1)$. Moreover, assume that f satisfies the Fisher–KPP assumptions

$$f(0) = f(1) = 0, \quad f|_{(0,1)} > 0, \quad \text{and} \quad f(u) \leqslant f'(0)u \quad \text{for all } u \in (0, 1). \tag{1.12}$$

Then, there exists a constant $s[\phi] \in \mathbb{R}$ (depending also on f) known as the *Bramson shift* corresponding to ϕ such that

$$u(t, x + m(t)) \to U(x - s[\phi]) \quad \text{as } t \to +\infty \tag{1.13}$$

uniformly on compact sets, with $c_* = 2\sqrt{f'(0)}$ and $\lambda_* = \sqrt{f'(0)}$ in the definition (1.2) of $m(t)$. In particular, (1.8) states that

$$s\left[1_{(-\infty,0]}\right] = \overline{x}_0. \tag{1.14}$$

This result is due, in increasing degrees of precision, to Fisher [8], Kolmogorov, Petrovskii, and Piskunov [14], Uchiyama [22], and Bramson [4, 5]. For more recent developments, see [9, 10, 16, 19, 20].

It is straightforward to check that $f(u)$ defined in (1.7) satisfies the Fisher–KPP property (1.12) with

$$f'(0) = N - 1 \quad \text{and} \quad f'(1) = -1. \tag{1.15}$$

Note, however, that the Fisher–KPP property is far more general: there exist many Fisher–KPP reactions that do not correspond to any branching process. Convergence in (1.13) holds for all nonlinearities of Fisher–KPP type, not just 'probabilistic' ones. One reflection of the rigidity of the class of probabilistic nonlinearities is that they satisfy

$$f'(1) = -1, \tag{1.16}$$

and the Fisher–KPP property is not related to $f'(1)$. Also, probabilistic nonlinearities are concave, which is not necessary for (1.12).

For later use, we introduce the decomposition

$$f(u) = (N - 1)u - F(u), \tag{1.17}$$

so that F denotes the 'nonlinear' part of the reaction f. By (1.12) and (1.15), $F(0) = 0$, $F'(0) = 0$ and $F(u) > 0$ for all $u \in (0, 1]$. When f is probabilistic, $f'' \leqslant 0$ and (1.16) imply that F and F' are increasing and

$$0 = F'(0) < F'(u) < F'(1) = N \quad \text{for all } u \in (0, 1). \tag{1.18}$$

The connection between BBM and the Fisher–KPP equation runs deeper than (1.4) and (1.5). Consider the measure-valued process that characterizes the BBM seen from the position $m(t)$:

$$\mathcal{X}_t = \sum_{k=1}^{n(t)} \delta_{x_k(t)-m(t)},$$

recalling that $n(t)$ is the total number of particles alive at time t and $x_1(t) \geqslant x_2(t) \geqslant \cdots \geqslant x_{n(t)}(t)$ are the positions of those particles. The works [1, 3, 6] show that the centered point process $\mathcal{X}_t$ converges in law to a limit $\mathcal{X}$:

$$\mathcal{X}_t \xrightarrow{\text{law}} \mathcal{X} = \sum_{k\geqslant 1} \delta_{\chi_k} \quad \text{as } t \to \infty. \tag{1.19}$$

In fact, the proof in [1] makes use of the fact that various functionals of $\mathcal{X}_t$ solve the Fisher–KPP equation. In the literature, $\mathcal{X}$ is often called the limiting extremal process (see [2, 3]).

In this paper, we examine the distance between the two leading particles $x_1(t)$ and $x_2(t)$. More precisely, we study the distance after the limit $t \to +\infty$:

$$d_{12} = \chi_1 - \chi_2.$$

The main result of this paper is the tail asymptotics of the random variable d_{12}.

Theorem 1.1 *As $a \to \infty$, we have*

$$\mathbb{P}(d_{12} > a) = \frac{A\gamma_*}{2\lambda_*^2\sqrt{\pi}}\left(\frac{a}{2\sqrt{N}}\right)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\left(\sqrt{N}+\sqrt{N-1}\right)(a+\bar{x}_0)}\left[1+O\left(a^{-1/2}\right)\right] \tag{1.20}$$

with A as in (1.10).

To prove theorem 1.1, we relate the distance $x_1(t) - x_2(t)$ to a derivative of the Fisher–KPP equation with respect to its initial data. The bulk of the proof is a detailed study of this derivative.

Weaker versions of the above result were already known. In [6], Derrida and the second author predicted the exponential order of (1.20) for binary BBM. Cortines, Hartung and Louidor confirmed this conjecture [7, theorem 1.4], again in the binary case $N = 2$:

$$\lim_{a\to\infty} a^{-1} \log \mathbb{P}(d_{12} > a) = -\left(\sqrt{2}+1\right). \tag{1.21}$$

(The limit given in [7] is in fact $-\left(\sqrt{2}+2\right)$, since the authors use Brownian motion of variance t rather than $2t$.) We note that their results easily extend to non-binary BBM. However, neither [6] nor [7] discuss the sub-exponential behavior of $\mathbb{P}(d_{12} > a)$. In particular, the algebraic pre-factor $a^{3\sqrt{N}/(2\lambda_*)}$ in theorem 1.1 is new. In appendix A, we present the probabilistic heuristics from [6, 7] and discuss their potential extension to (1.20). We emphasize that our own approach is primarily analytic, in contrast with the probabilistic focus of [6, 7].

More broadly, [6] is one of the main motivations for our study. It lays out numerous fascinating connections between the Bramson shifts of certain Fisher–KPP solutions and fine properties of BBM. Some of these predictions have been addressed elsewhere [18], but many have not yet been proven rigorously.

We close with a word on notation: throughout the paper, we let $C > 0$ and $c > 0$ be constants which may change from line to line. We think of C as large and c as small. These constants may depend on the branching probabilities $\{p_k\}_{k\geqslant 2}$ but not on t, x, or a. We write $f = O(g)$ if $|f| \leqslant C\,|g|$.

2. Outline of the proof of theorem 1.1

We begin by relating the law of d_{12} to the long-time behavior of a certain PDE solution.

2.1. An expression for the distribution of d_{12}

Consider a BBM shifted to start from a single particle at position $-a$ rather than 0. Recall that $x_1(t) \geqslant x_2(t) \geqslant \cdots \geqslant x_{n(t)}(t)$ are the positions of the $n(t)$ particles alive at time t. We introduce the (unnormalized) density

$$z(t, x; a)\,\mathrm{d}x := \mathbb{P}\left(x_1(t) \in [x, x + \mathrm{d}x], \quad x_2(t) < x - a\right) \tag{2.1}$$

under the convention that $x_2(t) = -\infty$ before the first branching event. Precisely:

$$\int_A z(t, x; a)\,\mathrm{d}x = \mathbb{P}\left(x_1(t) \in A, \quad x_2(t) < x_1(t) - a\right)$$

for every Borel set $A \subset \mathbb{R}$. The density z exists because $x_1(t)$ has a density, with c.d.f. given by the solution of the Fisher–KPP equation. We derive an evolution equation for z by considering the dependence of $z(t+\varepsilon, x; a)$ on the state of the system at the small time $\varepsilon > 0$. We implicitly assume that $z \in \mathcal{C}_t^1\mathcal{C}_x^2$. This assumption is justified *a posteriori* by (2.8).

At time ε, there have been no branching events with probability $1-\varepsilon+o(\varepsilon)$, one branching event with probability $\varepsilon + o(\varepsilon)$, and more than one branching event with probability $o(\varepsilon)$. We will only track terms of order ε or larger, so the last event can be safely discarded. More precisely, we denote the event defining $z(t+\varepsilon, x; a)$ by Z:

$$Z := \{x_1(t+\varepsilon) \in [x, x+\mathrm{d}x], x_2(t) < x-a\}.$$

Let b denote the (random) number of branching events by time ε. Then

$$z(t+\varepsilon, x; a)\,\mathrm{d}x = \mathbb{P}\left(Z|b=0\right)(1-\varepsilon) + \varepsilon\mathbb{P}\left(Z|b=1\right) + o(\varepsilon). \tag{2.2}$$

If $b = 0$, there is a single particle present at time ε. Its position is $-a+\eta\sqrt{\varepsilon}$, where η is a centered Gaussian of variance 2. Hence, by the Markov property,

$$\mathbb{P}\left(Z|b=0\right) = \mathbb{E}\left[z(t, x-\eta\sqrt{\varepsilon}; a)\right]\,\mathrm{d}x = \left[z(t,x;a) + \varepsilon\partial_x^2 z(t,x;a) + o(\varepsilon)\right]\,\mathrm{d}x. \tag{2.3}$$

When $b = 1$, there are $\varkappa$ particles (with $\varkappa > 1$ random) with positions $-a+\sqrt{\varepsilon}\eta_1, \ldots, -a+\sqrt{\varepsilon}\eta_\varkappa$. We further condition on the event $\varkappa = k$. Then $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_k)$ are the scaled positions of a time-ε BBM started from 0 and conditioned to have a singe branching event which produces k offspring. (In particular, each η_i is a centered Gaussian with variance 2). As $\varepsilon \to 0$, $\boldsymbol{\eta}$ converges in (joint) distribution to the positions at time 1 of the particles of a BBM with a single branching event at a uniform time on [0, 1] that produces k offspring. Conditioning on $\varkappa = k$ and $\boldsymbol{\eta}$, for Z to occur we need the descendant-BBM generated by one particle, say j, to satisfy the event defining

$$z\left(t, x-\sqrt{\varepsilon}\eta_j; a\right) = z(t,x;a) + O(\sqrt{\varepsilon}\eta_j).$$

Moreover, we need the descendant-BBMs generated by the remaining $k-1$ particles to *all* lie to the left of $x-a$ at time $t+\varepsilon$. Recalling the solution $H(t,x)$ to (1.5) and (1.6), the conditional probability density that Z occurs is

$$p_Z(t,x;a) = \sum_{j=1}^{k} z(t, x-\sqrt{\varepsilon}\eta_j; a)\prod_{i\neq j}\left[1 - H(t, x-\sqrt{\varepsilon}\eta_i)\right].$$

Recall the exponent β from (1.1). We claim that

$$p_Z(t,x;a) = kz(t,x;a)[1-H(t,x)]^{k-1} + O\left(\varepsilon^{\beta/4}k^{1+\beta/2}\,\max\{\|\boldsymbol{\eta}\|_\infty, 1\}\right). \tag{2.4}$$

As a first step, note that $0 \leqslant H \leqslant 1$ implies

$$p_Z(t,x;a) = z(t,x;a)\sum_{j=1}^{k}\prod_{i\neq j}\left[1-H(t,x-\sqrt{\varepsilon}\eta_i)\right] + O\left(\sqrt{\varepsilon}k\|\boldsymbol{\eta}\|_\infty\right). \tag{2.5}$$

This error is smaller than that in (2.4) because $\varepsilon^{\beta/4}k^{\beta/2} \geqslant \varepsilon^{1/2}$ when $\varepsilon < 1$. Next, we claim that

$$\prod_{i\neq j}\left[1 - H(t, x - \sqrt{\varepsilon}\eta_i)\right] = [1 - H(t,x)]^{k-1} + O\left(\min\left\{\sqrt{\varepsilon}k\|\boldsymbol{\eta}\|_\infty,\ 1\right\}\right). \tag{2.6}$$

The upper bound of 1 on the error is clear, as both products lie in $[0, 1]$. To obtain the other bound, we write $H(t, x - \sqrt{\varepsilon}\eta_i) = H(t,x) + O(\sqrt{\varepsilon}\|\boldsymbol{\eta}\|_\infty)$. Supposing for simplicity that $j = 1$, we use $0 \leqslant H \leqslant 1$ and induct on the following:

$$\prod_{i=2}^{k}\left[1 - H(t, x - \sqrt{\varepsilon}\eta_i)\right] = [1 - H(t,x)]\prod_{i=3}^{k}\left[1 - H(t, x - \sqrt{\varepsilon}\eta_i)\right] + O(\sqrt{\varepsilon}\|\boldsymbol{\eta}\|_\infty).$$

We now record an elementary bound for all $a, b > 0$ and $\gamma \in (0, 1)$:

$$\min\{ab, 1\} \leqslant \min\{a, 1\}\ \max\{b, 1\} \leqslant a^\gamma\ \max\{b, 1\}.$$

Taking $a = \sqrt{\varepsilon}k$, $b = \|\boldsymbol{\eta}\|_\infty$, and $\gamma = \beta/2$, we find

$$\min\left\{\sqrt{\varepsilon}k\|\boldsymbol{\eta}\|_\infty,\ 1\right\} \leqslant \varepsilon^{\beta/4}k^{\beta/2}\ \max\{\|\boldsymbol{\eta}\|_\infty, 1\}.$$

Summing over j in (2.6) and using (2.5), we obtain (2.4).

We now take expectation over $\boldsymbol{\eta}$ in (2.4). The maximum of k Gaussians is bounded in expectation by $\sqrt{\log k}$, so $\mathbb{E}_{\boldsymbol{\eta}}\left(k^{1+\beta/2}\ \max\{\|\boldsymbol{\eta}\|_\infty, 1\}\right) \leqslant Ck^{1+\beta}$ and

$$\begin{aligned}\mathbb{P}\left(Z|b = 1, \varkappa = k\right) &= \mathbb{E}_{\boldsymbol{\eta}}\, p_Z(t, x; a)\,\mathrm{d}x\\ &= \left[kz(t,x;a)[1 - H(t,x)]^{k-1} + O\left(\varepsilon^{\beta/4}k^{1+\beta}\right)\right]\mathrm{d}x.\end{aligned} \tag{2.7}$$

We can now take expectation over b and $\varkappa$. We have carefully tracked the k-dependence of the error because we only assume a limited number of moments on $\varkappa$. Gathering (2.2), (2.3), and (2.7), the moment bound (1.1) yields

$$\begin{aligned}z(t+\varepsilon, x; a) &= (1-\varepsilon)\left[z(t,x;a) + \varepsilon\partial_x^2 z(t,x;a)\right]\\ &\quad + \varepsilon\mathbb{E}\left[\varkappa z(t,x;a)[1 - H(t,x)]^{\varkappa-1} + O(\varepsilon^{\beta/4}\varkappa^{1+\beta})\right] + o(\varepsilon)\\ &= (1-\varepsilon)z(t,x;a) + \varepsilon\partial_x^2 z(t,x;a) + \varepsilon z(t,x;a)\sum_{k\geqslant 2}p_k k[1 - H(t,x)]^{k-1} + o(\varepsilon).\end{aligned}$$

Taking $\varepsilon \searrow 0$ and recalling the definition of f in (1.7), we obtain

$$\partial_t z(t,x;a) = \partial_x^2 z(t,x;a) + f'(H(t,x))z(t,x;a), \quad z(0,x;a) = \delta(x+a). \tag{2.8}$$

By the definition of z,

$$\mathbb{P}\left(x_1(t) - x_2(t) > a\right) = \int_{\mathbb{R}} z(t,x;a)\,\mathrm{d}x.$$

We observe that a is arbitrary; in particular, we could allow it to depend on t. We do not consider such dependence, however. Instead, we fix a and take $t \to \infty$ to obtain

$$\mathbb{P}(d_{12} > a) = \lim_{t\to\infty}\int_{\mathbb{R}} z(t,x;a)\,\mathrm{d}x. \tag{2.9}$$

To understand the long-time behavior of z, we observe that (2.8) resembles a derivative of the Fisher–KPP equation (1.5). Indeed, $\partial_x H$ solves (2.8) with a different initial condition. Bramson showed that H converges to a traveling wave in the m-moving frame. If we apply standard parabolic estimates to (1.8), we can conclude that

$$\partial_x H(t, x + m(t)) \to U'(x - \overline{x}_0)$$

uniformly in $x \in \mathbb{R}$ as $t \to \infty$. This suggests that $U'(x - \overline{x}_0)$ is a stable steady state at $t = \infty$ of (2.8) in the moving frame. Since (2.8) is linear, we might expect that z converges to a multiple of this state in the moving frame. The methods of [10, 19] can be adapted to confirm these heuristics:

Proposition 2.1 *For each $a > 0$, there exists $M(a) > 0$ such that*

$$z(t, x + m(t); a) \to -M(a)U'(x - \overline{x}_0) \tag{2.10}$$

uniformly in $x \in \mathbb{R}$ as $t \to \infty$.

The proof of this proposition is very similar to that of theorem 1.1 in [19]. There, the authors analyze the dynamics of the leading tail of a solution to the Fisher–KPP equation. They show that this tail converges to a multiple of $x\mathrm{e}^{-\lambda_* x}$ in an appropriate regime. In light of (1.10), this multiple determines the constant shift x_∞ in theorem 1.1 of [19]; in the present notation, this is the shift $s[\phi]$ in (1.13). In our case, the methods of [19] (particularly lemma 5.1) can be adapted to show that the leading tail of z converges to a multiple of the tail of U'. The arguments of section 4 in [19] then imply the full convergence in proposition 2.1. This adapted proof contains no new ideas, so we omit it. We also note that the convergence in proposition 2.1 can be extended to a wider class of initial data for z, e.g. to compactly-supported non-negative measures of finite mass.

To use (2.10) in (2.9), we need to commute the long-time limit of the spatial integral.

Lemma 2.2 *For all $a > 0$,*

$$\lim_{t\to\infty} \int_{\mathbb{R}} z(t, x + m(t); a)\,\mathrm{d}x = M(a). \tag{2.11}$$

Proof. Throughout, we fix $a > 0$. By the comparison principle and (2.8), $z \geqslant 0$. Since (2.10) is uniform in x, it suffices to show that no mass of z escapes to infinity. Recalling the definition (2.1) and (1.4), we have

$$z(t, x; a) \leqslant -\partial_x H(t, x + a).$$

Hence for $L > 0$, (1.8) and (1.10) imply that

$$\int_L^\infty z(t, x + m(t); a)\,\mathrm{d}x \leqslant H(t, L + m(t) + a) \leqslant 2U(L + a - \overline{x}_0) \leqslant C(a)\mathrm{e}^{-cL},$$

provided t is sufficiently large (depending on L and a). Similarly, (1.10) yields

$$\int_{-\infty}^{-L} z(t, x + m(t); a)\,\mathrm{d}x \leqslant C(a)\mathrm{e}^{-cL}$$

when t is large. Therefore (2.10) implies

$$\begin{aligned}\lim_{t\to\infty}\int_{\mathbb{R}} z(t,x+m(t);a)\,\mathrm{d}x &= \lim_{t\to\infty}\int_{-L}^{L} z(t,x+m(t);a)\,\mathrm{d}x + O\left(\mathrm{e}^{-cL}\right)\\ &= M(a)\left[U(-L-\overline{x}_0)-U(L-\overline{x}_0)\right]+O\left(\mathrm{e}^{-cL}\right)\\ &= M(a)+O\left(\mathrm{e}^{-cL}\right),\end{aligned}$$

where we have again used (1.10). Since $L>0$ was arbitrary, (2.11) follows. □

Combining (2.9) and (2.11), we obtain $\mathbb{P}(d_{12}>a)=M(a)$. Hence proposition 2.1 becomes

$$z(t,x+m(t);a)\to-\mathbb{P}(d_{12}>a)U'(x-\overline{x}_0)\quad\text{as } t\to\infty \tag{2.12}$$

uniformly in $x\in\mathbb{R}$. We will use this long-time limit to determine the the dependence of $\mathbb{P}(d_{12}>a)$ on a. First, however, we describe a connection with the Bramson shift introduced in (1.13).

2.2. *The distribution of d_{12} through Bramson shifts*

As noted above, (2.8) appears to be a derivative of (1.5). Precisely, it is the derivative with respect to a certain perturbation of the initial data $1_{(-\infty,0]}$. Consider the initial condition

$$u(0,x;y,a)=1_{(-\infty,0]}(x)-1_{(y-a,-a]}(x) \tag{2.13}$$

parameterized by $y<0$ and $a>0$. We emphasize that y is negative, so $y-a<-a$. We think of a as fixed, but allow y to vary. Note that

$$u(0,x;0,a)=1_{(-\infty,0]}(x) \tag{2.14}$$

for all $a>0$ and

$$\partial_y u(0,x;y,a)=\delta(x-y+a) \tag{2.15}$$

in the distributional sense. Now let $u(t,x;y,a)$ denote the solution to the Fisher–KPP equation (1.11) with initial data (2.13). In particular, (1.6) and (2.14) imply that

$$u(t,x;0,a)=H(t,x) \tag{2.16}$$

for all $a>0$.

