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A simple backward construction of branching Brownian motion with large displacement and applications

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Abstract. In this article, we study the extremal processes of branching Brownian motions conditioned on having an unusually large maximum. The limiting point measures form a one-parameter family and are the decoration point measures in the extremal processes of several branching processes, including branching Brownian motions with variable speed and multitype branching Brownian motions. We give a new, alternative representation of these point measures and we show that they form a continuous family. This also yields a simple probabilistic expression for the constant that appears in the large deviation probability of having a large displacement. As an application, we show that Bovier and Hartung's (*ALEA Lat. Am. J. Probab. Math. Stat.* **12** (2015) 261–291) results about variable speed branching Brownian motion also describe the extremal point process of branching Ornstein–Uhlenbeck processes.

Résumé. Le présent article a pour objet l'étude du processus extrémal du mouvement Brownien branchant conditionné à avoir une particule anormalement loin à droite. Ces mesures ponctuelles limites forment une famille à un paramètre et apparaissent dans les processus extrémaux de plusieurs processus de branchement tels que le mouvement Brownien branchant avec vitesse variable ou certains mouvement Brownien branchants multitype. Nous donnons une nouvelle représentation de ces mesures ponctuelles et nous montrons qu'elles forment une famille continue de lois. Nous obtenons ainsi une expression probabiliste simple de la constante qui apparaît dans le principe de grande déviation pour un déplacement anormal de la particule la plus à droite d'un mouvement Brownien branchant. Finalement, nous appliquons ces résultats pour montrer que les travaux de Bovier et Hartung (*ALEA Lat. Am. J. Probab. Math. Stat.* **12** (2015) 261–291) sur le mouvement Brownien branchant avec vitesse variable décrivent également le processus extrémal du processus d'Ornstein–Uhlenbeck branchant.

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1. Introduction

Spatial branching processes, and in particular, the behaviour of their extremal particles, have been at the centre of a wide research activity over the past few years, both in the physics [15,16,20] and in the mathematical literature [1,2,4,29]. These models have a rich and complex structure that is of intrinsic interest, but they are also representatives of an intriguing "universality" class, the so-called *log-correlated fields* which includes the two-dimensional Gaussian free field [9,14], Gaussian multiplicative chaos [31], random matrices [3] and others.

Perhaps the simplest model in this class is the *branching Brownian motion*, in which particles move in $\mathbb{R}$ as Brownian motions, branch into two particles at rate one and behave independently of each others. For the system started with a single particle at the origin, let $\mathcal{N}_t$ be the set of particles alive at time t and for $u \in \mathcal{N}_t$ let $X_t(u) \in \mathbb{R}$ be its position. For $s \leq t$ we will also write $X_s(u)$ for the position of the unique ancestor of u at time s so that $(X_s(u), s \leq t)$ is the path

followed by the particle u. Then, it was proved in [2,4] that the point measure

$$\mathcal{E}_t := \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - \sqrt{2}t + \frac{3}{2\sqrt{2}} \log t} \tag{1.1}$$

converges in law, as $t \to \infty$ toward a *random intensity decorated Poisson point process* (DPPP for short) $\mathcal{E}_\infty$.

In general, the law of a DPPP $\mathcal{E}$ is characterized by a pair $(\nu, \mathfrak{D})$ where ν is a random sigma-finite measure on $\mathbb{R}$ and $\mathfrak{D}$ is the law of a random point process on $\mathbb{R}$. The point measure $\mathcal{E}$ can be constructed, conditionally on ν, by first taking a realisation of a Poisson point process on $\mathbb{R}$ with intensity ν, whose atoms are listed as $(x_i, i \in I)$, and an independent family of i.i.d. point processes $(\mathcal{D}_i, i \in I)$ with law $\mathfrak{D}$. Then, each atom x_i is replaced by the point process $\mathcal{D}_i$, shifted by x_i (this action is called the *decoration* of x_i with a point process of law $\mathfrak{D}$). In other words, writing $(d_i^j, j \in J_i)$ the atoms of the point process $\mathcal{D}_i$, we have

$$\mathcal{E} = \sum_{i \in I} \sum_{j \in J_i} \delta_{x_i + d_i^j}. \tag{1.2}$$

We refer to [32] for an in-depth study of random intensity decorated Poisson point processes, and their occurrences as limit of extremal point measures.

With this notation, $\mathcal{E}_\infty := \lim_{t \to \infty} \mathcal{E}_t$ is the following DPPP

$$\mathcal{E}_\infty = \mathrm{DPPP}\big(\kappa Z_\infty e^{-\sqrt{2}x} \, dx, \mathfrak{D}^1\big) \tag{1.3}$$

where κ is an implicit constant, Z_∞ is the a.s. positive limit of the so-called *derivative martingale*

$$Z_t := \sum_{u \in \mathcal{N}_t} \big(\sqrt{2}t - X_t(u)\big) e^{\sqrt{2}X_t(u) - 2t}, \tag{1.4}$$

and where the *decoration law* $\mathfrak{D}^1$ is the law of a point measure supported on $(-\infty, 0]$, with an atom at 0 defined by the following weak limit

$$\mathfrak{D}^1(\cdot) := \lim_{t \to \infty} \mathbf{P}\bigg(\sum_{u \in \mathcal{N}_t} \delta_{\{X_t(u) - M_t\}} \in \cdot \,\big|\, M_t \geq \sqrt{2}t \bigg), \tag{1.5}$$

where $M_t := \max_{u \in \mathcal{N}_t} X_t(u)$. Moreover, it is well-known that $\max \mathcal{E}_t$ converges in distribution toward $\max \mathcal{E}_\infty$, where $\max \mathcal{E}$ is the position of the largest atom in a point process $\mathcal{E}$ (see Lalley and Selke [26]).

The decoration law $\mathfrak{D}^1$ belongs to the family $(\mathfrak{D}^\varrho, \varrho \in [1, \infty])$, defined, for $\varrho < \infty$ by the weak limits

$$\mathfrak{D}^\varrho(\cdot) := \lim_{t \to \infty} \mathbf{P}\bigg(\sum_{u \in \mathcal{N}_t} \delta_{\{X_t(u) - M_t\}} \in \cdot \,\big|\, M_t \geq \sqrt{2}\varrho t \bigg). \tag{1.6}$$

We denote by $\mathfrak{D}^\infty$ the law of the Dirac mass at 0. The family $\mathfrak{D}^\varrho(\cdot)$ was introduced by Bovier and Hartung [12] as the decorations appearing in the extremal processes of variable speed branching Brownian motions. A detailed statement of the result of Bovier and Hartung is given in Section 4. This convergence in law can be seen as an extension of the Yaglom limit [18, Theorem 3], that is the convergence of the number of particles above level $\sqrt{2}\varrho t$ conditionally on $M_t \geq \sqrt{2}\varrho t$. The decoration law $\mathfrak{D}^\varrho(\cdot)$ can also appear in the context of multitype branching Brownian motions [5].

Note that the law $\mathfrak{D}^\varrho$ is constructed by conditioning the branching Brownian motion on a large deviation event for its maximum. For $\varrho \in (1, \infty)$ we define

$$C(\varrho) := \varrho \lim_{t \to \infty} t^{1/2} e^{(\varrho^2 - 1)t} \mathbf{P}(M_t \geq \sqrt{2}\varrho t). \tag{1.7}$$

The asymptotic behaviour of $\mathbf{P}(M_t > \sqrt{2}\varrho t)$ was first studied in the seminal paper [18] (where the existence of the limit $C(\varrho)$ is implicit) and the function $C(\varrho)$ plays a key role in [12] where it is proven that $C(1) = 0$ and that $\lim_{\varrho \to \infty} C(\varrho) = (4\pi)^{-1/2}$. We plot in Figure 1 a graph of the function $\varrho \mapsto C(\varrho)$ based on Monte-Carlo simulations, using its representation from Theorem 1.1.

More recently, the same function $C(\cdot)$ is the focus of [20] where, in particular, the asymptotic behaviour of $C(\varrho)$ as $\varrho \to \infty$ and $\varrho \to 1$ are conjectured. They conjectured that $C(\varrho) \sim \alpha(\varrho - 1)$ as $\varrho \searrow 1$ using non-rigorous analytic
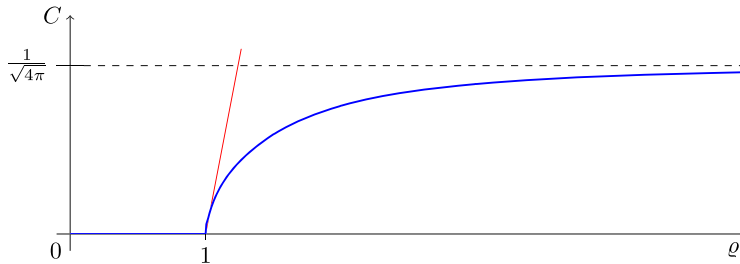
Fig. 1. Monte-Carlo approximation for the function C together with its right derivative at $\varrho = 1$.

computations. Here, using our notation

$$\alpha = 2 \lim_{z \to \infty} \frac{e^{\sqrt{2}z}}{\sqrt{2}z} \lim_{t \to \infty} \mathbf{P}\left(M_t \geq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + z \right). \tag{1.8}$$

The behavior of $C(\varrho)$ as $\varrho \to 1$ is deeply linked to moderate and large deviations of the maximal displacement of the branching Brownian motion, see [17,21,23] for further recent developments on this topic.

The goal of this article is to study both the function $\varrho \mapsto C(\varrho)$ and the family $(\mathfrak{D}^\varrho, \varrho \in (1, \infty])$. We provide a new construction of these quantities, that do not rely on the conditioning on a vanishing event but uses a *spine decomposition*. Recall that a sequence of random point measures $(\mathcal{P}_t)_{t \geq 0}$ on $\mathbb{R}$ converges to $\mathcal{P}$ in law for the topology of vague convergence if and only if, for every compactly supported continuous function φ, the real valued random variables

$$\langle \mathcal{P}_t, \varphi \rangle := \int \varphi(x) \mathcal{P}_t(dx) \tag{1.9}$$

converge in law to $\langle \mathcal{P}, \varphi \rangle$ as $t \to \infty$. We prove in this article that C is continuous on $[1, \infty]$, and that $\varrho \mapsto \mathfrak{D}^\varrho$ is continuous on $(1, \infty]$ for the topology of vague convergence. This can be used to extend the main theorem of [12].

Let $(B_t, t \geq 0)$ be a standard Brownian motion, $(\sigma_k, k \in \mathbb{N})$ be the ranked atoms of a Poisson point process with intensity $2dx$ on $\mathbb{R}_+$ and $(X_t^{(k)}(u), u \in \mathcal{N}_t^{(k)}, t \geq 0)$ for $k \in \mathbb{N}$ be i.i.d. branching Brownian motions. We shall assume that B, $(\sigma_k, k \geq 1)$ and $(X^{(k)}, k \geq 1)$ are independent of one another. Given $\varrho \in (1, \infty)$ and $t \geq 0$, we define the point process

$$\widetilde{\mathcal{D}}^\varrho = \delta_0 + \sum_{k \in \mathbb{N}} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{B_{\sigma_k} - \sqrt{2}\varrho\sigma_k + X_{\sigma_k}^{(k)}(u)}. \tag{1.10}$$

In words, $\widetilde{\mathcal{D}}^\varrho$ is the point process constructed using a Brownian motion with drift $-\sqrt{2}\varrho$, that spawns branching Brownian motions at rate 2. A branching Brownian motion spawned at time σ_k then starts evolving backward in time until it hits time 0, the particles alive at that time are added to the point process.

