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Exact solution and precise asymptotics of a Fisher–KPP type front

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Abstract
The present work concerns a version of the Fisher–KPP equation where the nonlinear term is replaced by a saturation mechanism, yielding a free boundary problem with mixed conditions. Following an idea proposed in Brunet and Derrida (2015 *J. Stat. Phys.* **161** 801), we show that the Laplace transform of the initial condition is directly related to some functional of the front position μ_t. We then obtain precise asymptotics of the front position by means of singularity analysis. In particular, we recover the so-called Ebert and van Saarloos correction (Ebert and van Saarloos 2000 *Physica* D **146** 1), we obtain an additional term of order $\log t/t$ in this expansion, and we give precise conditions on the initial condition for those terms to be present.

Keywords: front propagation, Fisher–KPP equation, partial differential equations, free boundary problem, velocity selection

1. Introduction

Since the seminal works of Fisher [3] and Kolmogorov, Petrovsky, Piscounov [4], travelling wave equations such as the Fisher–KPP equation

$$\partial_t H = \partial_x^2 H + H - H^2 \qquad \text{(Fisher–KPP)}, \tag{1}$$

have attracted uninterrupted attention [2, 5–7]. Although a lot is understood on the long time behaviour of these equations, explicit calculations are usually difficult because of the non-linearities. Recently [1, 8, 9], it has been shown that some front equations with the non-linearities replaced by a saturation at some moving boundary exhibit the same long time asymptotic as the Fisher–KPP equation (1).

In this paper, we study such a linearised Fisher–KPP equation which generalises [8, 9], and we analyse it by extending to the continuum the approach developed in [1] for a lattice version of the problem. We consider the joint evolution

$$\begin{cases} \partial_t h = \partial_x^2 h + h & \text{if } x > \mu_t, \\ h(\mu_t, t) = \alpha, \\ \partial_x h(\mu_t, t) = \beta, \end{cases} \tag{2}$$

of a boundary μ_t and of a function $h(x,t)$ defined for $x > \mu_t$ with a given initial condition $h_0(x)$ defined for $x \geqslant 0$. The parameters α and β are fixed. This is a free boundary problem [10–13] where both $h(x,t)$ and μ_t are unknown quantities to be determined. We limit our discussion to

$$\begin{cases} \alpha > 0, \\ \beta \in \mathbb{R}, \end{cases} \quad \text{or} \quad \begin{cases} \alpha = 0, \\ \beta > 0, \end{cases} \tag{3}$$

because the other cases can be obtained by changing the sign of h_0. By analogy with the Fisher–KPP equation where most studies focus on positive fronts, we also assume that

$$h_0(x) \geqslant 0. \tag{4}$$

In this case, the solution $h(x,t)$, whenever it exists, remains positive for all $t > 0$ and $x > \mu_t$.

One way to think about (2) is to first choose *a priori* a smooth boundary $t \mapsto \mu_t$ with $\mu_0 = 0$, and then to solve the system $\left\{\partial_t h = \partial_x^2 h + h \text{ if } x > \mu_t \text{ and } h(\mu_t, t) = \alpha\right\}$ with initial condition h_0, as in [8, 9]. Then comes the main difference from [8, 9]: out of all the possible choices for the trajectory $t \mapsto \mu_t$ of the boundary, we select (whenever it exists and is unique) the one for which the solution h satisfies $\partial_x h(\mu_t, t) = \beta$ at all times $t > 0$.

Whether there exists a solution (μ_t, h) to (2) for an arbitrary initial condition h_0 and whether such a solution is unique are not easy questions (see discussion in section 4). However, it is easy to show (see section 2) that (2) admits, for v large enough, positive travelling wave solutions, i.e. solutions of the form $\mu_t = vt$ and $h(x,t) = \omega_v(x - vt)$ for $x > vt$, where $\omega_v(z) \geqslant 0$ and $\omega_v(\infty) = 0$. For suitable choices of h_0, one could expect the solution to (2) to converge to one of these positive travelling waves, in a way similar to the Fisher–KPP equation, in the sense that

$$h(\mu_t + z, t) \underset{t\to\infty}{\longrightarrow} \omega_v(z) \quad \text{for } z > 0, \qquad \frac{\mu_t}{t} \to v. \tag{5}$$

This, however, is not always the case. For example, we show in section 4 that (5) does not hold in the $(\alpha = 0,\ \beta = 1)$ case when $\int \mathrm{d}z\, h_0(z) \neq 1$. See also [14] for an example in the Fisher–KPP case.