Now, standard parabolic estimates imply that u is differentiable in y and that its derivative satisfies the appropriate derivative of (1.11). That is, if

$$z(t,x;y,a)=\partial_y u(t,x;y,a), \tag{2.17}$$

then (2.15) yields

$$\begin{aligned}&\partial_t z(t,x;y,a)=\partial_x^2 z(t,x;y,a)+f'\left(u(t,x;y,a)\right)z(t,x;y,a),\\ &z(0,x;y,a)=\delta(x-y+a).\end{aligned}$$

In particular, (2.16) and uniqueness imply that $z(t,x;\ 0,a)=z(t,x;a)$. That is, the distribution of $x_1(t)-x_2(t)$ can be expressed via derivatives of the Fisher–KPP equation with respect to its initial data. This connection allows us to relate the law of d_{12} to the Bramson shift in (1.13).

Recall that for each fixed y and a, (1.13) implies that

$$u(t, x + m(t); y, a) \to U(x - s(y, a)) \tag{2.18}$$

uniformly in x as $t \to \infty$, where $s(y, a) = s\,[u(0, x; y, a)]$ is the Bramson shift associated with the initial condition $u(0, x; y, a)$. This is well defined because $u(0, x; y, a)$ is bounded, nonnegative, and compactly supported on the right. Let us formally suppose that s is differentiable in y and that ∂_y commutes with the long-time limit (2.18). Then (2.17) and (2.18) imply

$$\begin{aligned}\lim_{t\to\infty} z(t, x + m(t); a) &= \lim_{t\to\infty} \partial_y u(t, x + m(t); y, a)\Big|_{y=0}\\ &= \partial_y \left(\lim_{t\to\infty} u(t, x + m(t); y, a)\right)\Big|_{y=0}\\ &= \partial_y U(x - s(y, a))\big|_{y=0} = -\partial_y s(0, a) U'(x - \overline{x}_0).\end{aligned}$$

In the last line, we used the identity $s(0, a) = \overline{x}_0$, which follows from (1.14) and (2.14). Comparing with (2.12), we see that

$$\mathbb{P}(d_{12} > a) = \partial_y s(0, a). \tag{2.19}$$

That is, the gap between the two leaders in the BBM extremal process is encoded in the Bramson shifts $s(y, a)$.

This can be seen directly from ideas of [6, section 3] and [18, section 1]. Using (2.13) and McKean's identity [17], we have

$$\begin{aligned}1 - u(t, x + m(t); y, a) &= \mathbb{E}\left[\prod_{k=1}^{n(t)} (1 - u(0, x + m(t) - x_k(t); y, a))\right]\\ &= \mathbb{P}\,[x + m(t) - x_k(t) \in (y - a, -a] \cup (0, \infty) \quad \text{for all } 1 \leqslant k \leqslant n(t)]\\ &= \mathbb{P}\,[x_k(t) - m(t) \in [x + a, x + a - y) \cup (-\infty, x) \quad \text{for all } 1 \leqslant k \leqslant n(t)]\,.\end{aligned}$$

Taking the limit $t \to \infty$, (2.18) and (1.19) imply

$$\begin{aligned}1 - U(x - s(y, a)) &= \mathbb{P}\left[\chi_k \in [x + a, x + a + |y|) \cup (-\infty, x) \quad \text{for all } k \geqslant 1\right]\\ &= \mathbb{P}\,[\chi_1 < x] + \mathbb{P}\left[\chi_1 \in [x + a, x + a + |y|),\ \chi_2 < x\right] + O(y^2),\end{aligned} \tag{2.20}$$

where we have used $\chi_1 \geqslant \chi_2 \geqslant \ldots$ and $y < 0$. The $O(y^2)$ term contains all events with multiple particles in $[x + a, x + a + |y|)$. The first term in the right side of (2.20) is $1 - U(x - \overline{x}_0) = 1 - U(x - s(0, a))$. Moving this term to the left side and dividing by $|y|$, we obtain in the $y \nearrow 0$ limit

$$\begin{aligned}\partial_y U(x - s(y, a))\big|_{y=0} = -\partial_y s(0, a) U'(x - \overline{x}_0) &= \lim_{y\nearrow 0} \frac{\mathbb{P}\left[\chi_1 \in [x + a, x + a + |y|),\ \chi_2 < x\right]}{|y|}\\ &= \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{P}\,[\chi_1 < x + a,\ d_{12} > a]\,.\end{aligned}$$

Integrating over x, we recover (2.19). For other results of a similar flavor, see [6, 18].

2.3. Long-time asymptotics for z

The previous subsection puts our problem in a broader context, but it is not essential to our proof of theorem 1.1. Instead, we can proceed directly from (2.12), which states that $\mathbb{P}(d_{12} > a)$ is the coefficient of $-U'$ in the long-time limit of z in the m-moving frame. We must determine the dependence of this coefficient on a as $a \to \infty$. To do so, it helps to shift by m and remove the exponential decay of U'. We define

$$r(t, x) = (t + 1)^{-3/2}\mathrm{e}^{\lambda_*(x+a)}z(t, m(t) + x; a). \tag{2.21}$$

The factor of $(t + 1)^{-3/2}$ above is a matter of convenience which will be explained below. Of course, r depends on a as well, but explicit dependence will become cumbersome in the remainder of the paper, so we drop it.

Using (1.2) and (1.3), we see that the function $r(t, x)$ satisfies

$$\partial_t r = \partial_x^2 r - \frac{3}{2\lambda_*(t + 1)}\partial_x r - \left[N - 1 - f'\left(H(t, m(t) + x)\right)\right] r, \quad r(0, x) = \delta(x + a). \tag{2.22}$$

Recalling the decomposition (1.17) of f, we see that the coefficient of the reaction term is $-F'(H)$. We introduce the notation

$$V(t, x) = F'(H(t, m(t) + x)) \tag{2.23}$$

and write (2.22) as

$$\partial_t r = \partial_x^2 r - \frac{3}{2\lambda_*(t + 1)}\partial_x r - V(t, x)r, \quad r(0, x) = \delta(x + a). \tag{2.24}$$

By (1.18), $0 < V < N$ and

$$V(t, x) \to N \qquad \text{as } x \to -\infty,$$
$$V(t, x) \to 0 \qquad \text{as } x \to +\infty.$$

Moreover,

$$V(t, x) \to V(\infty, x) = F'\left(U(x - \overline{x}_0)\right) \quad \text{as } t \to \infty. \tag{2.25}$$

The potential V largely confines r to $\mathbb{R}_+$. Since the drift term in (2.24) decays in time, we can thus view (2.24) as a heat equation on $\mathbb{R}_+$ with (approximate) Dirichlet conditions at $x = 0$. It follows that the dynamics of r are largely driven at the diffusive scale $x \asymp \sqrt{t}$, by which we mean $c\sqrt{t} \leqslant x \leqslant C\sqrt{t}$. In particular, at that scale we should expect r to resemble a multiple of the fundamental solution to the Dirichlet heat equation on $\mathbb{R}_+$:

$$r(t, x) \sim \widetilde{M}(a)\frac{x}{(t + 1)^{3/2}}\mathrm{e}^{-x^2/4(t+1)} \quad \text{for } t \gg 1,\ x \asymp \sqrt{t} \tag{2.26}$$

and some $\widetilde{M}(a) > 0$.

Now, $f \in \mathcal{C}^1$, so (1.9) implies that $U \in \mathcal{C}^3$, and the asymptotics (1.10) are actually valid in $\mathcal{C}^3$. In particular, we can differentiate (1.10) to obtain:

$$\begin{aligned} U'(x - \overline{x}_0) &\sim -\lambda_* x\mathrm{e}^{-\lambda_*(x-\overline{x}_0)} && \text{as } x \to \infty, \\ U'(x - \overline{x}_0) &= -A\gamma_*\mathrm{e}^{\gamma_*(x-\overline{x}_0)} + O\left(\mathrm{e}^{(\gamma_*+c)x}\right) && \text{as } x \to -\infty. \end{aligned} \tag{2.27}$$

Thus, if x is large but fixed, (2.10), (2.21), and (2.27) suggest that

$$r(t,x) \approx \lambda_* \mathrm{e}^{\lambda_*(\overline{x}_0+a)} \mathbb{P}(d_{12} > a) \frac{x}{(t+1)^{3/2}} \quad \text{as } t \to \infty. \tag{2.28}$$

Comparing (2.26) and (2.28), we expect

$$\widetilde{M}(a) = \lambda_* \mathrm{e}^{\lambda_*(\overline{x}_0+a)} \mathbb{P}(d_{12} > a).$$

Once again, the methods of [19, section 5] confirm these calculations:

Proposition 2.3 *For each* $a > 0$ *and* $\gamma \in (0, 1/2)$, *there exist* $C(a,\gamma) > 0$ *and* $h : [0,\infty) \to \mathbb{R}$ *such that for all* $t \geqslant 0$ *and* $x \in \mathbb{R}$,

$$r(t,x) = \left[\lambda_* \mathrm{e}^{\lambda_*(\overline{x}_0+a)} \mathbb{P}(d_{12} > a) + h(t)\right] \frac{x_+}{(t+1)^{3/2}} \mathrm{e}^{-x^2/4(t+1)} + R(t,x) \tag{2.29}$$

with

$$|R(t,x)| \leqslant C(a,\gamma)(t+1)^{-(3/2-\gamma)} \mathrm{e}^{-x^2/[6(t+1)]} \quad \textit{and} \quad h(t) \to 0 \quad \textit{as } t \to \infty. \tag{2.30}$$

We have used the notation $x_+ = \max\{x, 0\}$ in (2.29). This proposition is very similar to lemma 5.1 in [19], so we omit the proof.

2.4. A moment computation

By (2.19) and (2.29), it now suffices to understand the long-time dynamics of r on the diffusive scale. We capture these dynamics in a certain moment of r. An elementary computation using (2.29) and (2.30) shows that

$$\mathbb{P}(d_{12} > a) = \frac{1}{2\lambda_* \sqrt{\pi}} \mathrm{e}^{-\lambda_*(a+\overline{x}_0)} \lim_{t\to\infty} \int_0^\infty x r(t,x)\, \mathrm{d}x. \tag{2.31}$$

This identity is the reason for introducing the $(t+1)^{-3/2}$ factor in (2.21). It is convenient to express this moment in terms of the function

$$\psi(x) = -U_0'(x) \mathrm{e}^{\lambda_* x},$$

where we have introduced the notation

$$U_0(x) = U(x - \overline{x}_0) \tag{2.32}$$

for the limiting shift of the front U. The function ψ is a positive time-independent solution to the leading-order part of (2.24):

$$0 = \psi'' - V(\infty, x)\psi = \psi'' - F'(U_0(x))\psi. \tag{2.33}$$

Here we have used (2.25) and (2.32) to express the long-time limit of the potential V introduced in (2.23). Recalling that $\gamma_* + \lambda_* = \sqrt{N}$, (2.27) implies that

$$\begin{aligned} &\psi(x) \sim \lambda_* \mathrm{e}^{\lambda_* \overline{x}_0} x && \text{as } x \to +\infty, \\ &\psi(x) = A\gamma_* \mathrm{e}^{-\gamma_* \overline{x}_0} \mathrm{e}^{\sqrt{N}x} + O\left(\mathrm{e}^{(\sqrt{N}+c)x}\right) && \text{as } x \to -\infty \end{aligned} \tag{2.34}$$

for some $c > 0$. For future use, we also record the asymptotics of ψ'. Recalling that the asymptotics in (1.10) are valid in $\mathcal{C}^3$, we can differentiate (2.34) to obtain

$$\begin{aligned} &\psi'(x) \sim \lambda_* \mathrm{e}^{\lambda_* \overline{x}_0} && \text{as } x \to +\infty, \\ &\psi'(x) = A\gamma_* \sqrt{N} \mathrm{e}^{-\gamma_* \overline{x}_0} \mathrm{e}^{\sqrt{N}x} + O\left(\mathrm{e}^{(\sqrt{N}+c)x}\right) && \text{as } x \to -\infty. \end{aligned} \tag{2.35}$$

It follows from (2.29), (2.30), and (2.34) that (2.31) can be written as

$$\mathbb{P}(d_{12} > a) = \frac{1}{2\lambda_*^2 \sqrt{\pi}} \mathrm{e}^{-\lambda_*(a+2\overline{x}_0)} \lim_{t \to +\infty} I(t), \tag{2.36}$$

where

$$I(t) = \int_{\mathbb{R}} \psi(x) r(t, x)\, \mathrm{d}x. \tag{2.37}$$

This completes our series of reductions: to control $\mathbb{P}(d_{12} > a)$, we determine the dependence of $I(\infty)$ on a.

Motivated by (2.25), we write (2.24) as

$$\partial_t r = \partial_x^2 r - \frac{3}{2\lambda_*(t+1)} \partial_x r - V(\infty, x) r + E(t, x) r, \quad r(0, x) = \delta(x + a), \tag{2.38}$$

with an error term

$$E(t, x) = V(\infty, x) - V(t, x). \tag{2.39}$$

If we multiply (2.38) by ψ and integrate, (2.33) yields

$$\frac{\mathrm{d}I}{\mathrm{d}t}(t) = \frac{3}{2\lambda_*(t+1)} \int_{\mathbb{R}} r(t, x) \psi'(x)\, \mathrm{d}x + \int_{\mathbb{R}} E(t, x) r(t, x) \psi(x)\, \mathrm{d}x. \tag{2.40}$$

We note that we can exchange the time derivative and the spatial integral because z is smooth and rapidly decaying in space.

We will see below that the second term in the right side of (2.40) is, indeed, an error term, so we focus on the first term. To this end, we describe the dynamics of (2.38) qualitatively, ignoring the error term Er. First note that the mass of the solution on $\mathbb{R}_-$ will decay exponentially in time under (2.38), due to absorption from the term $-F'(U_0)r$. After all, (1.18) implies $F'(U_0) > 0$ and, in particular, $F'(1) = N$. However, mass that escapes to $\mathbb{R}_+$ experiences almost no absorption because $F'(0) = 0$. This 'fugitive' mass diffuses, but gets absorbed whenever it returns to $\mathbb{R}_-$. Thus, as noted previously, (2.38) acts much like the heat equation on $\mathbb{R}_+$ with a Dirichlet boundary condition at $x = 0$. That said, there is initially no mass on $\mathbb{R}_+$, so we must include an initial time layer during which the mass escapes from $\mathbb{R}_-$ to $\mathbb{R}_+$.

The initial condition $r(0, x) = \delta(x + a)$ is 'deep in the large absorption territory,' and takes a while to diffuse to $\mathbb{R}_+$. If we neglect the drift in the right side of (2.38), then, in the absence of absorption, the proportion of mass that diffuses from position $-a$ to $\mathbb{R}_+$ at time t is roughly

$e^{-a^2/(4t)}$. Attrition by the absorbing potential approximately reduces this by the factor e^{-Nt}. Thus to leading order, the mass that escapes to $\mathbb{R}_+$ at time t is

$$\exp\left(-Nt - a^2/(4t)\right).$$

Therefore, the mass flux into $\mathbb{R}_+$ is maximized at the time

$$t_* = \frac{a}{2\sqrt{N}}, \tag{2.41}$$

and this is, roughly, the size of the initial time layer.

Let us now use this heuristic to approximate the first term in the right side of (2.40):

$$\frac{3}{2\lambda_*(t+1)}\int_{\mathbb{R}} r(t,x)\psi'(x)\,\mathrm{d}x = \frac{3}{2\lambda_*(t+1)}\int_{\mathbb{R}} \frac{\psi'(x)}{\psi(x)}\psi(x)r(t,x)\,\mathrm{d}x.$$

We see from (2.34) and (2.35) that

$$\frac{\psi'(x)}{\psi(x)} \sim \begin{cases} \sqrt{N} & \text{for } x \ll -1, \\ \dfrac{1}{x} & \text{for } x \gg 1. \end{cases}$$

The heuristic argument above suggests that most of the mass lies in $\mathbb{R}_-$ until t_*, when it transfers to $\mathbb{R}_+$. Therefore, we expect that

$$\int_{\mathbb{R}} \frac{\psi'(x)}{\psi(x)}\psi(x)r(t,x)\,\mathrm{d}x \approx \sqrt{N}\int_{\mathbb{R}} \psi(x)r(t,x)\,\mathrm{d}x \quad \text{for } t < t_*.$$

After t_*, the mass of r will largely stay in $\mathbb{R}_+$, and will spread to the diffusive scale $x \asymp \sqrt{t}$. Since $\frac{\psi'}{\psi} \approx \frac{1}{x}$, we should have

$$\int_{\mathbb{R}} \frac{\psi'(x)}{\psi(x)}\psi(x)r(t,x)\,\mathrm{d}x \ll \int_{\mathbb{R}} \psi(x)r(t,x)\,\mathrm{d}x \quad \text{for } t > t_*.$$

Thus, $I(t)$ satisfies

$$\frac{\mathrm{d}I}{\mathrm{d}t}(t) \approx \frac{3\sqrt{N}}{2\lambda_*(t+1)}I(t) \quad \text{for } t < t_*, \qquad \frac{\mathrm{d}I}{\mathrm{d}t}(t) \approx 0 \quad \text{for } t > t_*.$$

Integrating, we find

$$\lim_{t\to+\infty} I(t) \approx t_*^{3\sqrt{N}/(2\lambda_*)} I(0) = \left(\frac{a}{2\sqrt{N}}\right)^{3\sqrt{N}/(2\lambda_*)} I(0) \tag{2.42}$$

when $a \gg 1$. In the remainder of this paper, we justify (2.42).

Proposition 2.4 *We have*

$$\lim_{t\to+\infty} I(t) = \left(\frac{a}{2\sqrt{N}}\right)^{3\sqrt{N}/(2\lambda_*)} \left[1 + O\left(a^{-1/2}\right)\right] \int_{\mathbb{R}} \psi(x)r(0,x)\,\mathrm{d}x \quad \textit{as } a \to +\infty. \tag{2.43}$$

Theorem 1.1 is a consequence of proposition 2.4. Indeed, using (2.34), we see that the integral in the right side of (2.43) is

$$\int_{\mathbb{R}} \psi(x) r(0,x)\,\mathrm{d}x = \psi(-a) = A\gamma_* \mathrm{e}^{-\gamma_* \bar{x}_0} \mathrm{e}^{-\sqrt{N}a}\left(1 + o(\mathrm{e}^{-ca})\right) \tag{2.44}$$

for some $c > 0$. We then use (2.36) to write

$$\begin{aligned}
\mathbb{P}(d_{12} > a) &= \frac{1}{2\lambda_*^2\sqrt{\pi}}\mathrm{e}^{-\lambda_*(a+2\bar{x}_0)} \lim_{t\to+\infty} I(t) \\
&= \frac{A\gamma_*}{2\lambda_*^2\sqrt{\pi}}\mathrm{e}^{-(2\lambda_*+\gamma_*)\bar{x}_0}\left(\frac{a}{2\sqrt{N}}\right)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-(\lambda_*+\sqrt{N})a}\left[1 + O\left(a^{-1/2}\right)\right]\left(1 + o(\mathrm{e}^{-ca})\right) \\
&= \frac{A\gamma_*}{2\lambda_*^2\sqrt{\pi}}\left(\frac{a}{2\sqrt{N}}\right)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-(\sqrt{N}+\sqrt{N-1})(a+\bar{x}_0)}\left[1 + O\left(a^{-1/2}\right)\right].
\end{aligned}$$

This finishes the proof of theorem 1.1.

The rest of the paper contains the proof of proposition 2.4. The strategy is to estimate the ratio $\dot{I}(t)/I(t)$ on various times scales. Lemmas 4.1, 5.1, and 6.1 below express these estimates. At the end of section 6, we collect these results and prove proposition 2.4. Because we are interested in the regime $a \to \infty$, we always implicitly assume that $a \geqslant 1$.

3. The time scales and the correctors

We now turn to the proof of proposition 2.4. In this section, we discuss the time scales on which various approximations to the dynamics of (2.24) should be valid, and introduce the corresponding 'scattering decomposition' of the solution.