Theorem 1.1. *Let $C : [1, \infty] \mapsto \mathbb{R}_+$ be the function given by (1.7) and for $\varrho \geq 1$ let $\mathcal{D}^\varrho$ be a random point measure of law $\mathfrak{D}^\varrho$ as defined in (1.6). Then*

(i) $C(\varrho) = \frac{1}{\sqrt{4\pi}} \mathbf{P}(\widetilde{\mathcal{D}}^\varrho((0, \infty)) = 0)$ *for all $\varrho > 1$. The function C is continuous on $[1, \infty]$. It also satisfies $C(1) = 0$, $C(\varrho) > 0$ for $\varrho > 1$ and $C(\infty) = 1/\sqrt{4\pi}$.*

(ii) $\mathbf{P}(\mathcal{D}^\varrho \in \cdot) = \mathbf{P}(\widetilde{\mathcal{D}}^\varrho \in \cdot \mid \widetilde{\mathcal{D}}^\varrho((0, \infty)) = 0)$. *The family of point processes $(\mathcal{D}^\varrho, \varrho \in (1, \infty])$ is continuous in the space of Radon point measures equipped with the topology of vague convergence.*

The rest of the article is organized as follows. In Section 2 we introduce the *spinal decomposition* of the branching Brownian motion, and its application to the extremal process of the branching Brownian motion, seen from the rightmost particle. We then prove Theorem 1.1 in Section 3. Then, as an application of Theorem 1.1, in Section 4 we show how the results of Bovier and Hartung [11,12] about variable speed branching Brownian motion also describe the extremal point process of a branching Ornstein–Uhlenbeck. We conclude this article with some open questions.

2. Spinal decomposition at the maximum

We apply the so-called *spinal decomposition* of the branching Brownian motion to obtain the joint law of the maximum and the extremal process of the branching Brownian motion. The *spinal decomposition* is an alternative description of the

process constructed via the *probability tilting* by the additive martingale $W^{\sqrt{2}\varrho}$, which is defined for all $t \geq 0$ by

$$(2.1) \qquad W_t^{\sqrt{2}\varrho} := \sum_{u \in \mathcal{N}_t} e^{\sqrt{2}\varrho X_t(u) - (\varrho^2 + 1)t}.$$

This idea was already at the heart of the approach taken in [18] to study branching Brownian motions conditioned to have large displacements (i.e. the event $M_t > ct$ with $c > \sqrt{2}$). In particular, they give in [18, Theorem 5] a description of the law of the branching Brownian motion biased by W_t up to the first branching time. Spine decomposition techniques were expended by Lyons, Peamantle and Peres in [28] to study Galton–Watson processes, then generalized to branching random walks by Lyons [27] and to general branching processes in [8].

Let $(\mathcal{F}_t)$ be the natural filtration of the branching Brownian motion, defined by

$$\mathcal{F}_t = \sigma\left(\mathcal{N}_s, \left(X_s(u), u \in \mathcal{N}_s\right), s \leq t\right).$$

For $\varrho \in \mathbb{R}$ and $t \geq 0$, we introduce the size-biased law as

$$(2.2) \qquad \overline{\mathbf{P}}_\varrho|_{\mathcal{F}_t} = W_t^{\sqrt{2}\varrho} \cdot \mathbf{P}|_{\mathcal{F}_t},$$

and call X under $\overline{\mathbf{P}}_\varrho$ the size biased process. The law $\overline{\mathbf{P}}_\varrho$ is identified in [18, Theorem 4] as the limit law conditioning on the maximal displacement begin large at large times, precisely for all $\varrho \geq 1$,

$$\forall s \geq 0, \forall A \in \mathcal{F}_s, \quad \lim_{t \to \infty} \mathbf{P}(A|M_t \geq \sqrt{2}\varrho t) = \overline{\mathbf{P}}_\varrho(A).$$

The spinal decomposition links the size biased process with the so-called *branching Brownian motion with spine*. It describes the evolution of a branching particle system with a distinguished particle ξ_t, which behaves differently from the others. The system starts with the spine particle at position 0. This particle moves according to a Brownian motion with drift $\sqrt{2}\varrho$ and produces children at rate 2. Each of its children starts an independent (standard) branching Brownian motion from its birth place. We shall use the same notation $\mathcal{N}_t$ for the set of particles alive at time t in this process (it is not a Yule process anymore), and write $\xi_t \in \mathcal{N}_t$ for the label of the spine particle. The law of this branching Brownian motion with spine is denoted by $\widehat{\mathbf{P}}_\varrho$. The spinal decomposition can be stated as follows.

Theorem A (Spinal decomposition [6,18]). *For all $\varrho \in \mathbb{R}$, with the above notation we have $\overline{\mathbf{P}}_\varrho|_{\mathcal{F}_t} = \widehat{\mathbf{P}}_\varrho|_{\mathcal{F}_t}$ for all $t \geq 0$. Moreover, for all $u \in \mathcal{N}_t$,*

$$\widehat{\mathbf{P}}_\varrho(\xi_t = u \mid \mathcal{F}_t) = \frac{e^{\sqrt{2}\varrho X_t(u) - t(\varrho^2 + 1)}}{W_t^{\sqrt{2}\varrho}}.$$

In words: the law of the marked tree $((X_s(u), u \in \mathcal{N}_s), s \leq t)$ has same law under probability $\widehat{\mathbf{P}}$ and $\overline{\mathbf{P}}$. Moreover, conditionally on this marked tree, one can choose to distinguish at random an individual $u \in \mathcal{N}_t$ with probability proportional to $e^{\sqrt{2}\varrho X_t(u)}$ to construct the law of the branching Brownian motion with spine.

Using this result, we can describe the joint law of the extremal process and the maximal displacement of the branching Brownian motion.

Lemma 2.1. *Let $\varrho \geq 1$ and $t \geq 0$, we denote by*

$$\mathcal{E}_t^* = \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - M_t}$$

the extremal process of the branching Brownian motion seen from the rightmost individual, and we introduce the point process

$$(2.3) \qquad \widetilde{\mathcal{D}}_t^\varrho = \delta_0 + \sum_{k \in \mathbb{N}: \sigma_k \leq t} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{B_{\sigma_k} - \sqrt{2}\varrho\sigma_k + X_{\sigma_k}^{(k)}(u)},$$

where B is a Brownian motion, $(\sigma_k, k \geq 1)$ are the jump times of a Poisson process with intensity 2 and $(X_s^{(k)}(u), u \in \mathcal{N}_s, s \geq 0)_{k \in \mathbb{N}}$ are i.i.d. branching Brownian motions (see Figure 2). For all non-negative measurable function f, F, we
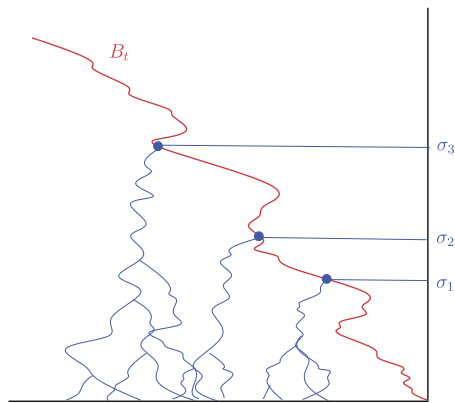
Fig. 2. Construction of the point process $\widetilde{\mathcal{D}}^{\varrho}$.

have

$$\mathbf{E}\big[F\big(\mathcal{E}_t^*\big)f(M_t - \sqrt{2}\varrho t)\big] = \mathrm{e}^{(1-\varrho^2)t}\mathbf{E}\big[\mathrm{e}^{\sqrt{2}\varrho B_t}f(-B_t)F\big(\widetilde{\mathcal{D}}_t^{\varrho}\big)\mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho}((0,\infty))=0\}}\big].$$

Proof. For $t \geq 0$, denote by $u_t^{\mathrm{tip}} \in \mathcal{N}_t$ the label of the largest particle alive at time t (which is a.s. unique). We observe that we can write

$$\mathbf{E}\big[F\big(\mathcal{E}_t^*\big)f(M_t - \sqrt{2}\varrho t)\big] = \mathbf{E}\bigg[\sum_{u \in \mathcal{N}_t} F\big(\mathcal{E}_t^*(u)\big)\mathbf{1}_{\{u=u_t^{\mathrm{tip}}\}}f(M_t - \sqrt{2}\varrho t)\bigg],$$

where $\mathcal{E}_t^*(u) := \sum_{v \in \mathcal{N}_t}\delta_{X_t(v)-X_t(u)}$ is the extremal point measure seen from particle $u \in \mathcal{N}_t$. Thanks to the spinal decomposition and using (2.2), the above reads

$$\mathbf{E}\big[F\big(\mathcal{E}_t^*\big)f(M_t - \sqrt{2}\varrho t)\big] = \overline{\mathbf{E}}_{\varrho}\bigg[\frac{1}{W_t^{\sqrt{2}\varrho}}\sum_{u \in \mathcal{N}_t} F\big(\mathcal{E}_t^*(u)\big)\mathbf{1}_{\{u=u_t^{\mathrm{tip}}\}}f(M_t - \sqrt{2}\varrho t)\bigg]$$

$$= \widehat{\mathbf{E}}_{\varrho}\big[\mathrm{e}^{-\sqrt{2}\varrho X_t(\xi_t)+(\varrho^2+1)t}F\big(\mathcal{E}_t^*(\xi_t)\big)\mathbf{1}_{\{\xi_t=u_t^{\mathrm{tip}}\}}f\big(X_t(\xi_t) - \sqrt{2}\varrho t\big)\big].$$

Next, we use the definition of the branching Brownian motion with spine to rewrite the above expression. For $s \in [0, t]$ let $B_s = Z_{t-s} - Z_t$ where $Z_s = X_t(\xi_s) - \sqrt{2}\varrho s$ and for all $k \in \mathbb{N}$, σ_k is the kth instant at which the spine gives birth to a new particle when running time backward from t (i.e. $t - \sigma_1$ is the last time before t at which the spine branches). Then, under $\widehat{\mathbf{E}}_{\varrho}$, B is a standard Brownian motion and (σ_k) are the atoms of a Poisson point process on $\mathbb{R}_+$ with intensity measure $2\mathrm{d}x$. For each branching event σ_k, the spine gives birth to a standard branching Brownian motion that we call $X^{(k)} \equiv (X_s^{(k)}(u), u \in \mathcal{N}_s^{(k)}; s \in \mathbb{R}_+)$. With these notation we then get

(2.4) $$\mathcal{E}_t^*(\xi_t) = \sum_{k \in \mathbb{N}:\sigma_k \leq t}\sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}}\delta_{B_{\sigma_k}-\sqrt{2}\varrho\sigma_k+X_{\sigma_k}^{(k)}(u)}.$$

All that is left to do is thus to note that under $\widehat{\mathbf{E}}_{\varrho}$, the pair of variables $(\mathcal{E}_t^*(\xi_t), X_t(\xi_t))$ jointly have the same law as $(\widetilde{\mathcal{D}}_t^{\varrho}, (-B_t + \sqrt{2}\varrho t))$ from (2.3). Thus substituting $\mathbf{1}_{\{\xi_t=u_t^{\mathrm{tip}}\}}$ by $\mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho}((0,\infty))=0\}}$ and $f(X_t(\xi_t) - \sqrt{2}\varrho t)$ by $f(-B_t)$ we conclude that

$$\mathbf{E}\big[F\big(\mathcal{E}_t^*\big)f(M_t - \sqrt{2}\varrho t)\big] = \mathrm{e}^{(1-\varrho^2)t}\mathbf{E}\big[\mathrm{e}^{\sqrt{2}\varrho B_t}f(-B_t)F\big(\widetilde{\mathcal{D}}_t^{\varrho}\big)\mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho}((0,\infty))=0\}}\big]. \qquad \square$$

We are now going to show that for all $\varrho > 1$, the point measure $\widetilde{\mathcal{D}}^{\varrho}$ given in (1.10) is well-defined as the increasing limit of $\widetilde{\mathcal{D}}_t^{\varrho}$ as $t \to \infty$. Recall that, with the notation of Lemma 2.1, we have

(2.5) $$\widetilde{\mathcal{D}}^{\varrho} = \delta_0 + \sum_{k \in \mathbb{N}}\sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}}\delta_{B_{\sigma_k}-\sqrt{2}\varrho\sigma_k+X_{\sigma_k}^{(k)}(u)}.$$

Our first step is to prove that $\widetilde{\mathcal{D}}^{\varrho}$ is a well-defined sigma-finite point measure, meaning that for every $a < b \in (-\infty, \infty]$, we have $\widetilde{\mathcal{D}}^{\varrho}([a, b]) < \infty$ a.s..