The key of our approach in the present paper is the following exact relation between μ_t and the initial condition h_0 : when (5) holds, for any r such that both sides converge,

$$\int_0^\infty \mathrm{d}z\, h_0(z) \mathrm{e}^{rz} = -\frac{\alpha}{r} + \left(\beta + \frac{\alpha}{r}\right) \int_0^\infty \mathrm{d}t\, \mathrm{e}^{r\mu_t - (1+r^2)t}. \tag{6}$$

This equation is a continuous version of a result obtained for a system defined on the lattice [1].

As in [1], one can use (6) to analyse how the large time asymptotics of μ_t depends on the initial condition h_0. For $\alpha + \beta > 0$ and when (5) holds we obtain that

$$\mu_t = 2t - \frac{3}{2}\log t + \text{Cst} + o(1) \qquad \text{iff} \int_0^\infty \mathrm{d}x\, h_0(x) x \mathrm{e}^x < \infty, \quad (7a)$$

$$\mu_t = 2t - \frac{3}{2}\log t + \text{Cst} - 3\frac{\sqrt{\pi}}{\sqrt{t}} + o(t^{-1/2}) \qquad \text{if} \int_0^\infty \mathrm{d}x\, h_0(x) x^2 \mathrm{e}^x < \infty, \quad (7b)$$

$$\mu_t = 2t - \frac{3}{2}\log t + \text{Cst} - 3\frac{\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + \mathcal{O}\Big(\frac{1}{t}\Big) \text{ if} \int_0^\infty \mathrm{d}x\, h_0(x) x^3 \mathrm{e}^x < \infty. \quad (7c)$$

(Here, and everywhere in this paper, 'Cst' stands for some constant term.) Observe that (7*a*) coincides with Bramson's result for the position of a Fisher–KPP front [5, 6], and that (7*b*) reproduces the prediction of Ebert and van Saarloos (recently proved in [15] for compactly supported initial conditions). This raises the question of whether the $\frac{\log t}{t}$ term in (7*c*) should be present for other Fisher–KPP like equations. The sufficient conditions on h_0 in (7*b*) and (7*c*) are close to be necessary; the precise necessary conditions are given in section 5.

More detailed asymptotics of μ_t for other initial conditions are given in section 5. In particular, we argue that when $\alpha + \beta < 0$, the solution to (2) behaves as a 'pushed front' [16] rather than a Fisher–KPP or 'pulled' front when $\alpha + \beta > 0$. (The case $\alpha + \beta = 0$ would correspond to some critical situation between the 'pulled' and 'pushed' cases.)

The long time asymptotics (7) are the same as in the lattice version considered in [1]:

$$\partial_t h(n,t) = \begin{cases} h(n,t) + h(n-1,t) & \text{if } h(n,t) < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The approach used in [1] relied on a relation similar to (6) which was, however, more complicated and limited to the 'pulled' case.

The particular ($\alpha = 0$, $\beta = 1$) case is related to a problem discussed in the mathematical literature [8, 9], where the question was how to choose μ_t for a given h_0 in such a way that the solution to

$$\begin{cases} \partial_t h = \partial_x^2 h + h & \text{if } x > \mu_t, \\ h(\mu_t, t) = 0, \end{cases} \tag{9}$$

converges to a travelling wave: $h(\mu_t + z, t) \to \omega_v(z) > 0$. It was shown that (7*a*) must hold for a fast decaying initial condition h_0 (see also [17]). By requiring furthermore that the convergence of $h(\mu_t + z, t)$ to $\omega_v(z)$ is fast, [8, 9] obtain results compatible with (7*b*).

A linear equation with a moving Dirichlet condition similar to (2) is expected to appear as the hydrodynamic limit of the N-BBM: consider a branching Brownian motion where N particles diffuse and branch independently. At each branching event, the leftmost particle is removed so that the population size N is kept constant. Call μ_t the position of the leftmost particle and $h(x,t)$ the empirical density of particles divided by N. Then, as $N \to \infty$, the evolution of h and μ_t becomes deterministic and satisfy [18, 19] the following system of equations:

$$\begin{cases} \partial_t h = \partial_x^2 h + h & \text{if } x > \mu_t, \\ h(\mu_t, t) = 0, \\ \int_{\mu_t}^\infty \mathrm{d}x\, h(x,t) = 1, \end{cases} \tag{10}$$

where both h and μ_t are unknown quantities to be solved for. (See also [20] who obtain the analogue of (10) for a specific N-branching random walk.) By differentiating the last line of

(10) with respect to t, one gets $0 = -h(\mu_t, t) + \int_{\mu_t}^{\infty} \mathrm{d}x\,(\partial_x^2 h + h) = -h(\mu_t, t) - \partial_x h(\mu_t, t) + \int_{\mu_t}^{\infty} \mathrm{d}x\, h$. With the conditions in (10), one obtains that $\partial_x h(\mu_t, t) = 1$ for all t. Therefore, (10) reduces to our problem (2) with $(\alpha = 0, \beta = 1)$.