3.1. The time scales

Let us first explain the time scales on which various effects will dominate. Our previous reasoning indicates that at times $t < t_*$, the heat equation has not had enough time to diffuse much mass from the initial position $x = -a$ to $x \geqslant 0$. The evolution of $r(t,x)$, the solution to (2.24), is thus dominated by its homogeneous part

$$\partial_t p = \partial_x^2 p - \frac{3}{2\lambda_*(t+1)}\partial_x p - Np, \quad p(0,x) = \delta(x+a), \tag{3.1}$$

as $F'(1) = N$. Its explicit solution is

$$p(t,x) = \frac{1}{\sqrt{4\pi t}}\exp\left\{-Nt - \frac{1}{4t}\left[x + a - \frac{3}{2\lambda_*}\log(t+1)\right]^2\right\}. \tag{3.2}$$

The corrector

$$q(t,x) = r(t,x) - p(t,x)$$

solves

$$\partial_t q = \partial_x^2 q - \frac{3}{2\lambda_*(t+1)}\partial_x q - V(t,x)q + (N - V(t,x))p, \quad q(0,x) = 0, \tag{3.3}$$

recalling that

$$V(t,x) = F'\left(H(t, x + m(t))\right).$$

Since $F'(u) \leqslant N$ by (1.18), $V \leqslant N$. Applying the comparison principle to (3.3), we see that

$$q(t,x) \geqslant 0. \tag{3.4}$$

We view (2.24) as a perturbation of the absorbing heat equation (3.1), so that $p(t,x)$ represents the free evolution and $q(t,x)$ accounts for the interaction with the potential $V(t,x)$. The role of p is to transport the mass from $x = -a$ to the half-line $\mathbb{R}_+$, and the role of q is to account for this escaped mass as $t \to \infty$. Indeed, we will show that

$$I(t) = \int_{\mathbb{R}} r(t,x)\psi(x)\,\mathrm{d}x \approx \int_{\mathbb{R}} p(t,x)\psi(x)\,\mathrm{d}x \quad \text{for } t \ll t_*$$

and

$$I(t) = \int_{\mathbb{R}} r(t,x)\psi(x)\,\mathrm{d}x \approx \int_{\mathbb{R}} q(t,x)\psi(x)\,\mathrm{d}x \quad \text{for } t \gg t_*.$$

In constructing our time scales, we must also consider the forcing term $(N - V)p$ in (3.3). In particular, we study its moment contribution

$$\int_{\mathbb{R}} (N - V(t,x))\psi(x)p(t,x)\,\mathrm{d}x. \tag{3.5}$$

To understand the time scales on which it may potentially play a role, note that the first two terms in the integrand decay on the left: $\psi(x)$ has the asymptotic behavior as in (2.34), and the first factor is controlled by the following lemma.

Lemma 3.1 *There exist $B > 0$ and $c > 0$ depending only on f such that*

$$0 \leqslant N - V(t,x) \leqslant \min\left\{Be^{\gamma_* x}, N\right\} \quad \textit{and} \quad 0 \leqslant V(t,x) \leqslant \min\left\{B\mathrm{e}^{-cx}, N\right\} \tag{3.6}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}$.

Proof. For $t > 0$, define the shift $\tilde{m}(t)$ by

$$H(t, \tilde{m}(t)) = U_0(0).$$

This is well-defined and continuous because $H(t, \cdot)$ strictly decreases from 1 to 0 and H is continuous when $t > 0$. By theorem 12 in [14],

$$U_0(x) \leqslant H(t, x + \tilde{m}(t)) \leqslant 1 \quad \text{for all } t > 0 \text{ and } x \leqslant 0 \tag{3.7}$$

and

$$0 \leqslant H(t, x + \tilde{m}(t)) \leqslant U_0(x) \quad \text{for all } t > 0 \text{ and } x \geqslant 0. \tag{3.8}$$

Bramson's work [4, 5] implies that $\tilde{m}(t) - m(t) \to 0$ as $t \to \infty$. Using (1.10) and (3.7), we find

$$0 \leqslant 1 - H(t, x + m(t)) \leqslant \min\left\{C\mathrm{e}^{\gamma_* x}, 1\right\} \quad \text{for all } t \geqslant 0 \text{ and } x \in \mathbb{R} \tag{3.9}$$

for some C depending only on f. Similarly, (1.10) and (3.8) imply

$$0 \leqslant H(t, x + m(t)) \leqslant \min\left\{C\mathrm{e}^{-\lambda_* x/2}, 1\right\} \quad \text{for all } t \geqslant 0 \text{ and } x \in \mathbb{R}. \tag{3.10}$$

Now, (1.1) and (1.7) imply that f is $\mathcal{C}^2$ away from 0 and $\mathcal{C}^{1,\beta}$ near 0. The decomposition (1.17) implies the same regularity for F. Using the definition (2.23) of V, the first bound in (3.6) follows from (1.18), (3.9), and $F' \in \mathcal{C}^1([1/2, 1])$. The second bound in (3.6) follows from (1.18), (3.10), and $F' \in \mathcal{C}^\beta([0, 1])$. □

To understand the spatial decay of $p(t, x)$, it helps to write it as

$$p(t, x) = \Lambda(t; a)\mathrm{e}^{-ax/(2t)}g(t, x) \tag{3.11}$$

with the factor

$$\Lambda(t; a) = \frac{(t+1)^{3a/(4\lambda_* t)}}{\sqrt{4\pi t}}\mathrm{e}^{-Nt - a^2/(4t)} \tag{3.12}$$

and the re-centered Gaussian

$$g(t, x) = \exp\left\{-\frac{1}{4t}\left[x - \frac{3}{2\lambda_*}\log(t+1)\right]^2\right. \tag{3.13}$$

Combining (2.34), (3.6) and (3.11), we see that it is straightforward to control the spatial decay of the integrand in (3.5) as $x \to -\infty$ for times t such that

$$\gamma_* + \sqrt{N} > \frac{a}{2t},$$

i.e.

$$t > \frac{a}{2(2\sqrt{N} - \sqrt{N-1})}.$$

Accordingly, we fix

$$\xi_- \in \left(\frac{1}{2(2\sqrt{N} - \sqrt{N-1})}, \frac{1}{2\sqrt{N}}\right), \qquad t_- = \xi_- a.$$

Let us decompose the solution to (3.3) as

$$q = q_\mathrm{e} + q_\mathrm{m},$$

where q_e is forced on the time interval $[0, t_-]$ and q_m on $[t_-, \infty)$:

$$\partial_t q_\mathrm{e} = \partial_x^2 q_\mathrm{e} - \frac{3}{2\lambda_*(t+1)}\partial_x q_\mathrm{e} - V(t, x)q_\mathrm{e} + (N - V(t, x))1_{[0,t_-]}(t)p \tag{3.14}$$

and

$$\partial_t q_\mathrm{m} = \partial_x^2 q_\mathrm{m} - \frac{3}{2\lambda_*(t+1)}\partial_x q_\mathrm{m} - V(t, x)q_\mathrm{m} + (N - V(t, x))1_{[t_-,\infty)}(t)p, \tag{3.15}$$

with $q_\mathrm{e}(0, \cdot) = q_\mathrm{m}(0, \cdot) = 0$.

As we have discussed, the product $(N - V)p$ is very small for $t \ll t_*$, and $t_- \ll t_*$ if $a \gg 1$. It follows that q_e should never form a significant part of r, and we think of it as *error*. In contrast, q_m is eventually the principal part of r, so we view it as the *main* part of q.

Although q_e should be irrelevant, it is challenging to estimate. We introduced the corrector q because it is easier to analyze the long-time behavior of adjoint-weighted mass which begins in $\mathbb{R}_+$, rather than deep in $\mathbb{R}_-$. As we argue above, the adjoint-weighted forcing $(N - V)\psi p$ for q_m is concentrated on $\mathbb{R}_+$. However, this is not the case for q_e, which is still primarily forced deep in $\mathbb{R}_-$. We appear to be no better off than when we started with a point mass at $-a$! And indeed, we will be forced to control q_e using a further corrector.

However, q_e is driven by the forcing $(N - V)p$, which is far smaller than p due to lemma 3.1. It follows that q_e is much smaller than the original solution r. We can therefore be less precise in our estimation of q_e. This wiggle room saves us from a futile infinite descent of correctors. Instead, two steps suffice. The details are rather technical, so we defer them to section 7. There, we show:

Lemma 3.2 *There exist $C, c > 0$ independent of a and t such that for all $a \geqslant 1$ and $t \geqslant 0$:*

$$\int_{\mathbb{R}} \psi(x) q_e(t,x)\,\mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a},$$

$$\int_{\mathbb{R}} \psi'(x) q_e(t,x)\,\mathrm{d}x \leqslant \frac{C}{\sqrt{t+1}}\mathrm{e}^{-(\sqrt{N}+c)a},$$

$$\int_{\mathbb{R}} \psi(x)\,|E(t,x)|\,q_e(t,x)\,\mathrm{d}x \leqslant \frac{C}{(t+1)^2}\mathrm{e}^{-(\sqrt{N}+c)a}.$$

(We note that ψ and ψ' are non-negative, so the first two bounds are effective.) These bounds feature an extra factor of e^{-ca} relative to the main term, which is of order $\mathrm{e}^{-\sqrt{N}a}$; cf (2.44). This justifies our treatment of q_e as error.

To prove lemma 3.2, we will make use of the following result, which will also be useful in subsequent sections:

Lemma 3.3 *Let $w(t,x)$ satisfy*

$$\partial_t w \leqslant \partial_x^2 w - \frac{3}{2\lambda_*(t+1)}\partial_x w - \alpha^2 1_{(-\infty,-K)}(x) w, \quad t > s,\ x \in \mathbb{R}, \tag{3.16}$$

$$w(s,x) \leqslant \begin{cases} e^{-\kappa_- x} & \text{for } x < 0, \\ e^{-\kappa_+ x}\mathrm{e}^{-x^2/(8s)} & \text{for } x \geqslant 0, \end{cases} \tag{3.17}$$

for some $\alpha > 0, K \geqslant 0, \kappa_- < \alpha$, and $\kappa_+ > 0$. Then there exists a constant $C > 0$ that depends on α, K, κ_-, and λ_* but not *on s or κ_+ such that for all $t \geqslant s$, we have*

$$\int_{\mathbb{R}} \psi(x) w(t,x)\,\mathrm{d}x \leqslant C \max\{\kappa_+^{-2}, 1\}, \tag{3.18}$$

$$\int_{\mathbb{R}} \psi'(x) w(t,x)\,\mathrm{d}x \leqslant C \max\{\kappa_+^{-2}, 1\}(t-s+1)^{-1/2}, \tag{3.19}$$

$$\int_{\mathbb{R}} |E(t,x)|\psi(x) w(t,x)\,\mathrm{d}x \leqslant C \max\{\kappa_+^{-2}, 1\}(t+1)^{-1/2}(t-s+1)^{-3/2}. \tag{3.20}$$

In particular, this lemma allows us to control the long-time behavior of the integrals in lemma 3.2, because q_e eventually satisfies the hypotheses of lemma 3.3 with an additional factor of $\mathrm{e}^{-(\sqrt{N}+c)a}$.

We prove lemma 3.3 in appendix B, but offer a heuristic explanation here. Roughly speaking, we may think of (3.16) as the heat equation on the half-line with Dirichlet boundary conditions, so that

$$w(t,x) \sim \frac{x}{(t-s+1)^{3/2}} \mathrm{e}^{-x^2/(4(t-s+1))} \int_0^\infty x w(s,x)\,\mathrm{d}x \leqslant \frac{C}{\kappa_+^2} \frac{x}{(t-s+1)^{3/2}} \mathrm{e}^{-x^2/(4(t-s+1))}. \tag{3.21}$$

The bounds in (3.18)–(3.20) come from the right side of (3.21).

Going forward, we separately consider three regimes delimited by t_* from (2.41) and a: the early times $0 \leqslant t \leqslant t_*$ in section 4, the middle times $t_* \leqslant t \leqslant a$ in section 5, and the late times $t \geqslant a$ in section 6. The cutoff a is more or less arbitrary: it simply allows us to assume that $t = O(a)$ in the middle regime. As stated above, proposition 2.4 and theorem 1.1 are immediate consequences of lemmas 4.1, 5.1, and 6.1.

4. The early times

In this section, we start analyzing the contributions of p and $q = q_{\mathrm{m}} + q_{\mathrm{e}}$ to $I(t)$ and $\dot{I}(t)$. We begin with the early times $t \leqslant t_*$. The terms involving the 'early' corrector q_{e} have already been bounded in lemma 3.2 and will turn out to be irrelevant—they are much smaller than the corresponding contributions of $p(t,x)$ in lemma 4.2 below. It remains to estimate the terms involving $p(t,x)$ and $q_{\mathrm{m}}(t,x)$. We will see that the terms involving p dominate for nearly the entire interval $t \in [0,t_*]$ (in particular, $q_{\mathrm{m}} \equiv 0$ for $t \leqslant t_-$). However, the contributions of p and q_{m} become comparable when $t_* - t = O\left(\sqrt{a}\right)$.

The main result of this section is the following lemma.

Lemma 4.1 *We have*

$$\frac{\dot{I}(t)}{I(t)} = \frac{3\sqrt{N}}{2\lambda_*(t+1)}\left[1 + O\left(\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}\right)\right] \tag{4.1}$$

for all $t \in [0,t_*]$.

We note that the error term in (4.1) is small over most of the time period $[t,t_*]$, but it becomes order 1 when $t_* - t = O\left(\sqrt{a}\right)$. After integration, lemma 4.1 implies that

$$\log \frac{I(t_*)}{I(0)} = \frac{3\sqrt{N}}{2\lambda_*}\log(t_*+1) + O(a^{-1/2}) = \frac{3\sqrt{N}}{2\lambda_*}\log\frac{a}{2\sqrt{N}} + O(a^{-1/2}), \tag{4.2}$$

where the error term is dominated by the region $t_* - t = O\left(\sqrt{a}\right)$.

4.1. The free contribution

We first look at the putative main term in this period.

Lemma 4.2 *There exist $c > 0$ and $C > 0$ so that for all $t \in [t_-, t_*]$, we have the asymptotics*

$$\int_{\mathbb{R}} \psi(x)p(t,x)\,\mathrm{d}x = A\gamma_* e^{-\gamma_*\overline{x}_0}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\left[1 + O\left(\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}\right)\right] \tag{4.3}$$

and

$$\int_{\mathbb{R}} \psi'(x)p(t,x)\,\mathrm{d}x = \sqrt{N}A\gamma_* e^{-\gamma_*\overline{x}_0}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\left[1 + O\left(\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}\right)\right], \tag{4.4}$$

as well as the error estimate

$$\int_{\mathbb{R}} |E(t,x)|p(t,x)\psi(x)\,\mathrm{d}x \leqslant \frac{C}{t+1}\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}\int_{\mathbb{R}} \psi'(x)p(t,x)\,\mathrm{d}x. \tag{4.5}$$

Moreover, for all $t \in [t_-, t_*]$,

$$I(t) \geqslant \int_{\mathbb{R}} \psi(x)p(t,x)\,\mathrm{d}x \geqslant c(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}. \tag{4.6}$$

The first estimate in this lemma permits $\int_{\mathbb{R}}\psi p$ to reach 0 when $t_* - t = O\left(\sqrt{a}\right)$, which could make $\dot{I}(t)/I(t)$ very large. The lower bound (4.6) ensures that this does not occur.

Proof. To prove (4.3), let us first re-write $p(t,x)$ in a more convenient form, starting from (3.2):

$$\begin{aligned} p(t,x) &= \frac{1}{\sqrt{4\pi t}}\exp\left\{-Nt - \frac{1}{4t}\left[x + a - \frac{3}{2\lambda_*}\log(t+1)\right]^2\right\} \\ &= \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{1}{4t}\left[x + a - \frac{3}{2\lambda_*}\log(t+1) - 2\sqrt{N}t\right]^2 - \sqrt{N}\left[x + a - \frac{3}{2\lambda_*}\log(t+1)\right]\right\} \\ &= \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-a\sqrt{N}}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}x}\exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\}. \end{aligned}$$

where we used the notation

$$\nu(t) = -a + \frac{3}{2\lambda_*}\log(t+1) + 2\sqrt{N}t.$$

It follows that

$$\int_{\mathbb{R}} \psi(x)p(t,x)\,\mathrm{d}x = \mathrm{e}^{-\sqrt{N}a}(t+1)^{3\sqrt{N}/(2\lambda_*)}\int_{\mathbb{R}} \zeta(x)\exp\left\{-\frac{1}{4t}[x-\nu(t)]^2\right\}\frac{\mathrm{d}x}{\sqrt{4\pi t}}, \tag{4.7}$$

with

$$\zeta(x) = \psi(x)\exp\left(-\sqrt{N}x\right).$$

Note that (2.34) gives

$$\zeta(x) = A\gamma_* \mathrm{e}^{-\gamma_* \overline{x}_0} 1_{\mathbb{R}_-}(x) + O\left(\mathrm{e}^{-c|x|}\right). \tag{4.8}$$

Note also that

$$\nu(t) = -a + \frac{3}{2\lambda_*}\, \log(t+1) + 2\sqrt{N}(t_* - (t_* - t)) \leqslant -2\sqrt{N}(t_* - t) + C\, \log\, a. \tag{4.9}$$

When $t_* - t \leqslant \sqrt{a}$, (4.3) follows simply from $0 \leqslant \zeta \leqslant C$, which is a consequence of (4.8). We may thus assume that $0 \leqslant t \leqslant t_* - \sqrt{a}$. Since $\log\, a \ll \sqrt{a}$ when a is large, we can in turn assume that (4.9) yields

$$\nu(t) \leqslant -c(t_* - t). \tag{4.10}$$

Using (4.8) and (4.9), we can write the integral in the right side of (4.7) as

$$\int_{\mathbb{R}} \zeta(x) \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} = A\gamma_* \mathrm{e}^{-\gamma_* \overline{x}_0} \int_{\mathbb{R}_-} \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}}$$
$$+ O\left(\int_{\mathbb{R}} \mathrm{e}^{-c|x|}\, \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}}\right). \tag{4.11}$$

We can write the main term on the right side as

$$\int_{\mathbb{R}_-} \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} = 1 - \int_{\mathbb{R}_+} \exp\left\{-\frac{1}{4t}\left[x + |\nu(t)|\right]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}}$$
$$= 1 + O\left(\mathrm{e}^{-\nu(t)^2/(4t)}\right) = 1 + O\left(\mathrm{e}^{-\frac{c(t_* - t)^2}{a}}\right). \tag{4.12}$$

We used (4.10) and $t \leqslant a$ in the last step. To bound the error in (4.11), we break the integral at the position $\nu(t)/2$. For $x \leqslant \nu(t)/2$ we write, on the one hand,

$$\int_{-\infty}^{\nu(t)/2} \mathrm{e}^{-c|x|}\, \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} \leqslant \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\nu(t)/2} \mathrm{e}^{cx}\, \mathrm{d}x = \frac{C}{\sqrt{t}} \mathrm{e}^{c\nu(t)/2}$$

and, on the other hand,

$$\int_{-\infty}^{\nu(t)/2} \mathrm{e}^{-c|x|}\, \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} \leqslant \mathrm{e}^{c\nu(t)/2} \int_{\mathbb{R}} \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} = \mathrm{e}^{c\nu(t)/2}.$$

Combining these two bounds gives

$$\int_{-\infty}^{\nu(t)/2} \mathrm{e}^{-c|x|}\, \exp\left\{-\frac{1}{4t}[x - \nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} \leqslant \frac{C}{\sqrt{1+t}} \mathrm{e}^{c\nu(t)/2} \leqslant \frac{C}{\sqrt{1+t}} \mathrm{e}^{-c(t_* - t)}$$
$$\leqslant \frac{C}{\sqrt{1+t}} \mathrm{e}^{-\frac{c(t_* - t)^2}{a}}. \tag{4.13}$$

For the the region $x \geqslant \nu(t)/2$ of the integral in the error term in (4.11), we have

$$\int_{\nu(t)/2}^{\infty} \mathrm{e}^{-c|x|} \exp\left\{-\frac{1}{4t}[x-\nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} \leqslant \frac{C}{\sqrt{t}} \mathrm{e}^{-\nu(t)^2/(16t)} \leqslant \frac{C}{\sqrt{1+t}} \mathrm{e}^{-\frac{c(t_*-t)^2}{a}}. \tag{4.14}$$

In the last step, we wrote $\nu(t)^2/(16t) \geqslant 1/(32t) + \nu(t)^2/(32t)$ and used

$$\frac{1}{\sqrt{t}} \exp\left(-\frac{1}{32t}\right) \leqslant \frac{C}{\sqrt{t+1}}.$$

Now, (4.3) follows from (4.7), (4.11)–(4.14). The proof of (4.4) is identical: recalling (2.35), we just need to use $\tilde{\zeta}(x) := \psi'(x)\exp(-\sqrt{N}x) = \sqrt{N}A\gamma_* \mathrm{e}^{-\gamma_* \overline{x}_0} 1_{\mathbb{R}_-}(x) + O\left(\mathrm{e}^{-c|x|}\right)$ in place of (4.8).

We now turn to (4.5). We rely on a quantitative form of (1.8), which essentially follows from theorem 1.3 in [20].