Lemma 2.2. *For all $\varrho > 1$, $\widetilde{\mathcal{D}}^\varrho$ is a well-defined point measure. Moreover, we have*

$$\lim_{t \to \infty} \widetilde{\mathcal{D}}_t^\varrho = \widetilde{\mathcal{D}}^\varrho \quad a.s. \text{ for the topology of the vague convergence.}$$

Remark 2.3. Note that the above result would not hold for $\varrho = 1$, as in that case one can prove that $\lim_{t \to \infty} \widetilde{\mathcal{D}}_t^1(0, 1) = \infty$ a.s.

Proof. Let $\varrho > 1$, the point measure $\widetilde{\mathcal{D}}^\varrho$ can be rewritten as

$$(2.6) \qquad \widetilde{\mathcal{D}}^\varrho = \delta_0 + \sum_{k \in \mathbb{N}} \sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{(B_{\sigma_k} - \sqrt{2}(\varrho-1)\sigma_k) + (X_{\sigma_k}^{(k)}(u) - \sqrt{2}\sigma_k)}.$$

We observe that $(B_{\sigma_k} - \sqrt{2}(\varrho - 1)\sigma_k, k \geq 0)$ is a random walk with negative drift $-\sqrt{2}(\varrho - 1)/2$. Moreover, for all $k \in \mathbb{N}$ the position of the largest atom in the point measure $\sum_{u \in \mathcal{N}_{\sigma_k}^{(k)}} \delta_{X_{\sigma_k}^{(k)}(u) - \sqrt{2}\sigma_k}$ is, for large values of k, typically around position $-\frac{3}{2\sqrt{2}} \log \sigma_k \approx -\frac{3}{2\sqrt{2}} \log k$. Thus, heuristically, if $\varrho > 1$, the random walk drifts to $-\infty$ at positive speed such that only a finite number of branching Brownian motions put particles in any given compact set. On the other hand, when $\varrho = 1$, the random walk B_{σ_k} has drift zero and we show that it implies that an infinite number of particles are to be found in any finite neighbourhood of 0.

To make the above argument rigorous, we write M_t for the maximal displacement at time t in a branching Brownian motion. Setting $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, It is well-known [2,4] that $(M_t - m_t, t \geq 0)$ is tight and has uniform exponential tails. More precisely, it is proved in [22] (in a much more general settings) there exists $C > 0$ and $\lambda > 0$ such that

$$(2.7) \qquad \mathbf{P}(|M_t - m_t| \geq x) \leq C e^{-\lambda x} \quad \text{for all } t, x > 0.$$

Given $k \in \mathbb{N}$, we denote by $M^{(k)} = \max_{u \in \mathcal{N}_{\sigma_k}^{(k)}} X_{\sigma_k}^{(k)}(u)$ the maximal displacement of $X^{(k)}$ at time σ_k. Using the bounds from (2.7), we observe immediately, using the Borel-Cantelli Lemma and the fact that $\sigma_k \sim_{k \to \infty} k/2$ that, with probability one,

$$(2.8) \qquad \limsup_{k \to \infty} \frac{|M^{(k)} - \sqrt{2}\sigma_k|}{\log k} \leq \frac{3}{2\sqrt{2}} + \lambda^{-1}.$$

In view of (2.8) and the law of large numbers, we deduce that

$$(2.9) \qquad \lim_{k \to \infty} \frac{1}{k}\left(B_{\sigma_k} + M^{(k)} - \sqrt{2}\varrho\sigma_k\right) = -\frac{\sqrt{2}(\varrho - 1)}{2} < 0 \quad \text{a.s.}$$

In particular, it implies that given $A > 0$ one can find a random $T \in \mathbb{R}_+$ such that

$$\forall \sigma_k \geq T, \quad B_{\sigma_k} + M^{(k)} - \sqrt{2}\varrho\sigma_k \leq -A,$$

in which case $\widetilde{\mathcal{D}}^\varrho((x, \infty)) = \widetilde{\mathcal{D}}_t^\varrho((x, \infty))$ for all $t > T$ and $x > -A$. This proves that $\widetilde{\mathcal{D}}^\varrho$ is locally finite a.s. and that $\widetilde{\mathcal{D}}_t^\varrho \nearrow \widetilde{\mathcal{D}}^\varrho$ as $t \to \infty$, as claimed. $\qquad \square$

Next we show the weak continuity of the family $(\widetilde{\mathcal{D}}^\varrho, \varrho > 1)$.

Lemma 2.4. *The family of point processes $(\widetilde{\mathcal{D}}^\varrho, \varrho > 1)$ is a.s. continuous in $\varrho > 1$. Moreover, for all $\varrho > 1$, $\mathbf{P}(\widetilde{\mathcal{D}}^\varrho((0, \infty) = 0) > 0$ and*

$$\lim_{t \to \infty} \mathbf{P}(\widetilde{\mathcal{D}}_t^1((0, \infty)) = 0) = 0 \quad and \quad \lim_{\varrho \to 1} \mathbf{P}(\widetilde{\mathcal{D}}^\varrho((0, \infty)) = 0) = 0.$$

Proof. To prove the a.s. continuity of $(\widetilde{\mathcal{D}}^\varrho, \varrho > 1)$, it is enough to show that for all continuous function φ with compact support, the function $\varrho \mapsto \langle \varphi, \widetilde{\mathcal{D}}^\varrho \rangle$ is continuous a.s. This is a direct consequence of the fact that there are only finitely many atoms in any compact interval, and that the position of these atoms in $\mathcal{D}^\varrho$ are decreasing and continuous with ϱ, by (1.10). Hence, for any $\varrho_0 > 1$, there is only a finite number of atoms to follow as ϱ increases to compute $\varrho \in [\varrho_0, \infty) \mapsto \langle \varphi, \widetilde{\mathcal{D}}^\varrho \rangle$. Hence this function is continuous, which completes the proof of the first statement. For the second statement, it suffices to observe that for $T > 0$ there is positive probability that $\sigma_1 > T$ and that $\widetilde{\mathcal{D}}^\varrho(0, \infty) - \widetilde{\mathcal{D}}_T^\varrho(0, \infty) = 0$.

We now focus on the case $\varrho = 1$. By law of iterated logarithms for the random walk, we have that

$$\limsup_{k \to \infty} k^{-1/2} B_{\sigma_k} = \infty \quad \text{a.s.},$$

which together with (2.8) yields

$$\limsup_{k \to \infty} \frac{B_{\sigma_k} + M^{(k)} - \sqrt{2}\sigma_k}{k^{1/2}} = \infty \quad \text{a.s.}$$

This shows that the event $\{B_{\sigma_k} + M^{(k)} - \sqrt{2}\sigma_k \geq a \text{ infinitely often}\}$ has probability 1 for every $a > 0$. In particular it implies that $\widetilde{\mathcal{D}}_t^1((a, \infty)) \uparrow \infty$ a.s. as $t \to \infty$.

To conclude the proof, we observe that for all $\varepsilon > 0$, there exists $t > 0$ such that $\mathbf{P}(\widetilde{\mathcal{D}}_t^1((0, \infty)) = 0) < \varepsilon$. At the same time it follows from (2.3) that $\widetilde{\mathcal{D}}_t^\varrho$ is continuous in $\varrho \in \mathbb{R}$, hence for all $\varrho > 1$ small enough, we have

$$\mathbf{P}\big(\widetilde{\mathcal{D}}^\varrho\big((0, \infty)\big) = 0\big) \leq \mathbf{P}\big(\widetilde{\mathcal{D}}_t^\varrho\big((0, \infty)\big) = 0\big) \leq 2\varepsilon,$$

which shows that $\lim_{\varrho \to 1} \mathbf{P}(\widetilde{\mathcal{D}}^\varrho((0, \infty)) = 0) = 0$, completing the proof. $\qquad \square$

3. Probabilistic representation of the extremal point process conditioned on a large maximum

In this section, we prove the weak continuity in ϱ of the cluster point process $\mathfrak{D}^\varrho$ as well as the continuity of the function $\varrho \mapsto C(\varrho)$ and their spine representation. To prove this, we show that the cluster law $\mathfrak{D}^\varrho$ and the function C can be computed as continuous functionals of $\widetilde{\mathfrak{D}}^\varrho$ defined in (1.10). This connection is based an application of Lemma 2.1 to the study of the extremal process of the branching Brownian motion conditioned on having a maximum larger than $\sqrt{2}\varrho$. Those results in combination complete the proof of Theorem 1.1.

We begin with the following computation of the extremal process of the branching Brownian motion conditioned on satisfying $\{M_t \geq \sqrt{2}\varrho t\}$.

Lemma 3.1. *Let $\varrho > 1$ and $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ be a continuous function whose support is bounded from the left. Then*

$$\lim_{t \to \infty} \varrho t^{1/2} e^{(\varrho^2 - 1)t} \mathbf{E}\big[e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\big] = \frac{1}{\sqrt{4\pi}} \mathbf{E}\big[e^{-\langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}^\varrho((0,\infty))=0\}}\big].$$

Proof. Fix φ as in the lemma and $\varrho > 1$. Using Lemma 2.1, we may write for $t > 0$

$$e^{(\varrho^2 - 1)t} \mathbf{E}\big[e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\big] = \mathbf{E}\big[e^{\sqrt{2}\varrho B_t} \mathbf{1}_{\{B_t \leq 0\}} e^{-\langle \widetilde{\mathcal{D}}_t^\varrho, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}_t^\varrho((0,\infty))=0\}}\big].$$

We compute the right hand side by first conditioning on $B_t = x$. Introducing the point measure $\widetilde{\mathcal{D}}_t^{\varrho,x}$ as $\widetilde{\mathcal{D}}_t^\varrho$ conditioned on $\{B_t = x\}$, one gets

$$e^{(\varrho^2 - 1)t} \mathbf{E}\big[e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mathbf{1}_{\{M_t \geq \sqrt{2}\varrho t\}}\big] = \int_{-\infty}^0 \frac{dx}{\sqrt{2\pi t}} e^{\sqrt{2}\varrho x - \frac{x^2}{2t}} \mathbf{E}\big[e^{-\langle \widetilde{\mathcal{D}}_t^{\varrho,x}, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho,x}((0,\infty))=0\}}\big].$$

We are going to show that, for any fixed $x < 0$,

$$(3.1) \qquad \lim_{t \to \infty} \mathbf{E}\big[e^{-\langle \widetilde{\mathcal{D}}_t^{\varrho,x}, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}_t^{\varrho,x}((0,\infty))=0\}}\big] = \mathbf{E}\big[e^{-\langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}^\varrho((0,\infty))=0\}}\big]$$

then, the lemma follows by a simple application of the dominated convergence Theorem.