The structure of this paper is the following:

- In section 2, we write the travelling wave solutions to (2) and discuss their similarity with the travelling waves of the Fisher–KPP equation when $\alpha + \beta > 0$.
- In section 3, we establish the main relation (6), on which the analysis of the long time asymptotics of μ_t are based.
- In section 4, we discuss for which initial conditions one can expect a travelling wave solution (5).
- In section 5 we obtain, from a singularity analysis of (6), precise asymptotics for the position μ_t of the front in the 'pulled' case $(\alpha + \beta > 0)$, in the 'pushed' case $(\alpha + \beta < 0)$ and in the 'critical' case $(\alpha + \beta = 0)$.
- In section 6, we briefly describe the long time behaviour of $h(x, t)$ and of μ_t when the front does not converge to a travelling wave. This allows to recover a recent result on a self-consistent method to find the typical position of the rightmost particle in a BBM [21].

2. Travelling waves

It is easy to determine all travelling wave solutions of (2). Writing

$$\mu_t = vt, \qquad h(x, t) = \omega_v(x - vt) \quad \text{if } x > vt, \tag{11}$$

one finds that ω_v satisfies

$$\omega_v'' + v\omega_v' + \omega_v = 0 \quad \text{for } z > 0, \qquad \omega_v(0) = \alpha, \qquad \omega_v'(0) = \beta. \tag{12}$$

With the extra condition that $\omega_v(z)$ goes to zero as $z \to +\infty$, a solution only exists for $v > 0$. Writing

$$v = \gamma + \frac{1}{\gamma}, \tag{13}$$

this gives

$$\omega_v(z) = \frac{(\alpha + \beta\gamma)\mathrm{e}^{-\gamma z} - \gamma(\alpha\gamma + \beta)\mathrm{e}^{-\frac{1}{\gamma}z}}{1 - \gamma^2} \text{ for } v \neq 2, \qquad \omega_2(z) = [\alpha + (\alpha + \beta)z]\mathrm{e}^{-z}. \tag{14}$$

For $v < 2$, the exponential rates γ and γ^{-1} are complex conjugates, and the travelling wave decays to zero with oscillations around zero. Such a travelling wave cannot be reached by a non-negative initial condition.

For $v > 2$, the exponential rates γ and γ^{-1} are real, one smaller than 1 and the other larger than 1. We always choose γ such that $0 < \gamma \leqslant 1 \leqslant \gamma^{-1}$.

For $v = 2$, the exponential decay rate is $\gamma = 1$, with a z in the prefactor.

Recalling (3), there are three subcases to be considered:

- If $\alpha + \beta > 0$. All the travelling waves (14) for $v \geqslant 2$ remain positive for all $z > 0$. They decay like $\mathrm{e}^{-\gamma z}$ as $z \to \infty$ for $v > 2$ and like $z\mathrm{e}^{-z}$ for $v = 2$. This is very similar to what is known for the Fisher–KPP case, which is often called the 'pulled' case [16].

- If $\alpha > 0$ and $\alpha + \beta < 0$. The travelling waves remain positive if and only if $v \geqslant v_*$ where
$$v_* = \gamma_* + \frac{1}{\gamma_*} \quad \text{with } \gamma_* = \frac{\alpha}{-\beta} \in (0, 1). \tag{15}$$
For $v > v_*$, the travelling waves decay like $A\mathrm{e}^{-\gamma z}$ with $A > 0$. For $v \in (2, v_*)$ they also decay like $A\mathrm{e}^{-\gamma z}$, but with $A < 0$. For $v = v_*$, the travelling wave is simply equal to $\alpha \exp(-\gamma_*^{-1} z)$; the decay is much faster than for any other velocity. This situation is sometimes called the 'pushed case'. [16]
- If $\alpha > 0$ and $\alpha + \beta = 0$. This case is critical between the pushed and the pulled case. All the travelling waves for $v \geqslant 2$ are positive, but the travelling wave for $v = 2$ decays as e^{-z} instead of $z\mathrm{e}^{-z}$.