Lemma 4.3 *There exist constants $C > 0$ and $c > 0$ such that*

$$|H(t, x+m(t)) - U_0(x)| \leqslant \frac{C\mathrm{e}^{-c|x|}}{\sqrt{t+1}} \quad \textit{for all}\,(t,x) \in [0,\infty) \times \mathbb{R}.$$

Recalling the definitions (2.39) of $E(t,x)$ and (2.23) of $V(t,x)$, and noting that F' is Lipschitz, we obtain

$$|E(t,x)| \leqslant C\,|H(t, x+m(t)) - U_0(x)| \leqslant \frac{C\mathrm{e}^{-c|x|}}{\sqrt{t+1}}. \tag{4.15}$$

In fact, [20] proves a bound with e^{-cx} rather than $\mathrm{e}^{-c|x|}$, so the original estimate deteriorates as $x \to -\infty$. However, it is not difficult to improve the spatial dependence when $x < 0$. To avoid cluttering the present argument, we prove lemma 4.3 in appendix C.

Using (4.15) in (4.5) and recalling that $\zeta(x) \leqslant C$, we find

$$\int_{\mathbb{R}} |E(t,x)|\,\psi(x)p(t,x)\,\mathrm{d}x \leqslant \frac{C}{\sqrt{t+1}} \mathrm{e}^{-\sqrt{N}a}(t+1)^{3\sqrt{N}/(2\lambda_*)} \times \int_{\mathbb{R}} \mathrm{e}^{-c|x|} \exp\left\{-\frac{1}{4t}\left[x+|\nu(t)|\right]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}}.$$

We estimated this integral above, assuming $t_* - t \geqslant \sqrt{a}$. Using (4.13) and (4.14), we obtain (4.5) for such times. When $t_* - t \leqslant \sqrt{a}$, we can bound the Gaussian in the above integral by 1 to obtain (4.5).

To obtain the lower bound in (4.6), we first recall (3.4), which implies $r \geqslant p$ and $I \geqslant \int_{\mathbb{R}} \psi p$. Next, we note that $\zeta \geqslant c 1_{\mathbb{R}_-}$. So, (4.7) implies

$$I(t) \geqslant \int_{\mathbb{R}} \psi(x)p(t,x)\,\mathrm{d}x \geqslant c(t+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a} \times \int_{\mathbb{R}_-} \exp\left\{-\frac{1}{4t}[x-\nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}}. \tag{4.16}$$

Changing variables via $\eta = \frac{x-\nu(t)}{\sqrt{t}}$, we have

$$\int_{\mathbb{R}_-} \exp\left\{-\frac{1}{4t}[x-\nu(t)]^2\right\} \frac{\mathrm{d}x}{\sqrt{4\pi t}} = \int_{-\infty}^{-\nu(t)/\sqrt{t}} \mathrm{e}^{-\eta^2/4} \frac{\mathrm{d}\eta}{\sqrt{4\pi}}. \tag{4.17}$$

By (4.9), for $t \geqslant t_-$,

$$-\frac{\nu(t)}{\sqrt{t}} \geqslant -\frac{C\,\log\,a}{\sqrt{a}} \geqslant -1,$$

assuming a is large. In light of (4.16) and (4.17) yields

$$I(t) \geqslant \int_{\mathbb{R}} \psi(x)p(t,x)\,\mathrm{d}x \geqslant c(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}.$$

This finishes the proof of lemma 4.2. □

4.2. The main corrector contribution

Next, we control the contributions of q_{m} to $I(t)$ and $\dot{I}(t)$. We only need to consider $t \geqslant t_-$ as $q_{\mathrm{m}} \equiv 0$ for $t < t_-$.

Lemma 4.4 *There exist $c > 0$ and $C > 0$ such that for all $t \in [t_-, t_*]$,*

$$\int_{\mathbb{R}} [\psi(x) + \psi'(x)]q_{\mathrm{m}}(t,x)\,\mathrm{d}x \leqslant C(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\mathrm{e}^{-\frac{c(t_*-t)^2}{a}} \tag{4.18}$$

and

$$\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x \leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}. \tag{4.19}$$

Recall that q_{m} satisfies (3.15):

$$\partial_t q_{\mathrm{m}} = \partial_x^2 q_{\mathrm{m}} - \frac{3}{2\lambda_*(t+1)}\partial_x q_{\mathrm{m}} - V(t,x)q_{\mathrm{m}} + (N - V(t,x))1_{[t_-,\infty)}(t)p, \quad q_{\mathrm{m}}(0,\cdot) = 0.$$

We use the Duhamel formula

$$q_{\mathrm{m}}(t,x) = \int_{t_-}^{t} q^s(t,x)\,\mathrm{d}s \tag{4.20}$$

to control q_{m}. Here, for $s \geqslant t_-$, the function q^s satisfies

$$\begin{aligned} &\partial_t q^s = \partial_x^2 q^s - \frac{3}{2\lambda_*(t+1)}\partial_x q^s - V(t,x)q^s, \quad t > s, \\ &q^s(s,x) = [N - V(s,x)]\,p(s,x). \end{aligned} \tag{4.21}$$

We can control this for all $t_- \leqslant s \leqslant t$.

Lemma 4.5 *For all $s \geqslant t_-$ and $t \geqslant s$,*

$$\begin{aligned} &\int_{\mathbb{R}} \psi(x)q^s(t,x)\,\mathrm{d}x \leqslant C\frac{s^2}{a^2}\Lambda(s;a), \\ &\int_{\mathbb{R}} \psi'(x)q^s(t,x)\,\mathrm{d}x \leqslant C\frac{s^2}{a^2}\Lambda(s;a)(t-s+1)^{-1/2}, \\ &\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q^s(t,x)\,\mathrm{d}x \leqslant C\frac{s^2}{a^2}\Lambda(s;a)(t+1)^{-1/2}(t-s+1)^{-3/2}. \end{aligned} \tag{4.22}$$

Proof. We first get rid of the logarithmic term in (3.11) and (3.13). We claim that

$$p(s,x) = \Lambda(s;a)\mathrm{e}^{-ax/(2s)}\ \exp\left[-\frac{1}{4s}\left(x - \frac{3}{2\lambda_*}\ \log(s+1)\right)^2\right]$$
$$\leqslant C\Lambda(s;a)\mathrm{e}^{-ax/(2s)}\ \exp\left(-\frac{x^2}{8s}\right).$$

After all, an elementary study of the quadratic polynomial shows that for all $\alpha > 0$ and $\varepsilon \in (0,1)$, there there exists C depending on α and ε such that

$$\frac{[z + \alpha\ \log(t+1)]^2}{4t} \geqslant (1-\varepsilon)\frac{z^2}{4t} - C \quad \text{for all } z \in \mathbb{R} \text{ and } t > 0. \tag{4.23}$$

By lemma 3.1, we obtain

$$\Lambda(s;a)^{-1}q^s(s,x) = \Lambda(s;a)^{-1}[N - V(s,x)]p(s,x) \leqslant \begin{cases} C\ \exp\left[-\left(\frac{a}{2s} - \gamma_*\right)x\right] & \text{for } x < 0, \\ C\ \exp\left(-\frac{ax}{2s}\right)\exp\left(-\frac{x^2}{8s}\right) & \text{for } x \geqslant 0. \end{cases} \tag{4.24}$$

Since $s \geqslant t_-$, we have

$$\frac{a}{2s} - \gamma_* \leqslant \frac{1}{2\xi_-} - \gamma_* =: \kappa_- \in \left(0, \sqrt{N}\right). \tag{4.25}$$

Let us fix $\alpha \in \left(\kappa_-, \sqrt{N}\right)$. By lemma 3.1, there exists $K \geqslant 0$ such that

$$V(t,x) \geqslant \alpha^2 1_{(-\infty,-K)}(x). \tag{4.26}$$

We see from (4.24) and (4.26) that $q^s/[C\Lambda(s;a)]$ satisfies the hypotheses of lemma 3.3, with κ_- given in (4.25) and $\kappa_+ = a/(2s)$. The bounds in (4.22) follow from lemma 3.3 and $s \geqslant t_-$. □

We can now control q_{m} in terms of q^s.

Proof of lemma 4.4. In light of (4.20), we need to integrate (4.22) over $s \in [t_-, t]$ for $t \in [t_-, t_*]$. For $s \geqslant t_-$, the definition (3.12) of Λ implies

$$\Lambda(s;a) \leqslant \frac{C}{\sqrt{a}}(s+1)^{3a/(4\lambda_* s)}\mathrm{e}^{-a\theta(s/a)} \tag{4.27}$$

for the rate function

$$\theta(\xi) = N\xi + \frac{1}{4\xi} \quad \text{for } \xi > 0. \tag{4.28}$$

This strictly convex function is minimized at $\xi_* = 1/(2\sqrt{N})$, so there exists $c > 0$ such that

$$\theta(\xi) = N\xi + \frac{1}{4\xi} \geqslant \theta(\xi_*) + c(\xi - \xi_*)^2 = \sqrt{N} + c(\xi - \xi_*)^2$$

for all $\xi \in (0, 1]$. Therefore,

$$\Lambda(s;a) \leqslant \frac{C}{\sqrt{a}}(s+1)^{3a/(4\lambda_* s)} \mathrm{e}^{-\sqrt{N}a - \frac{c(t_*-s)^2}{a}} \tag{4.29}$$

for all $s \in [t_-, a]$.

We next handle the polynomial prefactor, which we write as

$$(s+1)^{3a/(4\lambda_* s)} = \exp\left[\frac{3a}{4\lambda_* s}\log(s+1)\right].$$

For any $\varepsilon > 0$, we employ the Peter–Paul inequality $2AB \leqslant \varepsilon A^2 + B^2/\varepsilon$:

$$a\left(\frac{1}{s} - \frac{1}{t_*}\right)\log(s+1) \leqslant \frac{C(t_*-s)\log a}{a} \leqslant \frac{\varepsilon(t_*-s)^2}{a} + \frac{C^2\log^2 a}{4\varepsilon a} \leqslant \frac{\varepsilon(t_*-s)^2}{a} + C_\varepsilon.$$

Exponentiating, this implies that

$$(s+1)^{3a/(4\lambda_* s)} \leqslant C_\varepsilon (s+1)^{3a/(4\lambda_* t_*)} \mathrm{e}^{\frac{\varepsilon(t_*-s)^2}{a}} = C_\varepsilon (s+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{\frac{\varepsilon(t_*-s)^2}{a}}. \tag{4.30}$$

Taking $\varepsilon \ll 1$, we can absorb the last factor into the Gaussian term in (4.29). Thus, (4.29) and (4.30) yield

$$\Lambda(s;a) \leqslant \frac{C}{\sqrt{a}}(s+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a - \frac{c(t_*-s)^2}{a}} \leqslant \frac{C}{\sqrt{a}}(s+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a - \frac{c(t_*-s)^2}{a}} \tag{4.31}$$

for all $s \in [t_-, a]$. We now combine (4.22) and (4.31) to control the contribution of q_{m}. At these times, we do not need the distinction between ψ and ψ' in (4.22):

$$\int_{\mathbb{R}} [\psi(x) + \psi'(x)] q_{\mathrm{m}}(t,x)\,\mathrm{d}x = \int_{t_-}^{t}\int_{\mathbb{R}} [\psi(x) + \psi'(x)] q^s(t,x)\,\mathrm{d}x\,\mathrm{d}s \leqslant C\int_{t_-}^{t} \Lambda(s;a)\,\mathrm{d}s.$$

Using (4.31) and changing variables via $\eta = \frac{t_*-s}{\sqrt{a}}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} [\psi(x) + \psi'(x)] q_{\mathrm{m}}(t,x)\,\mathrm{d}x &\leqslant C(t+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a} \int_{(t_*-t)/\sqrt{a}}^{\infty} \mathrm{e}^{-c\eta^2}\,\mathrm{d}\eta \\ &\leqslant C(t+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a} \mathrm{e}^{-\frac{c(t_*-t)^2}{a}}.\end{aligned}$$

This is (4.18). Finally, the third line in (4.22) and (4.31) imply:

$$\int_{\mathbb{R}} |E(t,x)|\,\psi(x) q_{\mathrm{m}}(t,x)\,\mathrm{d}x \leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a} \times \int_{t_-}^{t} (t-s+1)^{-3/2} \mathrm{e}^{-c(t_*-s)^2/a}\,\mathrm{d}s. \tag{4.32}$$

Since $(|y|+1)^{-3/2}$ is integrable, the integral on the right side of (4.32) is bounded by $C\mathrm{e}^{-(t_*-t)^2/a}$. Hence

$$\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x \leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\mathrm{e}^{-c(t_*-t)^2/a}.$$

We have thus verified (4.19) and completed the proof of lemma 4.4. □

4.3. The proof of lemma 4.1

We wish to understand $I(t)$ and $\dot{I}(t)$ given by (2.37) and (2.40), respectively. Recall that $r = p + q_{\mathrm{e}} + q_{\mathrm{m}}$. We use lemmas 4.2, 3.2, and 4.4 to control the terms with p, q_{e}, and q_{m}, respectively. These lemmas yield

$$I(t) = \int_{\mathbb{R}} \psi(x)r(t,x)\,\mathrm{d}x = A\gamma_*\mathrm{e}^{-\gamma_*\overline{x}_0}(t+1)^{3\sqrt{N}/(2\lambda_*)} \times \mathrm{e}^{-\sqrt{N}a}\left[1 + O\left(\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}\right)\right]$$

and

$$\frac{\mathrm{d}I}{\mathrm{d}t}(t) = \frac{3}{2\lambda_*(t+1)}\int_{\mathbb{R}} \psi'(x)r(t,x)\,\mathrm{d}x + \int_{\mathbb{R}} E(t,x)\psi(x)r(t,x)\,\mathrm{d}x$$
$$= \frac{3}{2\lambda_*(t+1)}\sqrt{N}A\gamma_*\mathrm{e}^{-\gamma_*\overline{x}_0}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\left[1 + O\left(\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}\right)\right].$$

Finally, the lower bound (3.4) and (4.6) imply that $\dot{I}(t)/I(t)$ does not become singular as $t \to t_*$. Therefore

$$\frac{\dot{I}(t)}{I(t)} = \frac{3\sqrt{N}}{2\lambda_*(t+1)}\left[1 + O\left(\mathrm{e}^{-\frac{c(t_*-t)^2}{a}}\right)\right]$$

when $t \in [0, t_*]$, and the proof of lemma 4.1 is complete. □

5. Middle times

Here, we consider the time interval $t \in [t_*, a]$. Now, the story changes: the main corrector $q_{\mathrm{m}}(t,x)$ becomes the dominant term in $I(t)$ and $\dot{I}(t)$, though, of course, the homogeneous term $p(t,x)$ is comparable to it when $t - t_* = O\left(\sqrt{a}\right)$. Again, lemma 3.2 shows that the contributions of q_{e} are negligible relative to those of q_{m} in lemma 5.3 below.

We will prove the following.

Lemma 5.1 *There exist $c > 0$ and $C > 0$ such that for all $t \in [t_*, a]$,*

$$\frac{|\dot{I}(t)|}{I(t)} \leqslant \frac{C}{a}\left[(t - t_* + 1)^{-1/2} + \mathrm{e}^{-c(t-t_*)^2/a}\right]. \tag{5.1}$$

After integration, lemma 5.1 implies that

$$\log\frac{I(a)}{I(t_*)} = O(a^{-1/2}). \tag{5.2}$$

5.1. The free contribution

We first bound the contributions of the free term p.

Lemma 5.2 *There exist* $c > 0$ *and* $C > 0$ *such that for all* $t \in [t_*, a]$,

$$\int_{\mathbb{R}} [\psi(x) + \psi'(x)] p(t,x)\,\mathrm{d}x \leqslant C(t+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a} \mathrm{e}^{-\frac{c(t-t_*)^2}{a}}$$

and

$$\int_{\mathbb{R}} |E(t,x)|\, p(t,x)\,\mathrm{d}x \leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a} \mathrm{e}^{-\frac{c(t-t_*)^2}{a}}.$$

Proof. Recall expression (3.11) for $p(t,x)$. We consider $(\psi + \psi')p$ separately on $\mathbb{R}_\pm$. When $x \leqslant 0$, we use $\psi(x) + \psi'(x) \leqslant C \exp(\sqrt{N}x)$ from (2.34) and (2.35), and obtain

$$[\psi(x) + \psi'(x)] p(t,x) \leqslant C\Lambda(t;a) \exp\left[\left(\sqrt{N} - \frac{a}{2t}\right) x\right] g(t,x) \quad \text{for } x \leqslant 0. \tag{5.3}$$

Now $t \geqslant t_*$, so $\sqrt{N} - a/(2t) \geqslant 0$. Also, (3.13) shows that $\int_{\mathbb{R}} g(t,x)\,\mathrm{d}x = \sqrt{4\pi t}$. It follows that

$$\int_{\mathbb{R}_-} [\psi(x) + \psi'(x)] p(t,x)\,\mathrm{d}x \leqslant C\Lambda(t;a) \int_{\mathbb{R}} g(t,x)\,\mathrm{d}x \leqslant C\sqrt{t}\Lambda(t;a) \quad \text{for all } t \geqslant t_*. \tag{5.4}$$

When $x \geqslant 0$, we use $\psi(x) + \psi'(x) \leqslant C(1+x)$ from (2.34) and (2.35), and $t \leqslant a$ to obtain

$$[\psi(x) + \psi'(x)] p(t,x) \leqslant C(1+x)\Lambda(t;a)\mathrm{e}^{-ax/(2t)} \leqslant C(1+x)\Lambda(t;a)\mathrm{e}^{-x/2} \quad \text{for } x \geqslant 0. \tag{5.5}$$

So

$$\int_{\mathbb{R}_+} [\psi(x) + \psi'(x)] p(t,x)\,\mathrm{d}x \leqslant C\Lambda(t;a). \tag{5.6}$$

We now turn to the error term with E. Recalling the bound (4.15) on E, (5.3) and (5.5) imply

$$\int_{\mathbb{R}} |E(t,x)| \psi(x) p(t,x)\,\mathrm{d}x \leqslant \frac{C\Lambda(t;a)}{\sqrt{t+1}} \int_{\mathbb{R}} \mathrm{e}^{-c|x|}\,\mathrm{d}x \leqslant \frac{C\Lambda(t;a)}{\sqrt{t}}. \tag{5.7}$$

The lemma follows from (5.4), (5.6) and (5.7) using (4.29) with $a/t \leqslant a/t_* = 2\sqrt{N}$. □

5.2. The main corrector contribution

We now estimate the contribution of $q_{\mathrm{m}}(t,x)$ on the interval $[t_*, a]$.

Lemma 5.3 *There exist* $c > 0$ *and* $C > 0$ *such that for all* $t \in [t_*, a]$,

$$\int \psi'(x)q_{\rm m}(t,x)\,\mathrm{d}x \leqslant C(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a} \times \left[(t-t_*+1)^{-1/2} + \mathrm{e}^{-c(t-t_*)^2/a}\right], \tag{5.8}$$

$$\int |E(t,x)|\,\psi(x)q_{\rm m}(t,x)\,\mathrm{d}x \leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a} \times \left[(t-t_*+1)^{-1/2} + \mathrm{e}^{-c(t-t_*)^2/a}\right]. \tag{5.9}$$

Moreover, for all $t \geqslant t_*$,

$$I(t) \geqslant \int_{\mathbb{R}} \psi(x)q_{\rm m}(t,x)\,\mathrm{d}x \geqslant ca^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}. \tag{5.10}$$

We emphasize that the last bound holds for all $t \geqslant t_*$, and thus extends to the late times after a as well.

Proof. We again represent $q_{\rm m}(t,x)$ via the Duhamel formula (4.20) as an integral of q^s satisfying (4.21). By lemma 4.5,

$$\begin{aligned} &\int_{\mathbb{R}} \psi'(x)q_{\rm m}(t,x)\,\mathrm{d}x \leqslant C\int_{t_-}^{t} \Lambda(s;a)(t-s+1)^{-1/2}\,\mathrm{d}s,\\ &\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q_{\rm m}(t,x)\,\mathrm{d}x \leqslant C\int_{t_-}^{t} \Lambda(s;a)a^{-1/2}(t-s+1)^{-3/2}\,\mathrm{d}s. \end{aligned} \tag{5.11}$$

By (4.31),

$$\Lambda(s;a) \leqslant \frac{C}{\sqrt{a}}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\mathrm{e}^{-c(s-t_*)^2/a} \tag{5.12}$$

for $t_- \leqslant s \leqslant t \leqslant a$. In light of (5.11), we must bound integrals of the form

$$Z_\alpha(t) := \int_{t_-}^{t} (t-s+1)^{-\alpha}\mathrm{e}^{-c(s-t_*)^2/a}\,\mathrm{d}s \leqslant \int_0^\infty (s+1)^{-\alpha}\mathrm{e}^{-c(t-t_*-s)^2/a}\,\mathrm{d}s \tag{5.13}$$

for $\alpha \in \{1/2, 3/2\}$.