We shall couple the processes $\widetilde{\mathcal{D}}_t^{\varrho,x}$ and $\widetilde{\mathcal{D}}^\varrho$ in such a way, that for any fixed $x \in \mathbb{R}$,

$$(3.2) \qquad \lim_{t \to \infty} \langle \widetilde{\mathcal{D}}_t^{\varrho,x}, \varphi \rangle = \langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle \quad \text{a.s.}$$

Then, this gives (3.1) (and the lemma) by dominated convergence.

Fix $x \in \mathbb{R}$, recall that B is the Brownian underlying the construction of $\widetilde{\mathcal{D}}^\varrho$ and introduce for $0 \le s \le t$

$$\beta_s^{(t)} := B_s + \frac{s}{t}(x - B_t).$$

It is well-known that $(\beta_s^{(t)}; s \in [0,t])$ is a Brownian bridge from $\beta_0^{(t)} = 0$ to $\beta_t^{(t)} = x$.

Almost surely, there exists a random constant C such that

$$|B_s| \le 1 + Cs^{0.51} \quad \text{for all } s \ge 0.$$

Then, with the same constant C, one checks that we have the following uniform bound:

$$(3.3) \qquad \left| \beta_s^{(t)} \right| \le 2 + |x| + Cs^{0.51} + sCt^{-0.49} \quad \text{for all } t \ge 0 \text{ and all } s \in [0, t].$$

Recall that φ has bounded support on the left. Therefore, there exists $a \in \mathbb{R}$ such that $\varphi(x) = 0$ for all $x < a$. Let us fix $0 < \varepsilon < \sqrt{2}(\varrho - 1)$. For all t large enough so that $Ct^{-0.49} < \varepsilon$, observe that

$$\beta_s^{(t)} - \sqrt{2}\varrho s \le -(\sqrt{2}\varrho - \varepsilon)s + 2 + |x| + Cs^{0.51} \quad \text{for all } s \in [0, t].$$

As in the proof of Lemma 2.2, since $\sqrt{2}\varrho - \varepsilon > \sqrt{2}$, we conclude that there exists $T' < \infty$ a.s. such that *uniformly in* t, all the points in $\widetilde{\mathcal{D}}_t^{\varrho,x}$ on the right of a come from branching events on the spine that occurred at times $\sigma_k \le T'$.

Therefore, in computing $\langle \widetilde{\mathcal{D}}_t^{\varrho,x}, \varphi \rangle$, one only needs to consider finitely many points: those that branched from the spine at a time smaller than T'. These points converge, as $t \to \infty$ to the corresponding points in $\widetilde{\mathcal{D}}^\varrho$ (because $\beta_s^{(t)} \to B_s$ as $t \to \infty$) and, as φ is continuous, (3.2) holds and the lemma is proved. □

Using that last result, we now prove Theorem 1.1.

Proof of Theorem 1.1. We recall from (1.7) that for all $\varrho > 1$, we have

$$C(\varrho) = \varrho \lim_{t \to \infty} t^{1/2} e^{(\varrho^2 - 1)t} \mathbf{P}(M_t \ge \sqrt{2}\varrho t).$$

Therefore applying Lemma 3.1 with $\varphi \equiv 0$, we can rewrite $C(\varrho)$ as

$$(3.4) \qquad C(\varrho) = \frac{1}{\sqrt{4\pi}} \mathbf{P}\big(\widetilde{\mathcal{D}}^\varrho\big((0, \infty)\big) = 0\big).$$

We deduce from Lemma 2.4 that C is a continuous function on $[1, \infty)$ such that $C(1) = 0$. Additionally, it can be seen from the proof of [12, Lemma 3.3] that $\lim_{\varrho \to \infty} C(\varrho) = \frac{1}{\sqrt{4\pi}} =: C(\infty)$, which completes the proof of the first part of Theorem 1.1.

We now turn to the proof of the second part. We recall that by the definition (1.6), given $\mathcal{D}^\varrho$ a point process of law $\mathfrak{D}^\varrho$, for all continuous function φ with compact support, we have

$$\mathbf{E}\big[e^{-\langle \mathcal{D}^\varrho, \varphi \rangle}\big] = \lim_{t \to \infty} \mathbf{E}\big[e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mid M_t \ge \sqrt{2}\varrho t\big].$$

At the same time, by Lemma 3.1 we get

$$\lim_{t \to \infty} \mathbf{E}\big[e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mid M_t \ge \sqrt{2}\varrho t\big] = \lim_{t \to \infty} \frac{\mathbf{E}[e^{-\langle \mathcal{E}_t^*, \varphi \rangle} \mathbf{1}_{\{M_t \ge \sqrt{2}\varrho t\}}]}{\mathbf{P}(M_t \ge \sqrt{2}\varrho t)} = \frac{\mathbf{E}[e^{-\langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle} \mathbf{1}_{\{\widetilde{\mathcal{D}}^\varrho((0,\infty))=0\}}]}{\mathbf{P}(\widetilde{\mathcal{D}}^\varrho((0, \infty)) = 0)}.$$

This shows that for all continuous compactly supported function $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$

$$\mathbf{E}\big[e^{-\langle \mathcal{D}^\varrho, \varphi \rangle}\big] = \mathbf{E}\big[e^{-\langle \widetilde{\mathcal{D}}^\varrho, \varphi \rangle} \mid \widetilde{\mathcal{D}}^\varrho\big((0, \infty)\big) = 0\big],$$

proving that $\mathbb{P}(\mathcal{D}^\varrho \in \cdot) = \mathbb{P}(\widetilde{\mathcal{D}}^\varrho \in \cdot \mid \widetilde{\mathcal{D}}^\varrho((0, \infty)) = 0)$. The weak continuity of $\mathfrak{D}^\varrho$ for $\varrho \in (1, \infty)$ follows readily from Lemma 2.2 and the continuity of C. This concludes the proof. □

3.1. *An alternative proof for the first part of Theorem* 1.1

We sketch here an alternative proof for the representation of $C(\varrho)$ in terms of the point processes $\widetilde{\mathcal{D}}^{\varrho}$ defined in (1.10). This proof is based on PDE analysis rather than tight probabilistic estimates, and can thus be of independent interest.

Let M_t be the maximum at time t in a branching Brownian motion, and set $u(x, t) = \mathbf{P}(M_t > x)$ its tail distribution. We recall that u is solution to the Fisher-KPP equation $\partial_t u = \frac{1}{2}\partial_x^2 u + u - u^2$ with step initial condition $u(x, 0) = \mathbf{1}_{\{x < 0\}}$. We can thus compute $C(\varrho)$ from its definition (1.7) using the Feynman–Kac representation to evaluate $\mathbf{P}(M_t > \sqrt{2}\varrho t) = u(\sqrt{2}\varrho t, t)$.

Recall from Feynman–Kac that, given a function $K(x, t)$, the solution to $\partial_t h = \frac{1}{2}\partial_x^2 h + Kh$ can be written as

$$h(x, t) = \mathbf{E}^x\left[h(B_t, 0)\exp\left(\int_0^t ds\, K(B_s, t - s)\right)\right].$$

We apply this not to $u(x, t)$, but to $\partial_x u(x, t)$, the *derivative* of the solution to the Fisher-KPP equation, which is solution to $\partial_t[\partial_x u] = \frac{1}{2}\partial_x^2[\partial_x u] + (1 - 2u)[\partial_x u]$ with initial condition $\partial_x u(x, 0) = -\delta(x)$. This gives

$$\partial_x u(x, t) = -e^t \mathbf{E}^x\left[\delta(B_t)e^{-2\int_0^t ds\, u(B_s, t - s)}\right]$$

$$= -\frac{1}{\sqrt{2\pi t}}e^{t - \frac{x^2}{2t}}\mathbf{E}^{x\to 0}\left[e^{-2\int_0^t ds\, u(B_s, t - s)}\right]$$

where in the last expression B is a Brownian bridge from x to 0. We write $B_s = x(1 - \frac{s}{t}) - \widetilde{B}_{t-s}$, so that $\widetilde{B}$ is a Brownian bridge from 0 to 0, we make the change of variable $\widetilde{s} = t - s$ and we drop the tildas:

$$\partial_x u(x, t) = -\frac{1}{\sqrt{2\pi t}}e^{t - \frac{x^2}{2t}}\mathbf{E}^{0\to 0}\left[e^{-2\int_0^t ds\, u(x\frac{s}{t} - B_s, s)}\right].$$

Then, by setting $x = \sqrt{2}\varrho t + z$ and integrating over $z > 0$, one gets

$$u(\sqrt{2}\varrho t, t) = \frac{e^{(1 - \varrho^2)t}}{\sqrt{2\pi t}}\int_0^\infty dz\, e^{-\sqrt{2}\varrho z - \frac{z^2}{2t}}\mathbf{E}^{0\to 0}\left[e^{-2\int_0^t ds\, u(z\frac{s}{t} + \sqrt{2}\varrho s - B_s, s)}\right].$$

For $\varrho > 1$, the quantity $u(z\frac{s}{t} + \sqrt{2}\varrho s - B_s, s)$ goes exponentially fast to 0 as $s \to \infty$, (unless B has wild fluctuations, but these events have a vanishingly small probability). Then, using the fact that B_s (the value at time s of a Brownian bridge over a time t) looks, as $t \to \infty$ for fixed s, more and more like a Brownian motion at time s, it is not very difficult (and akin to what was done in the proof of Lemma 3.1) to show that

$$\lim_{t\to\infty}\mathbf{E}^{0\to 0}\left[e^{-2\int_0^t ds\, u(z\frac{s}{t} + \sqrt{2}\varrho s - B_s, s)}\right] = \mathbf{E}^0\left[e^{-2\int_0^\infty ds\, u(\sqrt{2}\varrho s - B_s, s)}\right] \quad \text{for } \varrho > 1,$$

where B on the right hand side is a Brownian motion. In fact, the convergence also holds for $\varrho = 1$, as one can check that the quantities on either side are then equal to zero. Then, by dominated convergence,

$$\int_0^\infty dz\, e^{-\sqrt{2}\varrho z - \frac{z^2}{2t}}\mathbf{E}^{0\to 0}\left[e^{-2\int_0^t ds\, u(z\frac{s}{t} + \sqrt{2}\varrho s - B_s, s)}\right]$$

$$\xrightarrow[t\to\infty]{} \frac{1}{\sqrt{2}\varrho}\mathbf{E}^0\left[e^{-2\int_0^\infty ds\, u(\sqrt{2}\varrho s - B_s, s)}\right] \quad \text{for } \varrho \geq 1$$

and

$$C(\varrho) = \frac{1}{\sqrt{4\pi}}\mathbf{E}^0\left[e^{-2\int_0^\infty ds\, u(\sqrt{2}\varrho s - B_s, s)}\right].$$

Observe that in the point process (1.10) the probability that there are no particles on the right of 0 is then

$$\mathbf{P}(\widetilde{\mathcal{D}}^{\varrho}((0, \infty)) = 0) = \mathbf{E}\left[\prod_k[1 - u(\sqrt{2}\varrho\sigma_k - B_{\sigma_k}, \sigma_k)]\right] = \mathbf{E}\left[e^{-2\int_0^\infty ds\, u(\sqrt{2}\varrho s - B_s, s)}\right]$$

and therefore $C(\varrho) = \frac{1}{\sqrt{4\pi}}\mathbf{P}(\widetilde{\mathcal{D}}^{\varrho}((0, \infty)) = 0)$, as claimed.

4. Application to branching Ornstein–Uhlenbeck processes

As an application of Theorem 1.1, we study the asymptotic behavior, as $t \to \infty$, of a branching Ornstein–Uhlenbeck process with a pulling parameter that decay to 0 as $t \to \infty$. The main motivation to study this process is the article of Cortines and Mallein [19], in which it is conjectured that such a process, when undergoing selection, should exhibit unusual behaviour. In particular, the genealogy of these processes could be given by Beta coalescents, a family that interpolates between the Kingman and Bolthausen–Sznitman coalescents. Let us begin by introducing the branching Ornstein–Uhlenbeck process.

An Ornstein–Uhlenbeck process X with spring constant μ is the solution of the stochastic differential equation

$$ \text{(4.1)} \qquad \mathrm{d}X_s^\mu = -\mu X_s^\mu \, \mathrm{d}s + \mathrm{d}B_s, $$

where B is a Brownian motion. It is well-known that Ornstein–Uhlenbeck processes may be represented, if $\mu > 0$, as a space-time scaled Brownian motion: given W a standard Brownian motion, the process defined by

$$ \text{(4.2)} \qquad \forall s \geq 0, \quad X_s^\mu = X_0 \mathrm{e}^{-\mu s} + \frac{\mathrm{e}^{-\mu s}}{\sqrt{2\mu}} W_{\mathrm{e}^{2\mu s}-1}, $$

is an Ornstein–Uhlenbeck with spring constant μ and initial condition X_0. Equation (4.2) shows that, if $\mu > 0$, the law of X_s, conditionally on $\{X_0 = x\}$, is $\mathcal{N}(x\mathrm{e}^{-\mu s}, \frac{1-\mathrm{e}^{-2\mu s}}{2\mu})$. In particular, X_s is then strongly recurrent and its invariant measure is $\mathcal{N}(0, \frac{1}{2\mu})$.

In a branching Ornstein–Uhlenbeck, since the genealogical structure of the process is independent of the motion of the particles, we continue to denote by $\mathcal{N}_t$ the set of particles alive in a branching Ornstein–Uhlenbeck process with spring constant μ and we write $(X_s^\mu(u), u \in \mathcal{N}_s)$ for the positions of such particles. It will be convenient to work with a normalized version $\widehat{X}_s^\mu(u)$ of $X_s^\mu(u)$ that has variance t so that things happen on the same scale as for the branching Brownian motion. This can be easily obtained by setting

$$ \text{(4.3)} \qquad \widehat{X}_s^\mu(u) = \sqrt{\frac{2\mu s}{1 - \mathrm{e}^{-2\mu s}}} X_s^\mu(u). $$

With this notation, we define the extremal point process:

$$ \text{(4.4)} \qquad \mathcal{E}_t^\mu = \sum_{u \in \mathcal{N}_t} \delta_{\widehat{X}_t^\mu(u) - \sqrt{2}t + \frac{1}{2\sqrt{2}} \log t}. $$

Note that here the logarithmic correction is $\frac{1}{2\sqrt{2}}$ instead of $\frac{3}{2\sqrt{2}}$ as in the branching Brownian motion case ($\mu = 0$, see (1.1)). The aim of this section is to study the asymptotic behaviour of $\mathcal{E}_t^\mu$ as $\mu \to 0$ and $t \to \infty$ simultaneously.

Throughout this section, we will choose the spring constant μ as depending on the time-horizon t at which we observe the positions of particles, in the sense that $\mu = \mu_t$ is kept fixed for the evolution of the branching process at all times $s \in [0, t]$. For reasons that will become clear later on, one should choose μ_t such that $\mu_t t \to \gamma \in (0, \infty]$ as $t \to \infty$, which trivially covers the standard case where μ is fixed for all t's.

The particular case $\mu_t = \gamma/t$ for some $\gamma \in (0, \infty)$ is a direct application of the results of Bovier and Hartung [12]. Hence we start by recalling their result.

4.1. *Extremal processes of variable speed branching Brownian motions and of branching Ornstein–Uhlenbeck processes*

For each $\sigma_b \in [0, 1)$ and $\sigma_e > 1$ let $\mathcal{E}_\infty^{\sigma_b, \sigma_e}$ be a decorated Poisson point process defined as

$$ \text{(4.5)} \qquad \mathcal{E}_\infty^{\sigma_b, \sigma_e} := \mathrm{DPPP}\big(\sqrt{2}C(\sigma_e)W_\infty^{\sqrt{2}\sigma_b} \mathrm{e}^{-\sqrt{2}x} \, \mathrm{d}x, \sigma_e \mathfrak{D}^{\sigma_e}\big), $$

with the parameters of the process being described as follows: Let $(X_t(u), u \in \mathcal{N}_t)$ be a branching Brownian motion and M_t its maximal displacement at time t. Then,

- W_∞^β is the limit of the *additive martingale*, previously defined in (2.1). As $(W_t^\beta, t \geq 0)$ is a non-negative martingale, it converges a.s. to a limit W_∞^β. Moreover, it is well known that a.s. $W_\infty^\beta > 0$ if, and only if, $\beta \in (-\sqrt{2}, \sqrt{2})$.

- The function C is the one defined in (1.7).
- The family of laws $(\mathfrak{D}^\varrho, \varrho \geq 1)$ is the family of point processes introduced in (1.6), and $c\mathfrak{D}^\varrho$ is the image measure of $\mathfrak{D}^\varrho$ by the application $\mathcal{D} \mapsto \sum_{d_j \in \mathcal{D}} \delta_{cd_j}$, scaling the positions of the atoms by a factor c.

Let us now introduce the *variable speed branching Brownian motion*. Let $A : [0, 1] \to [0, 1]$ be a twice differentiable increasing function with $A(0) = 0$ and $A(1) = 1$. Then, the variable speed branching Brownian motion with variance profile A and time horizon t is defined in the same way as a branching Brownian motion, except that particles move as Brownian motions with time-dependent variance $\sigma_t^2(s) = A'(s/t)$ where $s \in [0, t]$ is the time of the process. In particular, the position of a particle at time s is a Gaussian random variable with variance $t A(s/t)$.

The main result in [12] is the following:

Theorem B (Bovier and Hartung [12] Theorem 1.2). *Assume that the twice differentiable increasing function A : $[0, 1] \to [0, 1]$ satisfies*

1. $A(0) = 0$, $A(1) = 1$ *and* $A(x) < x$ *for all* $x \in (0, 1)$;
2. $\sigma_b^2 := A'(0) < 1$ *and* $\sigma_e^2 := A'(1) > 1$.

Let $(Y_s(u); u \in \mathcal{N}_s; s \in [0, t])$ *denote the variable speed branching Brownian motion with variance profile A and*

$$\overline{\mathcal{E}}_t^A = \sum_{u \in \mathcal{N}_t} \delta_{Y_t(u) - \sqrt{2}t + \frac{1}{2\sqrt{2}} \log t}$$

be its extremal point measure at time t. Then

(i) *the extremal process $\overline{\mathcal{E}}_t^A$ converges in law for the topology of the vague convergence to $\mathcal{E}_\infty^{\sigma_b, \sigma_e}$.*
(ii) *the maximal displacement of the process converges in law, and for all $x \in \mathbb{R}$,*

$$\lim_{t \to \infty} \mathbf{P}\big(\max \overline{\mathcal{E}}_t^A \leq x\big) = \mathbf{P}\big(\max \mathcal{E}_\infty^{\sigma_b, \sigma_e} \leq x\big).$$

This Theorem is the basis for obtaining a similar result for branching Ornstein–Uhlenbeck processes. More precisely, we will see that the case $\mu_t = \gamma/t$ is a direct consequence and that more generally the case $t\mu_t \to \gamma$ as $t \to \infty$ cane be deduced through comparison arguments. For each $\gamma > 0$, we define two constants, c_γ and d_γ by

$$(4.6) \qquad\qquad c_\gamma := \sqrt{\frac{2\gamma}{e^{2\gamma} - 1}} \quad \text{and} \quad d_\gamma := \sqrt{\frac{2\gamma}{1 - e^{-2\gamma}}}.$$

Now for $\gamma > 0$ let

$$(4.7) \qquad\qquad \mathcal{E}_\infty^\gamma := \mathcal{E}_\infty^{c_\gamma, d_\gamma}.$$

In the $\gamma = \infty$ case, we set $c_\infty = 0$ and $d_\infty = \infty$, thus $W_\infty^{\sqrt{2}c_\infty} = W_\infty^0$ is an exponential random variable with mean 1, the limit of the martingale associated to the Yule process $(\#\mathcal{N}_t, t \geq 0)$. As is shown in [12] (see also Theorem 1.1), $C(d_\infty) = C(\infty) = \frac{1}{\sqrt{4\pi}}$ and a point measure drawn from $\mathfrak{D}^{d_\infty} = \mathfrak{D}^\infty$ is a.s. δ_0.

We prove the following result in the rest of the section.

Theorem 4.1. *Assume that $\lim_{t \to \infty} t\mu_t = \gamma \in (0, \infty]$, then, with the above notations, we have that*

$$\lim_{t \to \infty} \big(\mathcal{E}_t^{\mu_t}, \max \mathcal{E}_t^{\mu_t}\big) = \big(\mathcal{E}_\infty^\gamma, \max \mathcal{E}_\infty^\gamma\big) \quad \text{jointly in law,}$$

where the convergence of the point process is in the sense of the topology of vague convergence.

Remark 4.2. We prove in the forthcoming Lemma 4.4 that the convergence in law of a random point measure (for the topology of vague convergence) jointly with that of its maximum is equivalent to the convergence in law of $\langle \mathcal{P}_t, \varphi \rangle$ to $\langle \mathcal{P}, \varphi \rangle$ for all continuous functions φ with support bounded from the left. This notion of convergence forms a thinner topology on the space of point measures.

Remark 4.3. In the simplest case where $\mu_t = \mu$ is a constant, the theorem with (4.3) and (4.4) implies the following behaviour for the non-normalised positions $X_t^\mu(u)$: the position of the rightmost particle is almost surely given by

$$\max_{u \in \mathcal{N}_t} X_t^\mu(u) = \sqrt{\frac{t}{\mu}} - \frac{\log t}{4\sqrt{\mu t}} + O\left(t^{-1/2}\right),$$

and the next particles are at distance of order $t^{-1/2}$ from the rightmost.

We shall call the case $t\mu_t \to \infty$ the *uncorrelated case*, because the extremal particles have the same distribution as the extremal particles of an i.i.d. sample of Gaussian random variables. Indeed, in this regime, the dilation factor $\sqrt{2\mu_t t / (1 - e^{-2\mu_t t})}$ diverges as $t \to \infty$, which prevents the existence of local correlations (decorations) in the limiting picture.

4.2. *The $\mu_t = \gamma/t$ case*

We start with the proof in the case $\mu_t = \gamma/t$, since it is a direct application of Theorem B.