We cut the integral in the right side of (5.13) at $s = (t-t_*)/2$:

$$\begin{aligned} \int_{\frac{t-t_*}{2}}^\infty (s+1)^{-\alpha}\mathrm{e}^{-c(t-t_*-s)^2/a}\,\mathrm{d}s &\leqslant \left(\frac{t-t_*}{2}+1\right)^{-\alpha}\int_{\frac{t-t_*}{2}}^\infty \mathrm{e}^{-c(t-t_*-s)^2/a}\,\mathrm{d}s\\ &\leqslant C\sqrt{a}(t-t_*+1)^{-\alpha} \end{aligned}$$

and

$$\begin{aligned} \int_0^{\frac{t-t_*}{2}} (s+1)^{-\alpha}\mathrm{e}^{-c(t-t_*-s)^2/a}\,\mathrm{d}s \leqslant \int_0^{\frac{t-t_*}{2}} \mathrm{e}^{-c(t-t_*-s)^2/a}\,\mathrm{d}s &\leqslant \int_{\frac{t-t_*}{2}}^\infty \mathrm{e}^{-cr^2/a}\,\mathrm{d}r\\ &\leqslant C\sqrt{a}\mathrm{e}^{-c(t-t_*)^2/(4a)}. \end{aligned}$$

We conclude that

$$Z_\alpha(t) \leqslant C\sqrt{a}\left[(t-t_*+1)^{-\alpha}+\mathrm{e}^{-c(t-t_*)^2/a}\right]. \tag{5.14}$$

When $\alpha > 1$, the integrability of $(s+1)^{-\alpha}$ gives an alternative bound. Still cutting at $s = (t - t_*)/2$, we now write

$$\int_{\frac{t-t_*}{2}}^{\infty}(s+1)^{-\alpha}\mathrm{e}^{-c(t-t_*-s)^2/a}\,\mathrm{d}s \leqslant \int_{\frac{t-t_*}{2}}^{\infty}(s+1)^{-\alpha}\,\mathrm{d}s \leqslant C(t-t_*+1)^{1-\alpha}$$

and

$$\begin{aligned}\int_0^{\frac{t-t_*}{2}}(s+1)^{-\alpha}\mathrm{e}^{-c(t-t_*-s)^2/a}\,\mathrm{d}s &\leqslant \mathrm{e}^{-c(t-t_*)^2/(4a)}\int_0^{\infty}(s+1)^{-\alpha}\,\mathrm{d}s\\ &\leqslant C\mathrm{e}^{-c(t-t_*)^2/(4a)}.\end{aligned}$$

We conclude that

$$Z_\alpha(t) \leqslant C\left[(t-t_*+1)^{1-\alpha}+\mathrm{e}^{-c(t-t_*)^2/a}\right] \quad \text{if } \alpha > 1. \tag{5.15}$$

Combining (5.11), (5.12), and (5.14) with $\alpha = 1/2$ gives

$$\begin{aligned}\int_{\mathbb{R}}\psi'(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x &\leqslant \frac{C}{\sqrt{a}}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}Z_{1/2}(t)\\ &\leqslant C(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\left[(t-t_*+1)^{-1/2}+\mathrm{e}^{-c(t-t_*)^2/a}\right],\end{aligned}$$

while (5.11), (5.12), and (5.15) with $\alpha = 3/2$ imply

$$\begin{aligned}\int_{\mathbb{R}}|E(t,x)|\,\psi(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x &\leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}Z_{3/2}(t)\\ &\leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\times\left[(t-t_*+1)^{-1/2}+\mathrm{e}^{-c(t-t_*)^2/a}\right].\end{aligned}$$

We have thus confirmed (5.8) and (5.9).

Finally, we need a lower bound on $I(t)$. With $p \geqslant 0$ and $q_{\mathrm{e}} \geqslant 0$, we can write for $t \geqslant t_*$

$$I(t) \geqslant \int_{\mathbb{R}}q_{\mathrm{m}}(t,x)\psi(x)\,\mathrm{d}x = \int_{t_-}^{t}\int_{\mathbb{R}}q^s(t,x)\psi(x)\,\mathrm{d}x\,\mathrm{d}s \geqslant \int_{t_-}^{t_*}\int_{\mathbb{R}_+}q^s(t,x)\psi(x)\,\mathrm{d}x\,\mathrm{d}s, \tag{5.16}$$

where we used again the Duhamel formula (4.20) and $q^s \geqslant 0$. We recall that q^s satisfies (4.21). We need a lower bound on q^s, so we look for a subsolution to (4.21) on $\mathbb{R}_+$.

We first focus on the initial condition $q^s(s,x) = [N - V(s,x)]p(s,x)$ for $x \in \mathbb{R}_+$. Recall the definition (2.23) of V. The comparison principle implies that H is decreasing in x, so

$$H(s, x+m(s)) \leqslant H(s, m(s)) \quad \text{for all } x \geqslant 0. \tag{5.17}$$

By (1.8) and $s \geqslant t_- = \xi\_a$, $H(s, x+m(s))$ is very close to $U_0(0) < 1$ provided a is sufficiently large. We can thus assume that $H(s, m(s)) \leqslant 1 - c$ for all $s \geqslant t_-$. Using (1.18) and (5.17), it follows that

$$N - V(s,x) = N - F'\left(H(t, x + m(s))\right) \geqslant N - \sup_{u\in[0,1-c]} F'(u) > 0$$

in this region. That is,

$$q^s(s,x) \geqslant cp(s,x) \quad \text{for all } s \geqslant t_-,\ x \geqslant 0. \tag{5.18}$$

Going back to (3.11), we see that for all $x \geqslant 0$ and $s \geqslant t_-$, we have

$$p(s,x) \geqslant \Lambda(s;a)\mathrm{e}^{-Cx} \exp\left\{-\frac{1}{4s}\left[x - \frac{3}{2\lambda_*}\log(s+1)\right]^2\right\}. \tag{5.19}$$

We are free to assume $s \geqslant t_- \geqslant 2$, so a variation on (4.23) yields

$$\frac{1}{4s}\left[x - \frac{3}{2\lambda_*}\log(s+1)\right]^2 \leqslant \frac{x^2}{8(1-\varepsilon)} + C(\varepsilon) \quad \text{for all } s \geqslant t_-,\ x \in \mathbb{R}$$

and $\varepsilon \in (0,1)$. Hence (5.19) and $\mathrm{e}^{-Cx} \geqslant C(\varepsilon)\mathrm{e}^{-\varepsilon x^2}$ implies

$$p(s,x) \geqslant c\Lambda(s;a)\mathrm{e}^{-Cx}\mathrm{e}^{-x^2/7} \geqslant c\Lambda(s;a)\mathrm{e}^{-x^2/6} \geqslant c\Lambda(s;a)x\mathrm{e}^{-x^2/4}, \tag{5.20}$$

where we have allowed $c > 0$ to change from expression to expression. We now define

$$\varphi(\lambda,x) = \frac{x}{\lambda^{3/2}}\exp\left(-\frac{x^2}{4\lambda}\right)$$

and note for later reference that

$$\int_{\mathbb{R}_+} x\varphi(\lambda,x)\,\mathrm{d}x = 2\sqrt{\pi} \quad \text{for all } \lambda > 0. \tag{5.21}$$

Combining (5.18) and (5.20), we can write

$$q^s(s,x) \geqslant c\Lambda(s;a)\varphi(1,x) \quad \text{for all } s \geqslant t_-,\ x \geqslant 0. \tag{5.22}$$

Now, we consider the PDE in (4.21). Since we are looking for a lower bound on $q^s(t,x)$, we cannot neglect the negative term $-V(t,x)q^s$ in the right side of (4.21). By lemma 3.1,

$$V(t,x) \leqslant B\mathrm{e}^{-cx} \tag{5.23}$$

for all $t > 0$ and $x > 0$. To obtain a subsolution, we are free to impose a Dirichlet condition at $x = 0$. We let $\underline{v}^s(t,x)$ solve

$$\begin{aligned}
&\partial_t \underline{v}^s = \partial_x^2 \underline{v}^s - \frac{3}{2\lambda_*(t+1)}\partial_x \underline{v}^s - B\mathrm{e}^{-cx}\underline{v}^s \quad \text{for } t > s \text{ and } x > 0,\\
&\underline{v}^s(t,0) = 0 \quad \text{for } t > s,\\
&\underline{v}^s(s,x) = \varphi(1,x) \quad \text{for } x > 0.
\end{aligned}$$

Applying the comparison principle to (4.21), (5.22) and (5.23) yield

$$q^s(t,x) \geqslant c\Lambda(s;a)\underline{v}^s(t,x) \quad \text{for all } t \geqslant s \geqslant t_-,\ x \geqslant 0.$$

Then, with (5.16),

$$I(t) \geqslant c \int_{t_-}^{t_*} \Lambda(s;a) \int_{\mathbb{R}_+} \underline{v}^s(t,x)\psi(x)\, \mathrm{d}x\, \mathrm{d}s. \tag{5.24}$$

The following lemma gives a lower bound on $\underline{v}^s(t,x)$.

Lemma 5.4 *There exists $c > 0$ such that for all $t \geqslant s$ and $x > 0$ we have*

$$\underline{v}^s(t,x) \geqslant c\,\varphi(t-s+1,x) + \underline{R}(t,s,x) \tag{5.25}$$

with

$$\begin{aligned} |\underline{R}(t,s,x)| &\leqslant C(t-s+1)^{-2} x \mathrm{e}^{-x^2/[8(t-s+1)]} \\ &= C(t-s+1)^{-1/2}\varphi\,(2(t-s+1),x)\,. \end{aligned} \tag{5.26}$$

Proof. This is nearly lemma 2.2 in [10]. The only difference is the exponentially decaying potential, which is negligible on the scale $x \sim \sqrt{t}$, where the analysis really happens. □

Using (2.34), we have $\psi(x) \geqslant cx$ for $x \geqslant 0$. Then (5.25), (5.26), and (5.21) imply

$$\begin{aligned} \int_{\mathbb{R}_+} \underline{v}^s(t,x)\psi(x)\,\mathrm{d}x &\geqslant c\int_{\mathbb{R}_+} x\varphi(t-s+1,x)\,\mathrm{d}x - C(t-s+1)^{-1/2} \\ &\quad \times \int_{\mathbb{R}} x\varphi\,(2(t-s+1),x)\ \mathrm{d}x \\ &\geqslant c - C(t-s+1)^{-1/2}, \end{aligned}$$

where we have allowed c to change from expression to expression. So $\int_{\mathbb{R}_+}\underline{v}^s\psi$ is uniformly positive once $t-s \geqslant C'$ for C' large. On the other hand, $\underline{v}^s$ is positive, so the integral is positive on the time interval $[s, s+C']$. By compactness, it follows that

$$\int_{\mathbb{R}_+} \underline{v}^s(t,x)\psi(x)\,\mathrm{d}x \geqslant c \quad \text{for all } t \geqslant s \geqslant t_-.$$

Now (5.24) yields

$$I(t) \geqslant c\int_{t_-}^{t_*} \Lambda(s;a)\,\mathrm{d}s \tag{5.27}$$

for all $t \geqslant t_*$. For $s \in [t_-, t_*]$, (3.12) implies

$$\Lambda(s;a) \geqslant \frac{c}{\sqrt{a}} a^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-a\theta(s/a)} \tag{5.28}$$

for $\theta(\xi) = N\xi + 1/(4\xi)$. We recall that θ is strictly convex and attains its minimum of $\sqrt{N}$ at $\xi_* = 1/(2\sqrt{N}) = t_*/a$. Also, $t_- = \xi_- a$. Since θ is smooth on the interval $[\xi_-, \xi_*]$, there exists $C > 0$ such that

$$\theta(\xi) \leqslant \sqrt{N} + C(\xi - \xi_*)^2 \quad \text{for all } \xi \in [\xi_-, \xi_*].$$

Then (5.28) yields

$$\Lambda(s;a) \geqslant \frac{c}{\sqrt{a}} a^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a} \mathrm{e}^{-C(t_*-s)^2/a}$$

and (5.27) implies

$$I(t) \geqslant ca^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\int_{t_-}^{t_*}\mathrm{e}^{-C(t_*-s)^2/a}\frac{\mathrm{d}s}{\sqrt{a}} \geqslant ca^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}$$

for all $t \geqslant t_*$. This completes the proof of lemma 5.3. □

5.3. The proof of lemma 5.1

Gathering together lemmas 3.2, 5.2, and 5.3, we obtain

$$\begin{aligned}|\dot{I}(t)| &\leqslant \frac{3}{2(t+1)}\int_{\mathbb{R}}\psi'(x)r(t,x)\,\mathrm{d}x + \int_{\mathbb{R}}|E(t,x)|\,\psi(x)r(t,x)\,\mathrm{d}x \\ &\leqslant \frac{C}{a}(t+1)^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\left[(t-t_*+1)^{-1/2}+\mathrm{e}^{-c(t-t_*)^2/a}\right].\end{aligned}$$

Taking into account (5.10), we see that (5.1) follows. □

6. The late times

We finish with the times $t \geqslant a$. In this regime, p and q_{e} should be exponentially negligible. However, since this time period is unbounded, we must take care to ensure that $\dot{I}(t)/I(t)$ is integrable in time, and, in fact, small. We will prove the following lemma.

Lemma 6.1 *There exists $C > 0$ such that for all $t \geqslant a$ we have*

$$\frac{|\dot{I}(t)|}{I(t)} \leqslant \frac{C}{t^{3/2}}. \tag{6.1}$$

After integration, lemma 6.1 implies that

$$\log\frac{I(\infty)}{I(a)} = O(a^{-1/2}). \tag{6.2}$$

Note that the lower bound (5.10) still holds, so it suffices to show that

$$|\dot{I}(t)| \leqslant \frac{C}{t^{3/2}}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a} \quad \text{for } t \geqslant a.$$

6.1. The free contribution

It is simple to control the free contribution at late times, since it has decayed into irrelevance.

Lemma 6.2 *There exist $c > 0$ and $C > 0$ such that for all $t \geqslant a$,*

$$\int_{\mathbb{R}}\left[\psi(x)+\psi'(x)+|E(t,x)|\,\psi(x)\right]p(t,x)\,\mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a}\mathrm{e}^{-ct}. \tag{6.3}$$

Proof. For $x \leqslant 0$ we recall that E is bounded and we simply use (4.27) and (5.4), which give

$$\int_{\mathbb{R}_-}\left[\psi(x)+\psi'(x)+|E(t,x)|\,\psi(x)\right]p(t,x)\,\mathrm{d}x \leqslant C\sqrt{t}\Lambda(t;a) \leqslant Ct^C\mathrm{e}^{-a\theta(t/a)} \tag{6.4}$$

for θ defined in (4.28). Now, $\min\theta = \sqrt{N}$ is uniquely attained at $\xi_* = 1/(2\sqrt{N}) < 1$, and $\theta(\xi) \geqslant N\xi$. If $c > 0$ is sufficiently small, it follows that

$$\theta(\xi) \geqslant \sqrt{N} + c + c\xi \quad \text{for all } \xi \geqslant 1. \tag{6.5}$$

Together with (6.4), this implies

$$\int_{\mathbb{R}_-} \left[\psi(x) + \psi'(x) + |E(t,x)|\,\psi(x)\right] p(t,x)\mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a}\mathrm{e}^{-ct}. \tag{6.6}$$

For $x \geqslant 0$, (2.34) and (2.35) imply $\psi(x) \leqslant C(1+x)$ and $\psi'(x) \leqslant C$. Then (3.11) and (4.27) imply

$$\int_{\mathbb{R}_+} \left[\psi(x) + \psi'(x) + |E(t,x)|\,\psi(x)\right] p(t,x)\,\mathrm{d}x$$
$$\leqslant C\Lambda(t;a)\int_0^\infty (1+x)\mathrm{e}^{-ax/(2t)}\,\mathrm{d}x \leqslant Ct^C\mathrm{e}^{-a\theta(t/a)}.$$

Again, (6.5) yields

$$\int_{\mathbb{R}_+} \left[\psi(x) + \psi'(x) + |E(t,x)|\,\psi(x)\right] p(t,x)\mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a}\mathrm{e}^{-ct}.$$

Combining this with (6.6), we obtain (6.3). □

6.2. The main corrector contribution

Next, we control q_{m} at late times.

Lemma 6.3 *There exist $c > 0$ and $C > 0$ such that for all $t \geqslant a$,*

$$\int_{\mathbb{R}} \psi'(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x \leqslant \frac{C}{\sqrt{t}}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a} \tag{6.7}$$

and

$$\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x \leqslant \frac{C}{t^2}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}. \tag{6.8}$$

Proof. Recall that q_{m} has a Duhamel representation (4.20) in terms of q^s, the solution to (4.21). To control the contributions of q_{m}, we integrate the bounds in lemma 4.5 over $s \in [t_-, t]$. We rely on two different estimates for Λ. When $s \in [t_-, a]$, (4.31) implies:

$$\frac{s^2}{a^2}\Lambda(s;a) \leqslant \frac{C}{\sqrt{a}}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\mathrm{e}^{-\frac{c(s-t_*)^2}{a}} \quad \text{for } s \in [t_-, a]. \tag{6.9}$$

This estimate cannot hold for $s \geqslant a$, since θ is not *uniformly* convex. Nonetheless, (4.27) and (6.5) yield:

$$\frac{s^2}{a^2}\Lambda(s;a) \leqslant Cs^C\mathrm{e}^{-a\theta(s/a)} \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a}\mathrm{e}^{-cs} \quad \text{for } s \in [a, \infty). \tag{6.10}$$

We first use lemma 4.5 and (6.9) to control the contributions of q^s when $s \in [t_-, a]$. Recalling the definition (5.13) of $Z_\alpha(t)$ and noticing from (5.14) that $Z_{1/2}(t) \leqslant C\sqrt{a/t}$ for $t \geqslant a$, we find

$$\int_{t_-}^{a}\int_{\mathbb{R}} \psi'(x)q^s(t,x)\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{C}{\sqrt{a}}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\int_{t_-}^{a}(t-s+1)^{-1/2}\mathrm{e}^{-c(s-t_*)^2/a}\,\mathrm{d}s$$
$$\leqslant \frac{C}{\sqrt{t}}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}. \tag{6.11}$$

Similarly, noticing from (5.14) that $Z_{3/2}(t) \leqslant C\sqrt{a}/t^{3/2}$, we obtain

$$\int_{t_-}^{a}\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q^s(t,x)\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{C}{\sqrt{at}}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}\int_{t_-}^{a}(t-s+1)^{-3/2}\mathrm{e}^{-c(s-t_*)^2/a}\,\mathrm{d}s$$
$$\leqslant \frac{C}{t^2}a^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a}. \tag{6.12}$$

Next, we control the contributions of q^s when $s \geqslant a$. To do so, we rely on the following bound: for each $\alpha \geqslant 0$ and $t > 0$,

$$\int_0^t \mathrm{e}^{-cs}(t-s+1)^{-\alpha}\,\mathrm{d}s \leqslant \int_0^{t/2}\mathrm{e}^{-cs}\left(\frac{t}{2}+1\right)^{-\alpha}\mathrm{d}s + \int_{t/2}^{t}\mathrm{e}^{-cs}\,\mathrm{d}s \leqslant C_\alpha(t+1)^{-\alpha}. \tag{6.13}$$

Thus, lemma 4.5 and (6.10) yield

$$\int_a^t\int_{\mathbb{R}} \psi'(x)q^s(t,x)\,\mathrm{d}x\,\mathrm{d}s \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a}\int_a^t (t-s+1)^{-1/2}\mathrm{e}^{-cs}\,\mathrm{d}s \leqslant \frac{C}{\sqrt{t}}\mathrm{e}^{-(\sqrt{N}+c)a} \tag{6.14}$$

and

$$\int_a^t\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q^s(t,x)\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{C}{\sqrt{t}}\mathrm{e}^{-(\sqrt{N}+c)a}\int_a^t (t-s+1)^{-3/2}\mathrm{e}^{-cs}\,\mathrm{d}s \leqslant \frac{C}{t^2}\mathrm{e}^{-(\sqrt{N}+c)a}. \tag{6.15}$$

We recall that

$$\int_{\mathbb{R}} \psi'(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x = \int_{t_-}^{t}\int_{\mathbb{R}} \psi'(x)q^s(t,x)\,\mathrm{d}x\,\mathrm{d}s,$$
$$\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q_{\mathrm{m}}(t,x)\,\mathrm{d}x = \int_{t_-}^{t}\int_{\mathbb{R}} |E(t,x)|\,\psi(x)q^s(t,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Therefore (6.11) and (6.14) imply (6.7), while (6.12) and (6.15) imply (6.8). □

6.3. The proof of lemma 6.1

First, we recall (5.10):

$$I(t) \geqslant ca^{3\sqrt{N}/(2\lambda_*)}\mathrm{e}^{-\sqrt{N}a} \quad \text{for } t \geqslant t_*. \tag{6.16}$$

Next, collecting lemmas 3.2, 6.2, and 6.3, we see that (2.40) gives

$$
\begin{aligned}
|\dot{I}(t)| &\leqslant \frac{3}{2\lambda_*(t+1)}\int_{\mathbb{R}} \psi'(x) r(t,x)\,\mathrm{d}x + \int_{\mathbb{R}} |E(t,x)|\, r(t,x)\psi(x)\,\mathrm{d}x \\
&\leqslant \frac{C}{t^{3/2}} a^{3\sqrt{N}/(2\lambda_*)} \mathrm{e}^{-\sqrt{N}a}.
\end{aligned} \tag{6.17}
$$

We obtain (6.1) from (6.16) and (6.17). □

6.4. Proof of the main result

As we have mentioned, proposition 2.4 is an immediate consequence of lemmas 4.1, 5.1, and 6.1. Combining (4.2), (5.2) and (6.2), we find

$$
\log \frac{I(\infty)}{I(0)} = \frac{3\sqrt{N}}{2\lambda_*} \log \frac{a}{2\sqrt{N}} + O\left(a^{-1/2}\right).
$$

Exponentiating, we obtain proposition 2.4. □

7. The early corrector—proof of lemma 3.2

Given $\varepsilon_0 \in (0,1)$, to be chosen later, lemma 3.1 ensures the existence of $K \geqslant 0$ such that

$$
N(1-\varepsilon_0) \leqslant V(t,x) \quad \text{for all } t > 0 \quad \text{and } x < -K.
$$

The same lemma implies that

$$
N - V(t,x) \leqslant B\mathrm{e}^{\gamma_* x}.
$$

Hence if $\tilde{q}$ solves

$$
\partial_t \tilde{q} = \partial_x^2 \tilde{q} - \frac{3}{2\lambda_*(t+1)}\partial_x \tilde{q} - N(1-\varepsilon_0)1_{(-\infty,-K)}(x)\tilde{q} + B\mathrm{e}^{\gamma_* x} 1_{[0,t_-]}(t)p, \quad \tilde{q}(0,x) = 0,
$$

comparison with (3.14) implies

$$
q_{\mathrm{e}}(t,x) \leqslant \tilde{q}(t,x).
$$

To estimate $\tilde{q}$, we decompose it into two parts, ρ and σ

$$
q_{\mathrm{e}}(t,x) \leqslant \tilde{q}(t,x) = \rho(t,x) + \sigma(t,x),
$$

where ρ is the solution to

$$
\partial_t \rho = \partial_x^2 \rho - \frac{3}{2\lambda_*(t+1)}\partial_x \rho - N(1-\varepsilon_0)\rho + B\mathrm{e}^{\gamma_* x} 1_{[0,t_-]}(t)p, \quad \rho(0,x) = 0, \tag{7.1}
$$

and σ the solution to

$$
\partial_t \sigma = \partial_x^2 \sigma - \frac{3}{2\lambda_*(t+1)}\partial_x \sigma - N(1-\varepsilon_0)1_{(-\infty,-K)}(x)\sigma + N(1-\varepsilon_0)1_{[-K,\infty)}(x)\rho, \ \sigma(0,x) = 0. \tag{7.2}
$$

We will be able to estimate ρ more or less explicitly. For σ, we emphasize that the forcing in the right side of (7.2) is supported on $[-K, \infty)$. That is, unlike r and q_{e}, σ is never forced deep in $\mathbb{R}_-$. There is thus no need to estimate it with a further corrector—we can control it with lemma 3.3.

Lemma 3.2 is a consequence of the two following lemmas:

Lemma 7.1 *There exist $C, c > 0$ independent of a and t such that for all $a \geqslant 1$ and $t \geqslant 0$:*

$$\int_{\mathbb{R}} \psi(x)\rho(t,x)\,\mathrm{d}x + \int_{\mathbb{R}} \psi'(x)\rho(t,x)\,\mathrm{d}x + \int_{\mathbb{R}} \psi(x)\,|E(t,x)|\,\rho(t,x)\,\mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a-ct}. \tag{7.3}$$

Lemma 7.2 *There exist $C, c > 0$ independent of a and t such that for all $a \geqslant 1$ and $t \geqslant 0$:*

$$\int_{\mathbb{R}} \psi(x)\sigma(t,x)\,\mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a},$$
$$\int_{\mathbb{R}} \psi'(x)\sigma(t,x)\,\mathrm{d}x \leqslant \frac{C}{\sqrt{t+1}}\mathrm{e}^{-(\sqrt{N}+c)a},$$
$$\int_{\mathbb{R}} \psi(x)\,|E(t,x)|\,\sigma(t,x)\,\mathrm{d}x \leqslant \frac{C}{(t+1)^2}\mathrm{e}^{-(\sqrt{N}+c)a}.$$

7.1. Proof of lemma 7.1

We use the Duhamel formula to write the solution to (7.1) as

$$\rho(t,x) = \int_0^{t\wedge t_-} \rho^s(t,x)\,\mathrm{d}s. \tag{7.4}$$

Here, $\rho^s(t,x)$ is the solution to

$$\partial_t\rho^s = \partial_x^2\rho^s - \frac{3}{2\lambda_*(t+1)}\partial_x\rho^s - N(1-\varepsilon_0)\rho^s \quad \text{for } t > s, \qquad \rho^s(s,x) = B\mathrm{e}^{\gamma_* x}p(s,x). \tag{7.5}$$

Recall the expression (3.2) of $p(t,x)$ and define

$$\mu(t) = -a + \frac{3}{2\lambda_*}\,\log(t+1). \tag{7.6}$$

Then we can write the initial condition $\rho^s(s,x)$ as a Gaussian:

$$\rho^s(s,x) = B\mathrm{e}^{\gamma_* x}p(s,x) = B\mathrm{e}^{\gamma_* x}\frac{1}{\sqrt{4\pi s}}\,\exp\left\{-Ns - \frac{1}{4s}[x-\mu(s)]^2\right\}$$
$$= \frac{B}{\sqrt{4\pi s}}\mathrm{e}^{\gamma_*(\mu(s)+\gamma_* s)}\,\exp\left\{-Ns - \frac{1}{4s}[x-\mu(s)-2\gamma_* s]^2\right\}.$$

One can then verify that

$$\rho^s(t,x) = \frac{B}{\sqrt{4\pi t}} \mathrm{e}^{-N\varepsilon_0 s} \mathrm{e}^{-N(1-\varepsilon_0)t} \mathrm{e}^{\gamma_*(\mu(s)+\gamma_* s)} \exp\left[-\frac{(x-\mu(t)-2\gamma_* s)^2}{4t}\right] \tag{7.7}$$

satisfies both the initial condition and (7.5). The Duhamel formula (7.4) implies

$$\int_{\mathbb{R}} \psi(x)\rho(t,x)\,\mathrm{d}x = \int_0^{t\wedge t_-} \int_{\mathbb{R}} \psi(x)\rho^s(t,x)\,\mathrm{d}x\,\mathrm{d}s \tag{7.8}$$

for all $t \geqslant 0$.

To bound the first term in (7.3), it suffices to show that for all $t \geqslant 0$ and $s \leqslant t \wedge t_-$,

$$\int_{\mathbb{R}} \psi(x)\rho^s(t,x)\,\mathrm{d}x \leqslant C(1+t)^C \mathrm{e}^{-(\sqrt{N}+c)a-ct} \tag{7.9}$$

for some $c > 0, C > 0$. Then, the integral over $s \leqslant t_- \leqslant a$ in (7.8) gives at most a factor a which can be absorbed, together with the $(1+t)^C$ factor, into in the exponential decay by making c smaller.

We now show (7.9). It follows from (2.34) that $\psi(x) \leqslant C\mathrm{e}^{\sqrt{N}x}$ for some C; then, in (7.7), we obtain

$$\psi(x)\rho^s(t,x) \leqslant \mathrm{e}^{\sqrt{N}x}\rho^s(t,x) \leqslant \frac{C}{\sqrt{t}} \mathrm{e}^{-N(1-\varepsilon_0)t} \mathrm{e}^{\gamma_*(\mu(s)+\gamma_* s)} \mathrm{e}^{\sqrt{N}x} \exp\left[-\frac{(x-\mu(t)-2\gamma_* s)^2}{4t}\right]. \tag{7.10}$$

Now,

$$\mathrm{e}^{\sqrt{N}x} \exp\left[-\frac{(x-\mu(t)-2\gamma_* s)^2}{4t}\right] = \mathrm{e}^{\sqrt{N}(\mu(t)+2\gamma_* s+\sqrt{N}t)} \exp\left[-\frac{(x-\mu(t)-2\gamma_* s-2\sqrt{N}t)^2}{4t}\right]. \tag{7.11}$$

Hence (7.10) yields

$$\begin{aligned} \int_{\mathbb{R}} \psi(x)\rho^s(t,x)\,\mathrm{d}x &\leqslant C \exp\left\{-N(1-\varepsilon_0)t + \gamma_*(\mu(s)+\gamma_* s) + \sqrt{N}\left(\mu(t)+2\gamma_* s+\sqrt{N}t\right)\right\} \\ &= C \exp\left\{\sqrt{N}\mu(t) + N\varepsilon_0 t + \gamma_*\mu(s) + \gamma_*\left(\gamma_*+2\sqrt{N}\right)s\right\} \\ &\leqslant C(1+t)^C \exp\left\{N\varepsilon_0 t - (\sqrt{N}+\gamma_*)a + \gamma_*\left(\gamma_*+2\sqrt{N}\right)s\right\}. \end{aligned} \tag{7.12}$$

(In the last line, we replaced $\mu(s)$ and $\mu(t)$ by their expression (7.6), and then used $s \leqslant t$ for the logarithmic terms.) Pick $c > 0$, and then ε_0 such that $N\varepsilon_0 \leqslant c$. We see that (7.9) holds if

$$ct - \gamma_* a + \gamma_*\left(\gamma_*+2\sqrt{N}\right)s \leqslant -ca - ct. \tag{7.13}$$

We now consider, until the end of this proof, the case where (7.13) does not hold, i.e. the case where

$$\gamma_*\left(\gamma_*+2\sqrt{N}\right)s > (\gamma_*-c)a - 2ct. \tag{7.14}$$

In that case, the bound $\psi(c) \leqslant C\mathrm{e}^{\sqrt{N}x}$ that we used to derive (7.12) is not good enough for $x > 0$. We now show that if $c > 0$ is small enough, the centering term in (7.11) satisfies

$$\mu(t) + 2\gamma_* s + 2\sqrt{N}t > ca. \tag{7.15}$$

Indeed, recall that $\mu(t) > -a$ and $\sqrt{N} > 1$. If $t > a$, then (7.15) is obvious. If $t < a$, then (7.14) and $t > s$ imply that

$$\mu(t) + 2\gamma_* s + 2\sqrt{N}t \geqslant -a + 2(\gamma_* + \sqrt{N})s \geqslant -a + \left[\frac{2(\gamma_* + \sqrt{N})}{\gamma_* + 2\sqrt{N}}\right] \frac{\gamma_* - 3c}{\gamma_*} a.$$

As the factor in square brackets is strictly larger than 1, it is possible to choose $c > 0$ small enough independent of a such that (7.15) holds. Thus when $x < 0$, (7.14) implies

$$\exp\left[-\frac{(x - \mu(t) - 2\gamma_* s - 2\sqrt{N}t)^2}{4t}\right] \leqslant \exp\left[-\frac{(\mu(t) + 2\gamma_* s + 2\sqrt{N}t)^2}{4t} + \frac{cax}{2t}\right].$$

Recalling (7.7), (7.10), and (7.11), we obtain

$$e^{\sqrt{N}x} \rho^s(t, x) \leqslant C\rho^s(t, 0) \exp\left(\frac{cax}{2t}\right) \quad \text{for } x < 0.$$

Therefore,

$$\int_{-\infty}^{0} \psi(x)\rho^s(t, x)\, dx \leqslant C\frac{t}{a}\rho^s(t, 0). \tag{7.16}$$

We now evaluate the integral over $x > 0$, still assuming (7.14). In this region, we use the bound $\psi(x) \leqslant C(1 + x)$ from (2.34). In the expression (7.7) of $\rho^s(t, x)$, we start by getting rid of the logarithmic terms that appear in $\mu(t)$ and $\mu(s)$. We employ (4.23) with some ε to be chosen later. Recalling that $N\varepsilon_0 \leqslant c$, (7.7) implies:

$$\begin{aligned} \rho^s(t, x) \leqslant \tilde{\rho}^s(t, x) \quad \text{with} \quad \tilde{\rho}^s(t, x) \\ = \frac{C}{\sqrt{t}} \exp\left[2ct - Nt - \gamma_*(a - \gamma_* s) - (1 - \varepsilon)\frac{(x + a - 2\gamma_* s)^2}{4t}\right]. \end{aligned} \tag{7.17}$$

The term $2ct$ has two contributions. On the one hand, $N\varepsilon_0 t \leqslant ct$. On the other, the logarithmic term in $\mu(s)$ is sublinear: $3\gamma_*/(2\lambda_*)\log(1 + s) \leqslant ct + C$.

Since $s \leqslant t_- \leqslant t_* = a/(2\sqrt{N})$ and $\gamma_* \leqslant \sqrt{N}$, we have $a - 2\gamma_* s \geqslant 0$ and

$$\tilde{\rho}^s(t, x) \leqslant \tilde{\rho}^s(t, 0) \exp\left[-(1 - \varepsilon)\frac{x^2}{4t}\right] \quad \text{for } x > 0.$$

Then

$$\begin{aligned} \int_0^\infty \psi(x)\rho^s(t, x)\, dx &\leqslant C\tilde{\rho}^s(t, 0) \int_0^\infty (1 + x) \exp\left[-(1 - \varepsilon)\frac{x^2}{4t}\right] dx \\ &\leqslant C(\sqrt{t} + t)\tilde{\rho}^s(t, 0). \end{aligned} \tag{7.18}$$

Combining (7.16) and (7.18), we have shown that, if (7.14) holds, we have

$$\int_{\mathbb{R}} \psi(x)\rho^s(t, x)\, dx \leqslant C(\sqrt{t} + t)\tilde{\rho}^s(t, 0).$$

We now evaluate $\tilde{\rho}^s(t,0)$ and show that if ε is chosen small enough, then there exists $C>0$ and $c>0$ such that, for all $t>0$ and all $s\leqslant t\wedge t_-$,

$$\tilde{\rho}^s(t,0)\leqslant\frac{C}{\sqrt{t}}\exp\left[-(\sqrt{N}+c)a-ct\right].\tag{7.19}$$

This will imply (7.9), and then lemma 7.1 for the term with ψ. We have from (7.17)

$$\begin{aligned}\tilde{\rho}^s(t,0)&=\frac{C}{\sqrt{t}}\exp\left[2ct-Nt-\gamma_*(a-\gamma_*s)-(1-\varepsilon)\frac{(a-2\gamma_*s)^2}{4t}\right]\\&=\frac{C}{\sqrt{t}}\exp\left[2ct-Nt-\gamma_*(a-\gamma_*s)-(1-\varepsilon)\left(\frac{a^2}{4t}-\frac{\gamma_*s}{t}(a-\gamma_*s)\right)\right]\\&\leqslant\frac{C}{\sqrt{t}}\exp\left[2ct-(1-\varepsilon)\left(N\frac{t}{a}+\frac{a}{4t}\right)a-\gamma_*(a-\gamma_*s)\left(1-(1-\varepsilon)\frac{s}{t}\right)\right]\\&\leqslant\frac{C}{\sqrt{t}}\exp\left[2ct-(1-\varepsilon)\theta\left(\frac{t}{a}\right)a-\gamma_*(a-\gamma_*s)\left(1-\frac{s}{t}\right)\right]\end{aligned}\tag{7.20}$$

with $\theta(\xi):=N\xi+1/(4\xi)$. Here we have used $a-\gamma_*s>0$. Notice that $\theta(\xi)$ reaches its minimum at $\xi_*=t_*/a=1/(2\sqrt{N})$ with $\theta(\xi_*)=\sqrt{N}$. Let $t_0=(t_-+t_*)/2$. Notice also that t_0/a does not depend on a and that $t_0/a<\xi_*$. We choose c small enough that $\theta(t_0/a)>\sqrt{N}+5c$, and then ε small enough that $(1-\varepsilon)\theta(t_0/a)>\sqrt{N}+4c$. Then, for $t\leqslant t_0$, we have $\theta(t/a)\geqslant\theta(t_0/a)$, so (7.20) yields

$$\tilde{\rho}^s(t,0)\leqslant\frac{C}{\sqrt{t}}\exp\left[2ct-(1-\varepsilon)\theta(t_0/a)a\right]\leqslant\frac{C}{\sqrt{t}}\exp\left[2ct-(\sqrt{N}+4c)a\right],$$

where we used $s\leqslant t$ and $s\leqslant t_-\leqslant t_*=a/(2\sqrt{N})\leqslant a/\gamma_*$. Since $t\leqslant t_0\leqslant a$, this implies (7.19).

We now consider $t\in[t_0,a]$. Using $\theta(t/a)\geqslant\theta(\xi_*)=\sqrt{N}$ and $s\leqslant t_-=\xi_-a$, (7.20) implies that

$$\tilde{\rho}^s(t,0)\leqslant\frac{C}{\sqrt{t}}\exp\left[2ct-(1-\varepsilon)\sqrt{N}a-a\gamma_*(1-\gamma_*\xi_-)\left(1-\frac{t_-}{t_0}\right)\right].$$

Notice that t_-/t_0 is independent of a and strictly smaller than one. We make c small enough that $\gamma_*(1-\gamma_*\xi_-)(1-t_-/t_0)>5c$ and then ε small enough that $(1-\varepsilon)\sqrt{N}\geqslant\sqrt{N}-c$. Then, since $t\leqslant a$, we obtain again the bound (7.19).

Finally, we consider $t\geqslant a$. As θ is convex, we have for $\xi\geqslant 1$

$$\theta(\xi)\geqslant\theta(1)+\theta'(1)(\xi-1)=\frac{1}{2}+\left(N-\frac{1}{4}\right)\xi\geqslant\sqrt{N}+\frac{1}{4}+\left(N-\sqrt{N}\right)\xi.$$

Hence (7.20) yields

$$\tilde{\rho}^s(t,0)\leqslant\frac{C}{\sqrt{t}}\exp\left[2ct-(1-\varepsilon)\left(\sqrt{N}+\frac{1}{4}\right)a-(1-\varepsilon)(N-\sqrt{N})t\right].$$

It is then clear that by taking c and ε small enough, we have again (7.19).

This concludes the proof that the first term in (7.3) is bounded as stated by lemma 7.1. The proof only relies on two properties of ψ: $\psi(x)\leqslant C\exp(\sqrt{N}x)$ for $x<0$ and $\psi(x)\leqslant C(1+x)$ for $x\geqslant 0$. Since ψ' satisfies the same two properties (by (2.35)) and $|E|\leqslant C$, the other terms in (7.3) are bounded in the same way as the first term and the proof of lemma 7.1 is complete. □

7.2. Proof of lemma 7.2

We now turn to the solution σ of (7.2).

Let us start by showing that

$$\int_{\mathbb{R}} \psi(x)\sigma(t,x)\,\mathrm{d}x = O\left(\mathrm{e}^{-(\sqrt{N}+c)a}\right).$$

We again represent $\sigma(t,x)$ via the Duhamel formula:

$$\sigma(t,x) = \int_0^t \sigma^s(t,x)\,\mathrm{d}s. \tag{7.21}$$

Here, $\sigma^s(t,x)$, defined for $0 < s < t$, is the solution to

$$\begin{aligned} &\partial_t\sigma^s = \partial_x^2\sigma^s - \frac{3}{2\lambda_*(t+1)}\partial_x\sigma^s - N(1-\varepsilon_0)1_{(-\infty,-K)}(x)\sigma^s, \quad t > s,\\ &\sigma^s(s,x) = N(1-\varepsilon_0)1_{[-K,\infty)}(x)\rho(s,x). \end{aligned} \tag{7.22}$$

Our first step will be to bound the initial condition in (7.22) with the following lemma.

Lemma 7.3 *There exist $C > 0$, $c > 0$, and $\kappa > 0$ such that for all $s \geqslant 0$ and all $x \in \mathbb{R}$:*

$$\sigma^s(s,x) \leqslant C\,\exp\left[-(\sqrt{N}+c)a - cs - \kappa\,\min\left\{1, \frac{a}{s}\right\}x_+ - \frac{x^2}{8s}\right]1_{[-K,\infty)}(x). \tag{7.23}$$

with $x_+ = \max\{x,0\}$.