Proof of Theorem 4.1 in the $\mu_t = \gamma/t$ case. Recall from (4.2) that, an Ornstein–Uhlenbeck $X_s^{\mu_t}$ at time s with spring-constant $\mu_t = \gamma/t$ started from 0 can be written as

$$X_s^{\mu_t} = \frac{e^{-\gamma s/t}}{\sqrt{2\gamma/t}} W_{e^{2\gamma s/t} - 1}.$$

For any $u \in \mathcal{N}_t$, we define $(Y_s(u), s \in [0, t])$ by

$$(4.8) \qquad Y_s(u) = \sqrt{\frac{2\gamma}{e^{2\gamma} - 1}} e^{\gamma s/t} X_s^{\gamma/t}(u).$$

Clearly, $Y_s(u)$ has variance $t \frac{e^{2\gamma s/t} - 1}{e^{2\gamma} - 1}$. It is easily checked that the whole process $(Y_s(u), s \le t)_{u \in \mathcal{N}_t}$ is then a variable speed branching Brownian motion, with variance profile $A(x) := \frac{e^{2\gamma x} - 1}{e^{2\gamma} - 1}$, which is a function satisfying the assumptions of Theorem B with

$$\sigma_b^2 = A'(0) = \frac{2\gamma}{e^{2\gamma} - 1} = c_\gamma^2, \qquad \sigma_e^2 = A'(1) = \frac{2\gamma}{1 - e^{-2\gamma}} = d_\gamma^2.$$

Therefore, the extremal point process $\sum_{u \in \mathcal{N}_t} \delta_{Y_t(u) - \sqrt{2}t + \frac{1}{2\sqrt{2}} \log t}$ converges in distribution as $t \to \infty$ to a

$$\text{DPPP}\left(\sqrt{2} C(\sigma_e) W_\infty^{\sqrt{2}\sigma_b} e^{-\sqrt{2}x} dx, \sigma_e \mathfrak{D}^{\sigma_e}\right),$$

and the maximal atom converges as well. Since $Y_t(u) = \widehat{X}_t^{\gamma/t}(u)$ by (4.3), and using the forthcoming Lemma 4.4, we conclude in the joint convergence $(\mathcal{E}_t^{\gamma/t}, \max \mathcal{E}_t^{\gamma/t})$ toward $(\mathcal{E}_\infty^\gamma, \max \mathcal{E}_\infty^\gamma)$ in law, completing the proof of Theorem 4.1 when $\mu_t = \gamma/t$. □

4.3. *Comparison of extremal processes of branching Ornstein–Uhlenbeck processes with different spring constants*

We use here Slepian-type computations to compare the extremal measures of Gaussian processes with different correlation structures. We begin with a general result on the joint convergence of point measures and their largest atom.

Lemma 4.4. *Let $(\mathcal{P}_t, \mathcal{P}_\infty)$ be point processes on $\mathbb{R}$ such that $\mathcal{P}_\infty((0, \infty)) < \infty$ a.s. The four following statements are equivalent: as $t \to \infty$,*

(i) *$(\mathcal{P}_t, \max \mathcal{P}_t) \to_d (\mathcal{P}_\infty, \max \mathcal{P}_\infty)$ jointly;*
(ii) *$\mathcal{P}_t \to_d \mathcal{P}_\infty$ and $\max \mathcal{P}_t \to_d \max \mathcal{P}_\infty$;*
(iii) *$\mathbf{E}[e^{-\langle \mathcal{P}_t, \varphi \rangle}] \to \mathbf{E}[e^{-\langle \mathcal{P}_\infty, \varphi \rangle}]$ for all continuous function φ with support bounded from the left.*
(iv) *$\mathbf{E}[e^{-\langle \mathcal{P}_t, \varphi \rangle}] \to \mathbf{E}(e^{-\langle \mathcal{P}_\infty, \varphi \rangle})$ for all C^∞ non-decreasing function φ with support bounded from the left and such that for some $a \in \mathbb{R}$, $\varphi(x)$ is constant for $x > a$.*

The proof of this lemma being rather classical and straightforward, we postpone it to the Appendix. A consequence of the above lemma is that to prove Theorem 4.1, it is enough to prove the convergence in distribution of random variables of the form $\langle \mathcal{E}^{\mu_t}, \varphi \rangle$, where φ is a generic non-decreasing bounded function with support bounded from the left.

We now recall that Kahane's theorem is a more general version of Slepian's lemma that allows to compare Gaussian processes with different variances. We refer to [10, Chapter 3.1] for a self-contained proof of Kahane's Theorem.

Theorem C (Kahane's Theorem [24]). *Let $(X_j, j \leq n)$, $(Y_j, j \leq n)$ be two centred Gaussian vectors. Let F be a twice differentiable function on $\mathbb{R}^n$ with bounded second derivatives, that satisfies*

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(x) \geq 0 \quad \text{if } \mathbf{E}(X_i X_j) > \mathbf{E}(Y_i Y_j)$$

$$\text{and} \quad \frac{\partial^2 F}{\partial x_i \partial x_j}(x) \leq 0 \quad \text{if } \mathbf{E}(X_i X_j) < \mathbf{E}(Y_i Y_j).$$

Then we have $\mathbf{E}(F(X)) \geq \mathbf{E}(F(Y))$.

From Kahane's Theorem C, we obtain Lemma 4.5 below, which is useful when comparing the Laplace transform of the extremal point measures of branching Ornstein–Uhlenbeck processes with different spring constants.

Lemma 4.5. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative non-decreasing function. Then, for all $\mu \leq \nu \leq \infty$ and $t > 0$, we have*

$$\mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\mu} \rangle\big)\big] \geq \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\nu} \rangle\big)\big],$$

where $\mathcal{E}_t^{\mu}$ and $\mathcal{E}_t^{\nu}$ are the normalized, centred extremal point measures of branching Ornstein–Uhlenbeck processes as defined in (4.4) when $\nu < \infty$, and $\mathcal{E}_t^{\infty}$ is the point measure defined as

$$\mathcal{E}_t^{\infty} = \sum_{u \in \mathcal{N}_t} \delta_{\widehat{X}_t^{\infty}(u) - \sqrt{2}t + \frac{1}{2\sqrt{2}} \log t},$$

where $(\widehat{X}_t^{\infty}(u), u \in \mathcal{N}_t)$ is a family of i.i.d. centred Gaussian random variables with variance t.

Remark 4.6. Note that as the spring constant μ increases toward ∞, the vector of normalized leaves $(\widehat{X}_t^{\mu}(u), u \in \mathcal{N}_t)$ converges in law toward i.i.d. Gaussian random variables with variance t. This can be checked by computing the covariance function of this vector, conditionally on $\mathcal{N}_t$. Therefore, we have $\lim_{\mu \to \infty} \mathcal{E}_t^{\mu} = \mathcal{E}_t^{\infty}$ in law, for the topology of weak convergence, justifying the notation.

Proof. Remember that, since the branching events are independent of the spatial displacements, one can construct a branching Ornstein–Uhlenbeck with spring constant μ by first drawing its genealogical Yule tree $(\mathcal{N}_s, s \geq 0)$ then, conditionally on $(\mathcal{N}_s, s \geq 0)$ the spatial positions $(X_s^{\mu}(u), u \in \mathcal{N}_s, s \geq 0)$. Thus, given two spring constants μ, ν, we can construct the two branching Ornstein–Uhlenbeck processes X^{μ} and X^{ν} using the same $(\mathcal{N}_s, s \geq 0)$. In the rest of the proof we work conditionally on $(\mathcal{N}_s, s \geq 0)$ to study the extremal processes.

For $u, v \in \mathcal{N}_t$, we denote by $\tau_{u,v}$ the time of the most recent common ancestor of u and v. The covariance matrix of the Gaussian vectors X^{μ} is given by

$$\mathbf{Var}\big(X_t^{\mu}(u)\big) = \frac{1 - e^{-2\mu t}}{2\mu} \quad \text{and} \quad \mathbf{Cov}\big(X_t^{\mu}(u), X_t^{\mu}(v)\big) = e^{-2\mu(t - \tau_{u,v})} \frac{1 - e^{-2\mu \tau_{u,v}}}{2\mu}.$$

Recall that we normalize positions to have variance t, setting as in (4.3)

$$\widehat{X}_t^{\mu}(u) = X_t^{\mu}(u) \sqrt{\frac{2\mu t}{1 - e^{-2\mu t}}}.$$

As a result, we have that

$$\mathbf{Cov}\big(\widehat{X}_t^{\mu}(u), \widehat{X}_t^{\mu}(v)\big) = t \frac{e^{2\mu \tau_{u,v}} - 1}{e^{2\mu t} - 1}. \tag{4.9}$$

Observe that when $\mu \leq \nu$ (including the case $\nu = \infty$), we have that

$$\mathbf{Cov}\big(\widehat{X}_t^\mu(u), \widehat{X}_t^\mu(v)\big) \geq \mathbf{Cov}\big(\widehat{X}_t^\nu(u), \widehat{X}_t^\nu(v)\big),$$

for all $u, v \in \mathcal{N}_t$. Indeed, it is easy to verify that for all $0 < s < t$ fixed the function $\mu \mapsto \frac{e^{2\mu s} - 1}{e^{2\mu t} - 1}$ is non-increasing in $\mu \in \mathbb{R}$.

We start by showing the result for $\varphi : \mathbb{R} \to \mathbb{R}$, a smooth non-negative non-decreasing function, such that φ' has compact support. Then the function

$$F : x \in \mathbb{R}^{\mathcal{N}_t} \mapsto \exp\Big(-\sum_{u \in \mathcal{N}_t} \varphi(x_u)\Big),$$

is twice differentiable and constant outside of a compact, hence its second derivatives are bounded. It satisfies

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(x) = \varphi'(x_i)\varphi'(x_j) \exp\Big(-\sum_{u \in \mathcal{N}_t} \varphi(x_u)\Big) \geq 0, \quad \text{for all } i \neq j \in \mathcal{N}_t,$$

by monotonicity of φ. Thus, we can apply Kahane's Theorem C, and we have that for all $\mu \leq \nu \leq \infty$,

$$\mathbf{E}\big[F\big(\widehat{X}_t^\mu\big) \mid \mathcal{N}_t\big] \geq \mathbf{E}\big[F\big(\widehat{X}_t^\nu\big) \mid \mathcal{N}_t\big].$$

Therefore, averaging over the genealogical tree $(\mathcal{N}_t, t \geq 0)$, we obtain that

$$\mu \in (-\infty, \infty] \mapsto \mathbf{E}\big(\exp\big(-\langle \mathcal{E}_t^\mu, \varphi \rangle\big)\big)$$

is non-increasing.

To conclude, note that any continuous non-decreasing non-negative function φ can be approached from below by a sequence $(\varphi_n, n \geq 1)$ of smooth non-decreasing functions with derivatives having compact support. Moreover,

$$\lim_{n \to \infty} \mathbf{E}\big[\exp\big(-\langle \mathcal{E}_t^\mu, \varphi_n \rangle\big)\big] = \mathbf{E}\big[\exp\big(-\langle \mathcal{E}_t^\mu, \varphi \rangle\big)\big]$$

by monotone convergence. Hence, we conclude that $\mu \mapsto \mathbf{E}(\exp(-\langle \mathcal{E}_t^\mu, \varphi \rangle))$ is non-increasing. $\qquad \square$

4.4. *Proof of Theorem* 4.1

We complete the proof of Theorem 4.1 in this section. We start with the observation that the family of limiting point measures $(\mathcal{E}_\infty^\gamma, \gamma \in (0, \infty])$, defined in (4.7), is continuous in distribution.

Proposition 4.7. *The family* $((\mathcal{E}_\infty^\gamma, \max \mathcal{E}_\infty^\gamma); \gamma \in (0, \infty])$ *is continuous in law. Otherwise said, as per Lemma* 4.4, *for all continuous* $\varphi : \mathbb{R} \to \mathbb{R}_+$ *non-decreasing with bounded support from the left, the function*

$$\gamma \in (0, \infty] \mapsto \mathbf{E}\big[e^{-\langle \mathcal{E}_\infty^\gamma, \varphi \rangle}\big] \quad \text{is continuous.}$$

Proof. Let φ be a continuous non-decreasing function, with support bounded from the left. For any $\gamma > 0$, by Campbell's formula, we have

$$(4.10) \qquad \mathbf{E}\big[e^{-\langle \mathcal{E}_\infty^\gamma, \varphi \rangle}\big] = \mathbf{E}\bigg[\exp\bigg(-\int_{\mathbb{R}} \mathbf{E}\big[1 - e^{-\langle \mathcal{D}^{d_\gamma}, \varphi(d_\gamma \cdot + z) \rangle}\big] \sqrt{2} C(d_\gamma) W_\infty^{\sqrt{2}c_\gamma} e^{-\sqrt{2}z} \, dz\bigg)\bigg].$$

We observe that $C(d_\gamma)$, $W_\infty^{\sqrt{2}c_\gamma}$ as well as $\mathbf{E}[1 - e^{-\langle \mathcal{D}^{d_\gamma}, \varphi(d_\gamma \cdot + z) \rangle}]$ are non-negative for all $\gamma > 0$ and hence the exponential term on the right-hand side of (4.10) is bounded by 1. Therefore, by dominated convergence, it is enough to prove that each of the above functions is continuous.

It is obvious from the definition that both functions $\gamma \mapsto c_\gamma$ and $\gamma \mapsto d_\gamma$ are continuous in γ with $c_\gamma \in (0, 1)$ and $d_\gamma > 1$ for all $\gamma > 0$. At the same time, Theorem 1.1 says that both

$$\gamma \mapsto C(\varrho) \quad \text{and} \quad \varrho \mapsto \mathbf{E}\big[1 - e^{-\langle \mathcal{D}^\varrho, \varphi(\varrho \cdot + z) \rangle}\big]$$

are continuous in $\varrho > 1$, by dominated convergence. Finally, Biggins [7] proved that the convergence of the additive martingale $W^{\sqrt{2\varrho}}$ is uniform on compact subsets of $(-1, 1)$, *i.e.* for all $\varepsilon \in (0, 1)$, we have

$$\lim_{t\to\infty} \sup_{\varrho\in[\varepsilon-1,1-\varepsilon]} \left| W_t^{\sqrt{2\varrho}} - W_\infty^{\sqrt{2\varrho}} \right| = 0 \quad \text{a.s.}$$

As a result, we deduce that $W_\infty^{\sqrt{2\varrho}}$ is continuous in ϱ, completing the proof. □

We now show that the point process $\mathcal{E}_t^\infty$ defined in Lemma 4.5 converges in law, as $t \to \infty$, to the Poisson point process $\mathcal{E}_\infty^\infty$ defined in (4.7), jointly with its maximum. Recall that

$$\mathcal{E}_t^\infty = \sum_{u\in\mathcal{N}_t} \delta_{\widehat{X}_t^\infty(u)-\sqrt{2}t+\frac{1}{2\sqrt{2}}\log t} \quad \text{and} \quad \mathcal{E}_\infty^\infty \sim \mathrm{PPP}\left(\frac{1}{\sqrt{2\pi}} W_\infty^0 e^{-\sqrt{2}x}\, dx\right),$$

where $(\widehat{X}_t^\infty(u), u \in \mathcal{N}_t)$ are i.i.d. centred Gaussian random variables with variance t.

Lemma 4.8. *We have*

$$\lim_{t\to\infty} \left(\mathcal{E}_t^\infty, \max \mathcal{E}_t^\infty\right) = \left(\mathcal{E}_\infty^\infty, \max \mathcal{E}_\infty^\infty\right) \quad \text{in law.}$$

Proof. Note this result can be straightforwardly deduced from standard extreme values theory for Gaussian processes. We include a direct self-contained proof which furthermore demonstrates how our toolbox can be used. Recall from Lemma 4.4 that to prove the joint convergence of $\mathcal{E}_t^\infty$ and its maximum, it is enough to prove the convergence of $\mathbf{E}(\exp(-\langle \mathcal{E}_t^\infty, f\rangle))$ for all non-decreasing continuous functions f with support bounded from the left.

Observe, by Campbell's formula for Poisson point processes, that

$$\mathbf{E}\big[\exp\big(-\langle\mathcal{E}_\infty^\infty, f\rangle\big) | W_\infty^0\big] = \exp\left(-W_\infty^0 \int \big(1 - e^{-f(y)}\big) \frac{\sqrt{2}e^{-\sqrt{2}y}}{\sqrt{4\pi}}\, dy\right).$$

Therefore, as W_∞^0 is distributed as a standard exponential random variable, we have

$$(4.11) \qquad \mathbf{E}\big[\exp\big(-\langle\mathcal{E}_\infty^\infty, f\rangle\big)\big] = \left(1 + \frac{1}{\sqrt{2\pi}} \int \big(1 - e^{-f(y)}\big) e^{-\sqrt{2}y}\, dy\right)^{-1}.$$

On the other hand, conditioning with respect to $\#\mathcal{N}_t$ the number of leaves at time t, and writing X_t for a Gaussian random variable with variance t and $m_t = \sqrt{2}t - \frac{1}{2\sqrt{2}}\log t$, we have

$$\mathbf{E}\big[\exp\big(-\langle\mathcal{E}_t^\infty, f\rangle\big)\big] = \mathbf{E}\big[\mathbf{E}\big(e^{-f(X_t-m_t)}\big)^{\#\mathcal{N}_t}\big].$$

As $\#\mathcal{N}_t$ is a geometric random variable with parameter e^{-t}, we have

$$(4.12) \qquad \begin{aligned} \mathbf{E}\big[\exp\big(-\langle\mathcal{E}_t^\infty, f\rangle\big)\big] &= \frac{e^{-t}\mathbf{E}[e^{-f(X_t-m_t)}]}{1 - (1-e^{-t})\mathbf{E}(e^{-f(X_t-m_t)})} \\ &= \frac{\mathbf{E}]e^{-f(X_t-m_t)}]}{e^t\mathbf{E}[1 - e^{-f(X_t-m_t)}] + \mathbf{E}[e^{-f(X_t-m_t)}]} \end{aligned}$$

Therefore, to complete the proof, it is enough to prove that

$$(4.13) \qquad \lim_{t\to\infty} e^t \mathbf{E}\big[1 - e^{-f(X_t-m_t)}\big] = \frac{1}{\sqrt{2\pi}} \int \big(1 - e^{-f(y)}\big) e^{-\sqrt{2}y}\, dy,$$

which implies that (4.12) converges to (4.11) as $t \to \infty$.

We now turn to the proof of (4.13). By change of variables, we have

$$\begin{aligned} \mathbf{E}\big[1 - e^{-f(X_t-m_t)}\big] &= \int \big(1 - e^{-f(x-m_t)}\big) \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\, dx \\ &= \int \frac{(1 - e^{-f(y)})}{\sqrt{2\pi t}} e^{-\sqrt{2}y} e^{-t+\frac{1}{2}\log t} e^{-\frac{y^2}{2t}+y\frac{\sqrt{2}\log t}{4t}-\frac{(\log t)^2}{16t}}\, dy. \end{aligned}$$

Hence, as the support of $y \mapsto 1 - e^{-f(y)}$ is bounded from the left, we can apply the dominated convergence theorem in the above equation yielding, as $t \to \infty$,

$$\mathbf{E}\big[1 - e^{-f(X_t - m_t)}\big] \sim \frac{e^{-t}}{\sqrt{2\pi}} \int \big(1 - e^{-f(y)}\big) e^{-\sqrt{2}y} \, dy$$

concluding the proof. □

Finally, we use the Kahane estimate to control the branching Ornstein–Uhlenbeck process with pulling strength μ_t by branching Ornstein–Uhlenbeck processes with pulling strength $(\gamma \pm \varepsilon)/t$.

Proof of Theorem 4.1. We denote by $(X_t^{\mu_t}(u), u \in \mathcal{N}_t)$ the positions at time t of a branching Ornstein–Uhlenbeck process with spring constant μ_t (recall that the spring constant μ_t remains constant throughout the process, up to time t) and assume that

$$\lim_{t \to \infty} t\mu_t = \gamma \in (0, \infty].$$

We first consider the case $\gamma < \infty$. Let $0 < \underline{\gamma} < \gamma < \overline{\gamma}$. For t large enough $\underline{\gamma}/t < \mu_t < \overline{\gamma}/t$. Thus, by Lemma 4.5,

$$\mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\underline{\gamma}/t}\rangle\big)\big] \leq \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\mu_t}\rangle\big)\big] \leq \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\overline{\gamma}/t}\rangle\big)\big],$$

for all φ continuous non-decreasing functions $\mathbb{R} \to \mathbb{R}_+$.

As a result, taking $t \to \infty$, and supposing furthermore that φ has bounded support on the left, combining Lemma 4.4 and Theorem B, we obtain that

$$\liminf_{t \to \infty} \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\mu_t}\rangle\big)\big] \geq \mathbf{E}\Big(\exp\big(-\langle \varphi, \mathcal{E}_\infty^{\underline{\gamma}}\rangle\big)\Big]$$

$$\limsup_{t \to \infty} \mathbf{E}[\exp\big(-\langle \varphi, \mathcal{E}_t^{\mu_t}\rangle\big)\big) \leq \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_\infty^{\overline{\gamma}}\rangle\big)\big].$$

Now, letting $\underline{\gamma} \uparrow \gamma$ and $\overline{\gamma} \downarrow \gamma$, using Proposition 4.7 we obtain

$$\lim_{t \to \infty} \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\mu_t}\rangle\big)\big] = \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}^{\gamma}\rangle\big)\big].$$

We conclude by Lemma 4.4 that $(\mathcal{E}_t^{\mu_t}, \max \mathcal{E}_t^{\mu_t})$ converge toward $(\mathcal{E}^{\gamma}, \max \mathcal{E}^{\gamma})$.

We now consider the case $\gamma = \infty$. If $\lim_{t \to \infty} t\mu_t = \infty$, then for all $\underline{\gamma} > 0$, one has $\mu_t \geq \underline{\gamma}/t$ for all t large enough. One the other hand, $(X_t^{\mu_t}(u), u \in \mathcal{N}_t)$ is straightforwardly "more correlated" than i.i.d. Gaussian random variables (formally corresponding to the case $\gamma = \infty$). Hence, using again Lemma 4.5, then Lemma 4.4 and Theorem B for the lower bound, and Lemma 4.8 for the upper bound, we obtain

$$\liminf_{t \to \infty} \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\mu_t}\rangle\big)\big] \geq \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_\infty^{\underline{\gamma}}\rangle\big)\big],$$

$$\limsup_{t \to \infty} \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_t^{\mu_t}\rangle\big)\big] \leq \mathbf{E}\big[\exp\big(-\langle \varphi, \mathcal{E}_\infty^{\infty}\rangle\big)\big]$$

for all smooth increasing function $\varphi : \mathbb{R} \to [0, 1]$ such that φ' has compact support. Letting $\underline{\gamma} \to \infty$ concludes the proof of Theorem 4.1. □

5. Open questions and future work

The cases $t\mu_t \to \gamma \in (0, \infty)$ interpolate between the *uncorrelated case* and the branching Brownian motion regime ($\mu_t = 0$). Notice, though, that the multiplicative factor of the logarithmic correction remains equal to $\frac{1}{2\sqrt{2}}$ (as in the uncorrelated case) and not $\frac{3}{2\sqrt{2}}$ (as in the branching Brownian motion). We believe that there is a second transition when $t\mu \to 0$ where one gradually goes from the $\frac{1}{2\sqrt{2}} \log t$ correction to $\frac{3}{2\sqrt{2}} \log t$ while the decoration measure always is $\mathcal{D}^1$, which is the decoration of the branching Brownian motion.