Proof. Recall from (7.22) that $\sigma^s(s,x) = C1_{[-K,\infty)}(x)\rho(s,x)$ with $\rho(t,x)$ given by (7.4):

$$\rho(t,x) = \int_0^{t\wedge t_-} \rho^s(t,x)\,\mathrm{d}s.$$

We use $\rho^s(t,x) \leqslant \tilde{\rho}^s(t,x)$ with $\tilde{\rho}^s(t,x)$ given in (7.17). It suffices to show that there exists $\kappa > 0$ such that, for all $s \leqslant t\wedge t_-$,

$$\tilde{\rho}^s(t,x) \leqslant \frac{C}{\sqrt{t}}\,\exp\left[-(\sqrt{N}+c)a - ct - \kappa\,\min\left\{(1, \frac{a}{t}\right\}x_+ - \frac{x^2}{8t}\right] \quad \text{for } x > -K. \tag{7.24}$$

Then $\rho(t,x) \leqslant C\sqrt{t}\,\exp[\dots]$, and the $\sqrt{t}$ can be absorbed in the exponential by making c smaller. We now show (7.24).

We first assume that $t \leqslant \delta a$, for some small $\delta > 0$ to be chosen. For such small times, the Gaussian term in (7.17) controls everything. We use the following result: for any $\varepsilon \in [0, 1/2)$, any $\kappa \geqslant 0$, any $\lambda \geqslant 0$ and any $K \geqslant 0$, there exists $\delta' > 0$ such that, if b is large enough,

$$(1-\varepsilon)\frac{(x+b)^2}{4t} \geqslant \frac{x^2}{8t} + \kappa x + \lambda b \quad \text{for all } t \leqslant \delta' b \quad \text{and } x \geqslant -K. \tag{7.25}$$

Indeed, taking $\varepsilon = 0$ for simplicity, (7.25) is equivalent to $x^2 + 2\beta x + \gamma \geqslant 0$ with $\beta = 2b - 4\kappa t$ and $\gamma = 2b^2 - 8\lambda bt$. By making δ' small enough, we have $\beta \in [1.9b, 2b]$ and $\gamma \in [1.9b^2, 2b^2]$ for $t \leqslant \delta' b$. Then, (7.25) holds for $x \geqslant -\beta + \sqrt{\beta^2 - \gamma}$, where $-\beta + \sqrt{\beta^2 - \gamma} < (-1.9 + \sqrt{4 - 1.9})b \approx -0.45b$, which is smaller than $-K$ for b large enough.

Then, in (7.17), we notice that $b := a - 2\gamma_* s \geqslant a - 2\gamma_* t_- = a(1 - 2\gamma_* \xi_-)$, with $1 - 2\gamma_* \xi_- > 0$. We use (7.25) with this b and $\lambda = (\sqrt{N} + c)/(1 - 2\gamma_* \xi_-)$ to obtain (7.24) for $t \leqslant \delta' b \leqslant \delta a$ with $\delta = \delta'(1 - 2\gamma_* \xi_-)$. (We wrote $2ct - Nt \leqslant -ct$ in (7.17) after assuming $c < N/3$, and we used $\kappa x \geqslant \kappa x_+ - \kappa K$ for $x \geqslant -K$.)

We now turn to $t \geqslant \delta a$. From (7.17) and (7.19), we have

$$\begin{aligned}\tilde{\rho}^s(t,x) &= \tilde{\rho}^s(t,0) \exp\left[-(1-\varepsilon)\frac{x^2}{4t} - (1-\varepsilon)\frac{a - 2\gamma_* s}{2t} x\right] \\ &\leqslant \frac{C}{\sqrt{t}} \exp\left[-(\sqrt{N}+c)a - ct - \frac{x^2}{8t} - (1-\varepsilon)\frac{a - 2\gamma_* s}{2t} x\right].\end{aligned} \tag{7.26}$$

Pick $\kappa = (1-\varepsilon)(1 - 2\gamma_* \xi_-)/2$. Since $s \in [0, \xi_- a]$ and $t \geqslant \delta a$, we have

$$(1-\varepsilon)\frac{a - 2\gamma_* s}{2t} \in \left[\frac{\kappa a}{t}, \frac{1}{2\delta}\right].$$

This implies that

$$(1-\varepsilon)\frac{a - 2\gamma_* s}{2t} x \geqslant \frac{\kappa a}{t} x_+ - \frac{K}{2\delta} \geqslant \kappa \min\left\{1, \frac{a}{t}\right\} x_+ - \frac{K}{2\delta} \quad \text{for all } x \geqslant -K.$$

Using this bound in (7.26) concludes the proof of (7.24) and of lemma 7.3. □

We now complete the proof of lemma 7.2. The solution to (7.22) with the initial condition (7.23) can be treated by lemma 3.3, with $\kappa_- = 0$ and $\kappa_+ = \kappa \min\{1, a/s\}$. For instance,

$$\int_{\mathbb{R}} \psi(x)\sigma^s(t,x)\ \mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a - cs} \max\left\{\left(\frac{s}{a}\right)^2, 1\right\}.$$

Since $a \geqslant 1$, we have $\max\{(s/a)^2, 1\} \leqslant C(\varepsilon)\mathrm{e}^{\varepsilon s}$ for any $\varepsilon > 0$. Thus, we can absorb $\max\{(s/a)^2, 1\}$ into the exponential factor e^{-cs} by reducing c. After this operation, lemma 3.3 yields:

$$\begin{aligned}&\int_{\mathbb{R}} \psi(x)\sigma^s(t,x)\ \mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a - cs}, \\ &\int_{\mathbb{R}} \psi'(x)\sigma^s(t,x)\ \mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a - cs}(t - s + 1)^{-1/2}, \\ &\int_{\mathbb{R}} |E(t,x)|\psi(x)\sigma^s(t,x)\ \mathrm{d}x \leqslant C\mathrm{e}^{-(\sqrt{N}+c)a - cs}(t+1)^{-1/2}(t - s + 1)^{-3/2}.\end{aligned} \tag{7.27}$$

In light of the Duhamel formula (7.21), we must integrate (7.27) over $s \in [0, t]$. Taking $\alpha \in \{1/2, 3/2\}$ in (6.13), (7.21) and (7.27) imply lemma 7.2. □

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Appendix A. Probabilistic intuition

In this appendix, we discuss at a heuristic level the connection between the present work and the more probabilistic approach used in [6, 7] to obtain the exponential decay of $\mathbb{P}(d_{12} > a)$. To simplify the discussion, we only consider binary BBM with $N = 2$, $\lambda_* = 1$ and $c_* = 2$.

The limit extremal point process $\mathcal{X}$ of the BBM defined in (1.19) has the following description [1–3]. We start with a Poisson point process of intensity $\mathrm{e}^{-(z-\theta)}\ \mathrm{d}z$, where θ is an independent random shift related to the limit of the so-called derivative martingale; see, for instance, [1] for details. We refer to the atoms of this point process as the 'leaders'. We then replace each leader by an independent copy of a certain random point measure, the 'decoration', shifted by the position of the leader. The decoration has an atom at 0 (to keep particles at the leaders' positions), and no mass to the right of 0. Informally, the leaders are the rightmost descendants of particles that branched early on, while the decorations consist of particles that branched off the leaders very recently.

Let d_{Poi} denote the distance between the two rightmost particles in the Poisson point process, and let d_{dec} denote the same for the decoration of the rightmost leader. Then $d_{12} = \min\{d_{\mathrm{Poi}}, d_{\mathrm{dec}}\}$. By the independence of the Poisson point process and the decoration, we have

$$\mathbb{P}(d_{12} > a) = \mathbb{P}(d_{\mathrm{Poi}} > a)\,\mathbb{P}(d_{\mathrm{dec}} > a) = \mathrm{e}^{-a}\mathbb{P}(d_{\mathrm{dec}} > a). \tag{A.1}$$

Here we have used properties of the Poisson point process to compute $\mathbb{P}(d_{\mathrm{Poi}} > a) = \mathrm{e}^{-a}$.

We now construct the decoration point process $\mathcal{D}$, following [1]. This 'backwards' construction lies at the heart of the arguments in [7]. For fixed $s > 0$, let $B_{t\in[0,s]}$ be a Brownian bridge from $B_0 = 0$ to $B_s = -2s$ (recalling that our Brownian motions have diffusivity 2). We call B_t the 'spine;' it represents the path of the leader traced in reverse time t. Next, let $\sigma_1 \leqslant \sigma_2 \leqslant \sigma_3 \leqslant \ldots$ be the successive atoms of a Poisson point process of intensity $2\mathrm{d}t$ on $[0, r_s]$, for some $1 \ll r_s \ll s$. These represent the times at which the leader branched. For each k, we start an independent BBM from the position B_{σ_k} and let it evolve for time σ_k (see figure 1).

Now let $\mathcal{D}_s$ denote the set of all the final positions in all the BBMs that branched from the spine, together with the singleton $\{0\}$. That is, if $x_1^{(k)}(\sigma_k) \geqslant x_2^{(k)}(\sigma_k) \geqslant \ldots \geqslant x_{n_k}^{(k)}(\sigma_k)$ denote the final positions of the particles of the BBM born at time σ_k, we have

$$\mathcal{D}_s = \delta_0 + \sum_{k\in\mathbb{N}\ |\ \sigma_k\leqslant r_s} \sum_{i\leqslant n_k} \delta_{B_{\sigma_k} + x_i^{(k)}(\sigma_k)}.$$

Finally, the decoration $\mathcal{D}$ is given by

$$\mathbb{P}\left[\mathcal{D} \in\ \cdot\ \right] = \lim_{s\to\infty} \mathbb{P}\left[\mathcal{D}_s \in\ \cdot\ |\mathcal{D}_s\left((0,\infty)\right) = 0\right]. \tag{A.2}$$

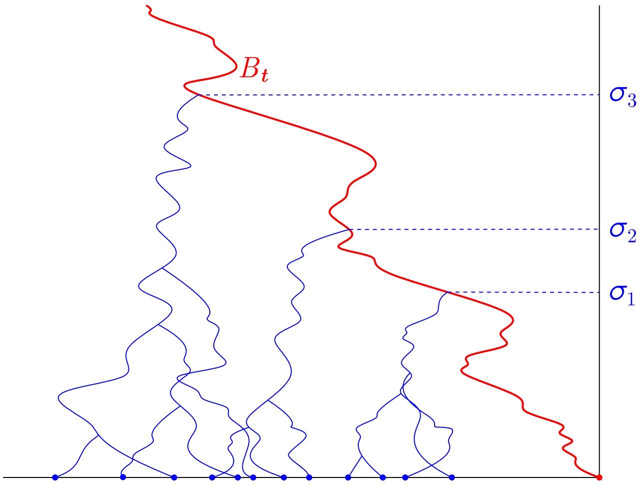

Figure 1. Backwards construction of the decoration point process.

We emphasize the conditioning in (A.2), which ensures that the spine ends up as the rightmost particle in its family. This conditioning affects the spine trajectory B_t, the point process of branching times σ_k, and the BBMs that branch off the spine.

Now, the event $d_{\text{dec}} > a$ is equivalent to all the BBMs ending to the left of $-a$. The rightmost particle of a BBM started from B_{σ_k} will typically have position near $B_{\sigma_k} + 2\sigma_k - \frac{3}{2}\log \sigma_k$ at time σ_k. So if $d_{\text{dec}} > a$, we should at least have $B_{\sigma_1} + 2\sigma_1 - \frac{3}{2}\log \sigma_1 < -a$. If we ignore the conditioning in (A.2) and all the other BBMs branching at $\sigma_2, \sigma_3, \ldots$, this suggests that

$$\mathbb{P}(d_{\text{dec}} > a) \approx \int_0^\infty \frac{\mathrm{d}\sigma}{\sqrt{\sigma}} \exp\left[-\frac{\left(a - \frac{3}{2}\log\sigma\right)^2}{4\sigma} - 2\sigma\right]$$
$$\approx \int_0^\infty \frac{\mathrm{d}\sigma}{\sqrt{\sigma}} \exp\left[-\frac{a^2}{4\sigma} - 2\sigma + \frac{3a\log\sigma}{4\sigma}\right] \tag{A.3}$$

for $a \gg 1$. In this calculation, we have approximated the Brownian bridge from $B_0 = 0$ to $B_s = -2s$ by a Brownian path with drift -2 (and, as usual, diffusivity 2). The $\mathrm{e}^{-2\sigma}$ term in (A.3) corresponds to the distribution of σ_1. The integral in (A.3) is dominated by σ close to $a/(2\sqrt{2})$, so

$$\mathbb{P}(d_{\text{dec}} > a) \approx \exp\left(-\sqrt{2}\,a + \frac{3}{\sqrt{2}}\log a\right). \tag{A.4}$$

If we keep the leading term $\exp\left(-\sqrt{2}\,a\right)$ and recall (A.1), we find to the exponential factor in (1.21). This is, in essence, the argument used in [7].

The subleading term in (A.4) corresponds to the polynomial term $a^{3/\sqrt{2}}$ in theorem 1.1, since $N = 2$. However, at this scale the conditioning in (A.2) cannot be ignored: it alters the trajectory of the spine and suppresses the branching rate. Indeed, to avoid producing particles to the right of 0, the spine is inclined to move further to the left than a typical Brownian motion with drift -2, and the effective branching rate along the spine will be less than 2 near the origin.

Moreover, we cannot ignore the other branchings σ_k along the spine. Without the conditioning in (A.2), the spine will often lie to the right of $-2t + \sqrt{t}$. A BBM branching from such a time would typically end with a particle near $-\frac{3}{2}\log t + \sqrt{t}$, well to the right of 0. The authors of [7] prove that the heuristic (A.3) remains correct at the exponential level (1.21) even when the conditioning and other branchings are accounted for. However, we do not believe that the argument extends easily to the subleading term.

To conclude this appendix, let us highlight that the heuristics in [6] are slightly different. There, the authors consider a BBM at a large time. The least unlikely scenario leading to $d_{12} > a \gg 1$ is as follows. A single particle starts from near the typical rightmost position and moves an anomalous distance $2\tau + a$ during a time τ without branching. Meanwhile, the rest of the BBM behaves typically, with rightmost particle moving at velocity 2. The probability of this event is roughly

$$\mathbb{P}(d_{12} > a) \approx \mathbb{P}(d_{12} \approx a) \approx \mathrm{e}^{-\frac{(2\tau+a)^2}{4\tau} - \tau}.$$

The maximizing τ is $a/(2\sqrt{2})$, which yields the correct probability $\mathbb{P}(d_{12} > a) \approx \mathrm{e}^{-(1+\sqrt{2})a}$. However, there does not seem to be an easy way (even at the heuristic level) to obtain the polynomial prefactor in theorem 1.1 from this perspective.

Appendix B. The proof of lemma 3.3

To prove lemma 3.3, we argue that the absorption on $(-\infty, -K)$ in (3.16) acts similarly to a Dirichlet boundary condition.

We begin by constructing a family of supersolutions based on the Dirichlet problem. Recall that we defined

$$\varphi(\lambda, x) = \frac{x}{\lambda^{3/2}} \exp\left(-\frac{x^2}{4\lambda}\right).$$

Now let $v(t, x;\ s)$, with $s \geqslant 0$, be the solution to

$$\begin{cases} \partial_t v = \partial_x^2 v - \dfrac{3}{2\lambda_*(t+s+1)}\partial_x v & \text{for } t > 0 \text{ and } x > 0, \\ v(t,0) = 0 & \text{for } t > 0, \\ v(0,x) = \varphi(\lambda, x) & \text{for } x > 0. \end{cases} \tag{B.1}$$

We define the 'λ-adapted' self-similar variables

$$\tau = \log\left(\frac{t+\lambda}{\lambda}\right) \quad \text{and} \quad \eta = \frac{x}{\sqrt{t+\lambda}}.$$

Let v denote v in these coordinates:

$$v(\tau, \eta) := v\left(\lambda(\mathrm{e}^{\tau} - 1),\ \sqrt{\lambda}\mathrm{e}^{\tau/2}\eta\right).$$

Then v satisfies

$$\partial_\tau v = \partial_\eta^2 v + \frac{\eta}{2}\partial_\eta v - m(\tau)\partial_\eta v, \quad v(0, \eta) = \lambda^{-1}\varphi(1, \eta)$$

with a drift

$$m(\tau) = \frac{3}{2\lambda_*} \frac{\sqrt{\lambda}\mathrm{e}^{\tau/2}}{\lambda(\mathrm{e}^{\tau} - 1) + s + 1}.$$

We wish to argue that the drift term $m\partial_\eta v$ is negligible, so that $\varphi(\lambda + t, x)$ is an approximate solution to (B.1). Precisely, we want $\sup m$ and $\int_0^\infty m(\tau)\, \mathrm{d}\tau \ll 1$ to be small. Suppose $\lambda \geqslant 1$ and $s \geqslant \beta\lambda$ for some $\beta > 0$. Then we will have $m \leqslant C\beta^{-1/2}s^{-1/2}$ when $\tau \in [0, 1]$, and

$$m(\tau) \leqslant Cs^{-1/2} \min\left\{\left(\lambda s^{-1}\mathrm{e}^{\tau}\right)^{1/2}, \left(\lambda s^{-1}\mathrm{e}^{\tau}\right)^{-1/2}\right\} \quad \text{for } \tau \geqslant 1.$$

It follows that both $\sup m$ and $\int_0^\infty m(\tau)\, \mathrm{d}\tau$ are bounded by $C\beta^{-1/2}s^{-1/2}$.

Combining the proof of lemma 2.2 in [10] with methods from [19, section 5], we obtain the following analogue of lemma 5.4.

Lemma B.1 *For each $\beta > 0$, there exists $C(\beta) > 0$ such that for all $\lambda \geqslant 1$ and $s \geqslant \beta\lambda$, the solution $v(t, x;\ s)$ to (B.1) satisfies*

$$v(t, x; s) = \varphi(t + \lambda, x)\, [1 + h_0(t, x; s)] + R_0(t, x; s)$$

and

$$\partial_x v(t, x; s) = \partial_x\varphi(t + \lambda, x)\, [1 + h_1(t, x; s)] + R_1(t, x; s)$$

with error terms satisfying $|h_0| + |h_1| \leqslant Cs^{-1/2}$ and

$$|R_0| + (t + \lambda)^{1/2}\, |R_1| \leqslant Cs^{-1/2}(t + \lambda)^{-3/2} \exp\left[-\frac{x^2}{6(t + \lambda)}\right]$$

for all $t, x > 0$.

This lemma enables the construction of supersolutions for more complicated equations. If we have an absorbing potential on the left rather than a Dirichlet condition, we can join v to a decaying exponential. We illustrate the construction with a simple example. Consider the equation

$$\partial_t \nu = \partial_x^2 \nu - \frac{3}{2(t + s + 1)}\partial_x \nu - 1_{(-\infty,10)}(x)\nu. \tag{B.2}$$

It is easy to check that

$$A_\lambda(t, x) = \frac{1}{2}(t + \lambda)^{-3/2} \exp\left(\frac{x}{2}\right)$$

is a supersolution to (B.2) on $(-\infty, 10)$, provided $\lambda \geqslant 2$. We can glue $A_\lambda(t, x)$ on the left to the solution to the Dirichlet problem (B.1) on the right to form a global super-solution to (B.2). For this 'hybrid' to be a supersolution itself, we need its slope to decrease at the joint, which ensures that in a neighborhood of the joint, it is the minimum of two supersolutions, and thus a supersolution. We take some care to achieve this.

To emphasize the dependence on λ, let v_λ denote the solution to (B.1). If $\lambda \geqslant 2$, we can verify numerically that the graphs of $A_\lambda(t, \cdot)$ and $\varphi(t + \lambda, \cdot)$ intersect twice in $(0, 10)$. By lemma B.1, the same is true of $A_\lambda(t, x)$ and $v_\lambda(t, x;\ s)$ when s is sufficiently large. Let $j(t)$ denote their rightward point of intersection. Then the 'hybrid' function

$$\Psi(t, x) = \begin{cases} A_\lambda(t, x) & \text{for } x \leqslant j(t) \\ v_\lambda(t, x; s) & \text{for } x > j(t) \end{cases}$$

is a supersolution to (B.2).

We now generalize this construction to produce supersolutions of (3.16) in lemma 3.3. By 'supersolution,' we mean a function that satisfies the opposite inequality to that in (3.16). Recall the parameters $\alpha > 0$, $\kappa_- < \alpha$, and $K \geqslant 0$ from lemma 3.3. Define $b := (\kappa_- + \alpha)/2 \vee 0$, so that $\kappa_- \vee (-\alpha) < b < \alpha$.

Lemma B.2 *For each $\beta > 0$, there exist $C_0(\beta) \geqslant 1$ and $B, L > 0$ depending also on α, κ_-, K, and λ_* such that the following holds for all $\lambda \geqslant C_0$ and $s \geqslant \beta\lambda$. There exists a continuous function $j : [0, \infty) \to (-K-1, -K)$ depending on α, κ_-, K, λ_*, λ, and s such that if*

$$\Phi(t,x) := \begin{cases} B(t+\lambda)^{-3/2}\mathrm{e}^{-bx} & \text{for } x \leqslant j(t), \\ v_\lambda(t, x+L; s) & \text{for } x > j(t), \end{cases} \tag{B.3}$$

then Φ is continuous, $\partial_x \Phi$ decreases at $x = j(t)$, and $\Phi(t-s, x)$ is a supersolution to (3.16).

Proof. Let $C_1 := \frac{3}{2(\alpha^2 - b^2)}$, which is positive because $0 \leqslant b < \alpha$. Then if

$$A(t,x) := (t+\lambda)^{-3/2} \cosh\left[b(x+K+1)\right]$$

and $\lambda \geqslant C_1$, one can easily check that

$$\partial_t A \geqslant \partial_x^2 A - \alpha^2 A.$$

Moreover, if $C_2 := \frac{3b}{\lambda_*(\alpha^2 - b^2)}$, we have

$$\partial_t A \geqslant \partial_x^2 A - \frac{3}{2\lambda_*(t+s+1)} \partial_x A - \alpha^2 A$$

provided $s \geqslant C_2$. Thus $A(t-s, x)$ is a supersolution of (3.16) when $x < -K$.

We call a function $f \in \mathcal{C}^1([-K-1, -K])$ *good* if it intersects $\cosh[b(\cdot + K + 1)]$ exactly twice in $(-K-1, -K)$, both intersections are transverse, and

$$\partial_x f(j) < \partial_x \left\{\cosh\left[b(x+K+1)\right]\right\}\big|_{x=j},$$

where j denotes the rightward intersection.

Now, $\cosh[b(x+K+1)]$ is convex and increasing on $[-K-1, -K]$. It follows that there exists a good affine function $\ell(x) := B^{-1}(x+L)$ with $B > 0$ and $L > K+1$ depending on b and K. Moreover, it is easy to check that goodness is an open condition in $\mathcal{C}^1$. Now note that $\lambda^{3/2}\varphi(\lambda, x) \to x$ in $\mathcal{C}^1(\mathcal{K})$ as $\lambda \to \infty$ for every compact $\mathcal{K} \subset \mathbb{R}$. Moreover, by lemma B.1,

$$(t+\lambda)^{3/2} \| v_\lambda(t, \cdot\,; s) - \varphi(t+\lambda, \cdot)\|_{\mathcal{C}^1(\mathcal{K})} \leqslant C(\beta, \mathcal{K})\lambda^{-1/2}$$

for every compact $\mathcal{K} \subset \mathbb{R}$, provided $s \geqslant \beta\lambda$. Thus

$$B^{-1}(t+\lambda)^{3/2} v_\lambda(t, x+L; s) \to \ell(x)$$

in $\mathcal{C}^1([-K-1, -K])$ as $\lambda \to \infty$. Since goodness is open, there exists $C_3(\beta) > 0$ depending also on b, K, and λ_* such that $B^{-1}(t+\lambda)^{3/2} v_\lambda(t, x+L;\ s)$ is good for all $t \geqslant 0$, $\lambda \geqslant C_3$, and $s \geqslant \beta\lambda$.

Therefore, $BA(t, \cdot)$ and $v_\lambda(t, \cdot + L;\ s)$ intersect exactly twice in $(-K-1, -K)$ and $\partial_x v_\lambda < B\partial_x A$ at the rightward intersection, which we call $j(t)$. Because the intersections are transverse, j is continuous in t. We checked above that BA is a supersolution of (3.16) when $x < -K$ (as the equation is linear), and (B.1) implies that v_λ is a supersolution of (3.16) when

$x > -L$. Defining Φ by (B.3), we see that it is a supersolution of (3.16) on the whole line, because it is locally a minimum of supersolutions.

The above conclusions hold under the conditions $s \geqslant \beta\lambda$, $\lambda \geqslant C_1$, and $s \geqslant C_2 \vee C_3$. We therefore take $C_0 := \max\{C_1, \beta^{-1}C_2, \beta^{-1}C_3, 1\}$, so that the above holds when $s \geqslant \beta\lambda$ and $\lambda \geqslant C_0$. □

In the sequel, we say that a function is *hybrid* if it is a positive multiple of a function Φ provided by lemma B.2. Let $\mathcal{H}$ denote the set of hybrids. We can now prove lemma 3.3.

Proof of lemma 3.3. Consider w as in lemma 3.3. Let $\overline{w}$ denote the solution to (3.16) and (3.17) with inequalities replaced by equalities. Then, by the comparison principle, we have $w \leqslant \overline{w}$, and it suffices to control $\overline{w}$. To bound $\overline{w}$, we use a sum of hybrid supersolutions to cover different spatial regions of its initial condition.

For the moment, assume that $s \geqslant 2^{-4}C_0$. Recall that there is a time shift between (3.16) and (B.1). We thus want to bound $\overline{w}(s,x)$ by various hybrid functions $\Phi(0,x)$. Define

$$k_0 := \lceil \log_4 C_0 \rceil \quad \text{and} \quad k_1 := \lceil \log_4 s \rceil + 1. \tag{B.4}$$

By (3.17), $\overline{w}(s,x) \lesssim \mathrm{e}^{-\kappa_- x}$ on $(-\infty, 2^{k_0} - L)$, where L is given by lemma B.2. Since $b > \kappa_-$, we also have $\overline{w}(s,x) \lesssim \cosh\left[b(x + K + 1)\right]$ on $(-\infty, 2^{k_0} - L)$. Thus there exists a hybrid $\Phi_- \in \mathcal{H}$ with parameter $\lambda = C_0$ such that

$$\overline{w}(s,x) \leqslant \Phi_-(0,x) \quad \text{for all} \;\; x \leqslant 2^{k_0} - L. \tag{B.5}$$

This hybrid Φ_- can be chosen independently of κ_+.

We are left with the decaying tail of $\overline{w}(s,x)$ for $x \geqslant 2^{k_0} - L$. We cover this tail by a sequence of ever-wider Gaussians. Note that

$$\lambda\varphi(\lambda,x) \geqslant \frac{1}{2} \quad \text{for all} \;\; x \in \left[\sqrt{\lambda}, 2\sqrt{\lambda}\right]. \tag{B.6}$$

We claim that

$$\overline{w}(s, x - L) \leqslant 2\sum_{k=k_0}^{k_1} \exp\left(-2^k\kappa_+\right) 4^k \varphi(4^k, x) \tag{B.7}$$

for all $x \geqslant 2^{k_0}$. By (3.17) and (B.6), (B.7) holds for all $x \in [2^{k_0}, 2^{k_1+1}]$. Moreover, for all $x \geqslant 2^{k_1}$, (3.17) and the definition (B.4) of k_1 imply that

$$\begin{aligned}
\overline{w}(s,x) &\leqslant \mathrm{e}^{-\kappa_+ x}\mathrm{e}^{-x^2/8s} \leqslant \exp\left(-\kappa_+ 2^{k_1}\right)\frac{x}{2^{k_1}}\exp\left(-\frac{x^2}{4^{k_1+1}}\right) \\
&= \exp\left(-2^{k_1}\kappa_+\right) 4^{k_1}\varphi(4^{k_1}, x).
\end{aligned}$$

The final expression is the final term in the sum in (B.7), so (B.7) holds for all $x \geqslant 2^{k_0}$ as claimed. By the definition (B.4) of k_0, we have only used parameters $\lambda = 4^k \geqslant 4^{k_0} \geqslant C_0$ in (B.7). Moreover, $\lambda \leqslant 4^{k_1} \leqslant 2^4 s$, so $s \geqslant \beta\lambda$ with $\beta = 2^{-4}$. Thus by (B.1) and lemma B.2, there exist hybrids $\Phi_k \in \mathcal{H}$ independent of κ_+ such that

$$\Phi_k(0, x - L) \geqslant \varphi(4^k, x),$$

with equality when $x \geqslant L - K$. Then (B.5) and (B.7) imply:

$$\overline{w}(s,x) \leqslant \Phi_-(0,x) + 2\sum_{k=k_0}^{k_1} \exp\left(-2^k\kappa_+\right) 4^k\Phi_k(0,x).$$

By lemma B.2, $\Phi_-(t-s,x)$ and $\Phi_k(t-s,x)$ are supersolutions of (3.16). Thus the comparison principle implies that

$$\overline{w}(t,x) \leqslant \Phi_-(t-s,x) + 2\sum_{k=k_0}^{k_1} \exp\left(-2^k\kappa_+\right) 4^k\Phi_k(t-s,x) \tag{B.8}$$

for all $t \geqslant s$ and $x \in \mathbb{R}$.

We can finally control various integrals of $\overline{w}$. Notice that

$$\int_0^\infty \varphi(\lambda,x)\,\mathrm{d}x \sim \lambda^{-\frac{1}{2}} \quad \text{and} \quad \int_{\mathbb{R}_+} x\varphi(\lambda,x)\,\mathrm{d}x \sim 1.$$

If Φ is a hybrid with parameter $\lambda \geqslant C_0$, it is a multiple of $Cv_\lambda(t,x+L;\ s)$ to the right of $j(t)$. By lemma B.1, the main term in $v_\lambda(t,x+L;\ s)$ is bounded by a multiple of $\varphi(t+\lambda,x+L)$ as $t \to \infty$. Using (2.34) and (2.35), lemma B.1, $\lambda \geqslant C_0 \geqslant 1$, and $s \geqslant 2^{-4}\lambda$, we can check that

$$\begin{aligned}
&\int_{\mathbb{R}} \psi(x)\Phi(t,x)\,\mathrm{d}x \leqslant C,\\
&\int_{\mathbb{R}} \psi'(x)\Phi(t,x)\,\mathrm{d}x \leqslant C(t+1)^{-1/2},\\
&\int_{\mathbb{R}} \mathrm{e}^{-c|x|}\psi(x)\Phi(t,x)\,\mathrm{d}x \leqslant C(t+1)^{-3/2}.
\end{aligned} \tag{B.9}$$

The sum of the coefficients $4^k\exp\left(-2^k\kappa_+\right)$ in (B.8) is $O\left(\kappa_+^{-2}\right)$. Therefore, (B.8) and (B.9) imply

$$\begin{aligned}
&\int_{\mathbb{R}} \psi(x)\overline{w}(t,x)\,\mathrm{d}x \leqslant C\max\{\kappa_+^{-2},1\}\\
&\int_{\mathbb{R}} \psi'(x)\overline{w}(t,x)\,\mathrm{d}x \leqslant C\max\{\kappa_+^{-2},1\}(t-s+1)^{-1/2}\\
&\int_{\mathbb{R}} \mathrm{e}^{-c|x|}\psi(x)\overline{w}(t,x)\,\mathrm{d}x \leqslant C\max\{\kappa_+^{-2},1\}(t-s+1)^{-3/2}
\end{aligned}$$

for all $t \geqslant s$, with a constant C depending on α, K, κ_-, and λ_* but not on s or κ_+. Now (3.18)–(3.20) follow from (4.15).

In all the above calculations, we assumed that $s \geqslant 2^{-4}C_0$, to satisfy the hypotheses of lemma B.2. Now suppose that $s \in [0, C_0]$. By (3.17),

$$\overline{w}(s,x) \leqslant \mathrm{e}^{-\kappa_- x}1_{\mathbb{R}_-}(x) + \mathrm{e}^{-x^2/8s}.$$

We claim that

$$\overline{w}(t,x) \leqslant C\mathrm{e}^{-\kappa_- x}1_{\mathbb{R}_-}(x) + C\mathrm{e}^{-x^2/8t} \tag{B.10}$$

for all $t \in [s, C_0]$ and some $C > 0$ depending on κ_-, λ_*, and C_0. Then the estimates (3.18)–(3.20) will hold for $t \in [s, C_0]$. Moreover, $\overline{w}(C_0,x)$ will satisfy (3.17) up to a multiplicative factor e^{CC_0}, so our prior analysis will apply to $t \geqslant C_0$.

It remains to check (B.10). Let W_1 denote the solution to the heat equation

$$\partial_t W_1 = \partial_x^2 W_1, \quad W_1(s,x) = \mathrm{e}^{-\kappa_- x} 1_{\mathbb{R}_-}(x).$$

Because $\mathrm{e}^{-\kappa_- x + \kappa_-^2 t}$ solves the heat equation, we certainly have

$$W_1(t,x) \leqslant \mathrm{e}^{-\kappa_- x + \kappa_-^2 t}.$$

We can do better on $\mathbb{R}_+$ using the fundamental solution $G_t(x) := (4\pi t)^{-1/2} \exp\left(-\frac{x^2}{4t}\right)$. An explicit computation yields

$$W_1(t,x) = [G_t * W_1(0,\cdot)](x) \leqslant \exp\left(\kappa_-^2 t - \frac{x^2}{4t}\right)$$

when $x \geqslant 0$. So

$$W_1(t,x) \leqslant \mathrm{e}^{\kappa_+^2 t}\left[\mathrm{e}^{-\kappa_- x} 1_{\mathbb{R}_-}(x) + \exp\left(-\frac{x^2}{4t}\right)\right].$$

On the other hand, if W_2 solves

$$\partial_t W_2 = \partial_x^2 W_2, \quad W_2(s,x) = \mathrm{e}^{-x^2/8s},$$

Gaussian identities imply that

$$W_2(t,x) \leqslant \mathrm{e}^{-x^2/4(t+s)}$$

for $t > s$. So if W solves the heat equation with $W(s,x) = \overline{w}(s,x)$, we have

$$W \leqslant W_1 + W_2 \leqslant \mathrm{e}^{\kappa_+^2 t}\left[\mathrm{e}^{-\kappa_- x} 1_{\mathbb{R}_-}(x) + \exp\left(-\frac{x^2}{4t}\right)\right] + \exp\left(-\frac{x^2}{4(t+s)}\right). \tag{B.11}$$

If we incorporate the drift term in (3.16), then W is a supersolution of (3.16). That is,

$$\overline{w}(t,x) \leqslant W\left(t, x - \frac{3}{2\lambda_*}\log\left(\frac{t+1}{s+1}\right)\right).$$

Using (B.11), we can easily verify (B.10). This completes the proof of lemma 3.3. □

Appendix C. The proof of lemma 4.3

In this appendix, we adapt the main result in [20] to incorporate exponential decay when $x < 0$. Although [20] only explicitly handles the traditional quadratic KPP nonlinearity $u - u^2$, its results extend to general KPP nonlinearities f of the type considered here. The following bound follows directly from theorem 1.3 in [20].

Theorem C.1 *There exist $C_0, c_0 > 0$ such that*

$$|H(t, x + m(t)) - U_0(x)| \leqslant \frac{C_0 \mathrm{e}^{-c_0 x}}{\sqrt{t+1}} \quad \textit{for all } (t,x) \in [0,\infty) \times \mathbb{R}.$$

We now replace $e^{-c_0 x}$ by $e^{-c|x|}$ in this bound.

Proof of lemma 4.3. Our KPP nonlinearity satisfies $f'(1) = -1$ and $f' \in \mathcal{C}^\beta$. Hence there exists $\theta \in (0, 1)$ such that

$$\inf_{s\in[\theta,1]} \left[-f'(s)\right] \geqslant \frac{1}{2}.$$

Define

$$m_\theta(t) := \sup\left\{x \in \mathbb{R} | H(t, x) \geqslant \theta\right\}.$$

By theorem 12 in [14],

$$H(t, x + m_\theta(t)) \geqslant U_0(x + U_0^{-1}(\theta)) \quad \text{for all } (t, x) \in [0, \infty) \times (-\infty, 0]. \tag{C.1}$$

On the other hand, theorem C.1 implies that

$$m_\theta(t) - m(t) \to U_0^{-1}(\theta) \quad \text{as } t \to \infty.$$

Hence there exists $T \geqslant 1$ such that

$$\left|m(t) - [m_\theta(t) - U_0^{-1}(\theta)]\right| \leqslant 1 \quad \text{for all } \ t \geqslant T. \tag{C.2}$$

Now note that H is always decreasing in x. Combining (C.1) and (C.2), we obtain

$$H(t, x + m(t)) \geqslant H(t, x + 1 - U_0^{-1}(\theta) + m_\theta(t)) \geqslant U_0(x + 1) \geqslant \theta \tag{C.3}$$

for all $t \geqslant T$ and $x \leqslant x_\theta := U_0^{-1}(\theta) - 1$. Now define

$$H^m(t, x) := H(t, x + m(t)) \quad \text{and} \quad w(t, x) := H^m(t, x) - U_0(x).$$

Using (1.5) and (1.9), we can check that w satisfies

$$\partial_t w = \partial_x^2 w + c_* \partial_x w - \frac{3}{2\lambda_*(t + 1)}(\partial_x w + U_0') + f(H^m) - f(U_0). \tag{C.4}$$

We study w on the domain $\mathcal{Q} := [T, \infty) \times (-\infty, x_\theta]$. There, (C.3) and $x_\theta = U_0^{-1}(\theta) - 1$ imply that $\theta \leqslant H^m, U_0 < 1$. It follows that $-f' \geqslant 1/2$ on the interval between H^m and U_0. Thus by the mean value theorem,

$$Q := -\frac{f(H^m) - f(U_0)}{w} \geqslant \frac{1}{2} \quad \text{in } \mathcal{Q}. \tag{C.5}$$

We can then write (C.4) as

$$\partial_t w = \partial_x^2 w + c_* \partial_x w - Qw - \frac{3}{2\lambda_*(t + 1)}(\partial_x w + U_0'). \tag{C.6}$$

Before analyzing (C.6), we state two *a priori* bounds on w. First, the second inequality in (C.3) implies that $U_0(x + 1) \leqslant H^m \leqslant 1$ in $\mathcal{Q}$. In light of (1.10), both $1 - U_0$ and $1 - H^m$ decay exponentially as $x \to -\infty$. In particular, there exists $C_T > 0$ such that

$$|w(T,x)| \leqslant C_T \mathrm{e}^{\gamma_* x} \quad \text{for all } \; x \leqslant x_\theta. \tag{C.7}$$

This is an 'initial' bound on w. Using theorem C.1, we can control w on the boundary $x = x_\theta$:

$$|w(t,x_\theta)| \leqslant \frac{C_0 \mathrm{e}^{-c_0 x_\theta}}{\sqrt{t+1}} \quad \text{for all } \; t \geqslant T. \tag{C.8}$$

Together, these bounds control w on $\partial\mathcal{Q}$.

Now define the affine operator

$$\mathcal{A}W := \partial_t W - \partial_x^2 W - c_* \partial_x W + QW + \frac{3}{2\lambda_*(t+1)}(\partial_x W + U_0').$$

We let

$$W(t,x) := \frac{M}{\sqrt{t+1}} \mathrm{e}^{\alpha x},$$

and choose $M \in \mathbb{R}_+$ and $\alpha \in (0,\gamma_*)$ so that $\mathcal{A}W \geqslant 0$ and $\mathcal{A}(-W) \leqslant 0$. We compute

$$\mathcal{A}W = \left[-\frac{1}{2(t+1)} - \alpha^2 - c_*\alpha + Q + \frac{3\alpha}{2\lambda_*(t+1)}\right] W + \frac{3}{2\lambda_*(t+1)} U_0'.$$

By (2.27) and $\alpha < \gamma_*$, there exists $C_U \in \mathbb{R}_+$ such that

$$\frac{3}{2\lambda_*(t+1)} |U_0'| \leqslant C_U \mathrm{e}^{\alpha x}.$$

Using (C.5) and our assumption $t \geqslant T \geqslant 1$, we find

$$(\mathcal{A}W)\mathrm{e}^{-\alpha x} \geqslant \left[\frac{1}{4} - \alpha^2 - c_*\alpha\right] M - C_U. \tag{C.9}$$

Choose $\alpha \in (0,\gamma_*)$ so that

$$\alpha^2 + c_*\alpha < \frac{1}{4}.$$

Then using (C.7) and (C.8), we choose $M > 0$ sufficiently large that

$$\left[\frac{1}{4} - \alpha^2 - c_*\alpha\right] M - C_U \geqslant 0$$

and $W \geqslant |w|$ on $\partial\mathcal{Q}$. In particular, (C.9) implies that $\mathcal{A}W \geqslant 0$.

The same argument shows that $\mathcal{A}(-W) \leqslant 0$, so W is a supersolution of (C.6) and $-W$ is a subsolution. Since $|w| \leqslant W$ on $\partial\mathcal{Q}$, the comparison principle implies that $|w| \leqslant W$ in $\mathcal{Q}$. In light of theorem C.1, lemma 4.3 follows. □

ORCID iDs

Julien Berestycki https://orcid.org/0000-0001-8783-4937
Cole Graham https://orcid.org/0000-0001-5734-7257

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