More precisely, it is predicted in [20] that $C(\varrho) \sim \kappa(\varrho - 1)$ as $\varrho \to 1$, with the same constant κ as in (1.3). Note that $\kappa \approx 1.18$ is also the constant such that $\lim_t \mathbf{P}(M_t \geq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + y) \sim \frac{\kappa}{\sqrt{2}} y e^{-\sqrt{2}y}$, as $y \to \infty$. This constant is proved to exist for all branching random walks in [1, Proposition 4.1]. Note that in [20] the function Φ defined by

$$u(ct, t) \sim \frac{e^{-t(c^2/4-1)}}{\sqrt{4\pi t}} \Phi(c) \quad \text{as } t \to \infty$$

where u is the solution of the Fisher-KPP equation $\partial_t u = \partial_x^2 u + u(1 - u)$ started from the Heavyside initial condition is the analogue of C. The exact correspondence between the functions Φ and C is

$$C(\varrho) = \frac{\varrho}{\sqrt{4\pi}} \Phi(2\varrho).$$

Our factor κ is thus given by the constant denoted 2α in [20] (see Equation (73) there).

On the other hand, we also know from [29], that for the additive martingale W^β

$$\lim_{\beta \to \sqrt{2}-} \frac{W_\infty^\beta}{\sqrt{2} - \beta} = \sqrt{2}Z_\infty,$$

with Z_∞ the limit of the derivative martingale. Since $d_\gamma \simeq 1 + \gamma/2$ and $c_\gamma \simeq 1 - \gamma/2$ when $\gamma \to 0$, we see that

$$C(d_\gamma)W_\infty^{\sqrt{2}c_\gamma} \simeq \frac{\kappa \gamma^2}{2} Z_\infty \quad \text{as } \gamma \to 0.$$

Since $\gamma^2 e^{-\sqrt{2}x} = e^{-\sqrt{2}(x-\sqrt{2}\log\gamma)}$, the extremal point process $\mathcal{E}_\infty^\gamma$ is roughly $\mathcal{E}_\infty$, the centred extremal point process of the standard branching Brownian motion see (1.3), shifted to the left by $\sqrt{2}|\log\gamma| + \mathcal{O}(1)$ (as $\gamma \to 0$). This might suggest that the above-mentioned intermediate logarithmic corrections between $\frac{1}{2\sqrt{2}}$ and $\frac{3}{2\sqrt{2}}$ should appear for $\mu_t = t^{-\alpha}$ with $\alpha \in (1, 3/2)$, and the extremal point measure would be the same as for the branching Brownian motion as soon as $\mu_t = o(t^{-3/2})$. This would complement the recent work [13] on a similar phenomenon for branching Brownian motion with piecewise constant variance.

It may be worth noting that our model is notably different from the one studied by Kiestler and Schmidt [25] which yields a different interpolation between the uncorrelated case and the branching Brownian motion. In that later model, the extremal model is a Poisson point process without decoration, but the logarithmic correction of the median of the maximal displacement interpolates between $-\frac{1}{2\sqrt{2}}$ and $\frac{-3}{2\sqrt{2}}$. On the contrary, in our case, the decoration of the extremal processes interpolate continuously between the absence of decoration of the uncorrelated case and the decoration of the branching Brownian motion. However, the logarithmic correction does not interpolate continuously on the scale of parameters we are considering.

The case $\mu < 0$ is also interesting and is not covered in the present work. Notice that in the case $\mu > 0$ we rely heavily on the results from Bovier and Hartung [12]. However we think that the $\mu < 0$ case corresponds to that of decreasing variances for the variable speed branching Brownian motion for which results concerning the position of the maximum are known (see e.g. Maillard and Zeitouni [30]), but not concerning the full extremal point process.

Appendix: Proof of Lemma 4.4

Proof. Obviously, (i) implies (ii) and (iii) implies (iv). It remains to proove that (ii) implies (iii) and (iv) implies (i).

We start by proving that (ii) implies (iii). First consider the case of a non-negative continuous function φ with support bounded from the left, and introduce for $A \in \mathbb{R}$

$$\varphi^A : x \mapsto \begin{cases} \varphi(x) & \text{if } x < A \\ (A + 1 - x)\varphi(A) & \text{if } x \in [A, A+1] \\ 0 & \text{if } x > A + 1. \end{cases}$$

The function φ^A is continuous compactly supported, hence by (ii) we have

$$\lim_{t \to \infty} \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}\big) = \mathbf{E}\big(e^{-\langle \mathcal{P}_\infty, \varphi^A \rangle}\big).$$

By triangular inequality,

$$\left| \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi \rangle}\big) - \mathbf{E}\big(e^{-\langle \mathcal{P}_\infty, \varphi \rangle}\big) \right| \leq \left| \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi \rangle}\big) - \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}\big) \right|$$
$$+ \left| \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}\big) - \mathbf{E}\big(e^{-\langle \mathcal{P}_\infty, \varphi^A \rangle}\big) \right| + \left| \mathbf{E}\big(e^{-\langle \mathcal{P}_\infty, \varphi^A \rangle}\big) - \mathbf{E}\big(e^{-\langle \mathcal{P}_\infty, \varphi \rangle}\big) \right|.$$

Moreover, as φ is non-negative, we have for all $t \geq 0$ and also for $t = \infty$:

$$\left| \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi \rangle}\big) - \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi^A \rangle}\big) \right| \leq \mathbf{P}(\max \mathcal{P}_t \geq A).$$

Hence, by convergence of $\max \mathcal{P}_t$, we have

$$\limsup_{t \to \infty} \left| \mathbf{E}\big(e^{-\langle \mathcal{P}_t, \varphi \rangle}\big) - \mathbf{E}\big(e^{-\langle \mathcal{P}_\infty, \varphi \rangle}\big) \right| \leq 2\mathbf{P}(\max \mathcal{P}_\infty \geq A).$$

As the right hand side goes to zero as $A \to \infty$, we have proved (iii) for non-negative functions. Now consider an arbitrary continuous function φ with support bounded on the left, and write

$$\varphi = \varphi_+ - \varphi_- \quad \text{where } \varphi_+(x) = \max\big(\varphi(x), 0\big) \text{ and } \varphi_-(x) = \max\big(-\varphi(x), 0\big).$$

Then, for any $\alpha, \beta \geq 0$, the function $\alpha \varphi_+ + \beta \varphi_-$ is continuous non-negative with support bounded on the left and, therefore,

$$\lim_{t \to \infty} \mathbf{E}\big(e^{-\alpha \langle \mathcal{P}_t, \varphi_+ \rangle - \beta \langle \mathcal{P}_t, \varphi_- \rangle}\big) = \mathbf{E}\big(e^{-\alpha \langle \mathcal{P}_\infty, \varphi_+ \rangle - \beta \langle \mathcal{P}_\infty, \varphi_- \rangle}\big).$$

We conclude that $(\langle \mathcal{P}_t, \varphi_+ \rangle, \langle \mathcal{P}_t, \varphi_- \rangle)$ jointly converge in law toward $(\langle \mathcal{P}_\infty, \varphi_+ \rangle, \langle \mathcal{P}_\infty, \varphi_- \rangle)$. Therefore, $\langle \mathcal{P}_t, \varphi \rangle$ converges as well toward $\langle \mathcal{P}_\infty, \varphi \rangle$, which implies that (iii) holds.

We now prove that (iv) implies (i). Let f be a $\mathcal{C}^\infty$ non-decreasing function such that $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 1$. For any $y \in \mathbb{R}$ and $\varepsilon > 0$, we set $f_{\varepsilon, y}(x) = f(\varepsilon^{-1}(x - y))$.

Noting that $f_{\varepsilon, y}(x) \leq \mathbf{1}_{\{x > y\}} \leq f_{\varepsilon, y - \varepsilon}(x)$, we have for all $(y_1, \ldots, y_n) \in \mathbb{R}^n$, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n$ and $\varepsilon > 0$:

$$\mathbf{E}\big(e^{-\sum_i \lambda_i \langle \mathcal{P}_t, f_{\varepsilon, y_i - \varepsilon} \rangle}\big) \leq \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}\big) \leq \mathbf{E}\big(e^{-\sum_i \lambda_i \langle \mathcal{P}_t, f_{\varepsilon, y_i} \rangle}\big).$$

As $t \to \infty$, the two bounds converge by (iv) applied to the functions $\sum_i \lambda_i f_{\varepsilon, y_i}$ and $\sum_i \lambda_i f_{\varepsilon, y_i - \varepsilon}$

$$\mathbf{E}\big(e^{-\sum_i \lambda_i \langle \mathcal{P}_\infty, f_{\varepsilon, y_i - \varepsilon} \rangle}\big) \leq \liminf_{t \to \infty} \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}\big)$$
$$\leq \limsup_{t \to \infty} \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}\big) \leq \mathbf{E}\big(e^{-\sum_i \lambda_i \langle \mathcal{P}_\infty, f_{\varepsilon, y_i} \rangle}\big).$$

Note that $f_{\varepsilon, y}(x) \to \mathbf{1}_{\{x > y\}}$ and $f_{\varepsilon, y - \varepsilon}(x) \to \mathbf{1}_{\{x \geq y\}}$ as $\varepsilon \to 0$. Hence one gets

(A.1)
$$\mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_\infty([y_i, \infty))}\big) \leq \liminf_{t \to \infty} \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}\big)$$
$$\leq \limsup_{t \to \infty} \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))}\big) \leq \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_\infty((y_i, \infty))}\big).$$

We conclude that $(\mathcal{P}_t((y_i, \infty)), i \leq n)$ jointly converge in law to $(\mathcal{P}_\infty((y_i, \infty)), i \leq n)$ as $t \to \infty$, except at discontinuity points y_i where $\mathcal{P}_\infty(\{y_i\}) > 0$ with positive probability. Hence, $\mathcal{P}_t$ converges in law to $\mathcal{P}_\infty$ for the topology of vague convergence.

In (A.1), add one extra pair (λ, y) to the λ_i, y_i, and send λ to infinity. Noticing that for $A \geq 0$ that

$$\mathbf{E}(A\mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}) \leq \mathbf{E}\big(Ae^{-\lambda \mathcal{P}_t((y, \infty))}\big) \leq \mathbf{E}(A\mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}) + e^{-\lambda} \mathbf{E}(A),$$

one gets

$$\mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_\infty([y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_\infty \leq y\}}\big) \leq \liminf_{t \to \infty} \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}\big)$$
$$\leq \limsup_{t \to \infty} \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_t((y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_t \leq y\}}\big)$$
$$\leq \mathbf{E}\big(e^{-\sum_i \lambda_i \mathcal{P}_\infty((y_i, \infty))} \mathbf{1}_{\{\max \mathcal{P}_\infty \leq y\}}\big).$$

Hence $(\mathcal{P}_t, \max \mathcal{P}_t)$ converges to $(\mathbf{P}_\infty, \max \mathcal{P}_\infty)$ in law jointly. $\qquad \square$

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