3. Derivation of the main relation (6)

In this section, we establish the relation (6) between the initial condition h_0 and the position μ_t of the solution to (2). With $h(x, t)$ the solution to (2), let us introduce
$$g(r, t) = \int_0^\infty \mathrm{d}z\, h(\mu_t + z, t)\mathrm{e}^{rz}, \tag{16}$$
where r is real small enough for the integral to converge. (r can be negative if needed. We do not discuss here initial conditions h_0 that increase so fast that no such r exists.) Differentiating with respect to t and replacing $\partial_t h$ by its expression $\partial_t h = \partial_x^2 h + h$, one gets
$$\partial_t g(r, t) = \int_0^\infty \mathrm{d}z \left[\dot{\mu}_t \partial_x h(\mu_t + z, t) + \partial_x^2 h(\mu_t + z, t) + h(\mu_t + z, t)\right]\mathrm{e}^{rz}. \tag{17}$$
Integration by parts with $h(\mu_t, t) = \alpha$ and $\partial_x h(\mu_t, t) = \beta$ yields
$$\int_0^\infty \mathrm{d}z\, \partial_x h(\mu_t + z, t)\mathrm{e}^{rz} = -\alpha - rg(r, t), \qquad \int_0^\infty \mathrm{d}z\, \partial_x^2 h(\mu_t + z, t)\mathrm{e}^{rz} = \alpha r + r^2 g(r, t) - \beta. \tag{18}$$
Then
$$\partial_t g(r, t) = \left[r^2 - r\dot{\mu}_t + 1\right] g(r, t) + \alpha r - \beta - \alpha\dot{\mu}_t. \tag{19}$$
This can of course be integrated:
$$g(r, t) = -\frac{\alpha}{r} + \left[C_r - \left(\beta + \frac{\alpha}{r}\right)\int_0^t \mathrm{d}s\, \mathrm{e}^{-(r^2+1)s + r\mu_s}\right]\mathrm{e}^{(r^2+1)t - r\mu_t}, \tag{20}$$
with C_r a constant of integration. By taking the limit $t \to 0^+$,
$$C_r = \frac{\alpha}{r} + \int_0^\infty \mathrm{d}z\, h_0(z)\mathrm{e}^{rz}. \tag{21}$$

Up to now, we made no assumption on the long time behaviour of the solution h of (2), and (20) with (21) is always valid as long as the solution exists. From now on, we restrict ourselves to the case where the solution of (2) converges to a travelling wave with some velocity $v = \lim_{t\to\infty} \frac{\mu_t}{t}$, as in (5). As in section 2, using that $v \geqslant 2$, we write
$$v = \gamma + \frac{1}{\gamma} \quad \text{with } \gamma \in (0, 1], \tag{22}$$

It is then clear that $\exp\left[(r^2+1)t - r\mu_t\right]$ in (20) diverges as $t\to\infty$ for $r<\gamma$ or $r>\gamma^{-1}$, and goes to zero for $r\in(\gamma,\gamma^{-1})$. Furthermore, the left hand side $g(r,t)$ of (20) converges when $t\to\infty$ to $\int \mathrm{d}z\,\omega_v(z)\mathrm{e}^{rz}$ which is finite when $r<\gamma$, as can be checked with (14) from the large z behaviour of the travelling waves ω_v.

Then, for $r<\gamma$, the left hand side of (20) converges and the outer exponential in the right hand side diverges, so the expression in square brackets must vanish in the $t\to\infty$ limit:

$$C_r = \left(\beta + \frac{\alpha}{r}\right)\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-(r^2+1)s + r\mu_s} \qquad \text{for } r<\gamma. \tag{23}$$

Combining (21) and (23) leads to our main relation (6).

A similar equation was obtained in [22] for the Stefan problem, which is another free boundary problem.

4. Some remarks on the solutions to (2)

In this section, we give some conditions on h_0 for the solution of (2) to converge to a travelling wave.

By an explicit construction of the solution, we first argue that in the ($\alpha=1$, $\beta=0$) case, the solution to (2) converges to a travelling wave for a large class of initial conditions h_0; this class includes all the decreasing functions smaller than 1 which decay exponentially fast at infinity. In fact, as explained in section 4.1, the construction works as long as $h(x,t)$ remains below 1 for all $t>0$ and $x>\mu_t$.

Then, we show in section 4.2 that one can relate the solution of (2) for arbitrary (α, β) and initial condition h_0 to a solution of the same problem (2) but with parameters ($\alpha=1,\beta=0$) and an initial condition η_0 computed from h_0. This mapping leads to a necessary condition for the front to converge to a travelling wave:

$$\beta \leqslant 0 \qquad \text{or} \qquad \int_0^\infty \mathrm{d}z\,h_0(z)\mathrm{e}^{-\frac{\alpha}{\beta}z} = \beta. \tag{24}$$

It turns out, in the ($\alpha=0$, $\beta>0$) case, that (24) is also sufficient as η_0 is then decreasing, as shown below in (30). On the other hand, when (24) fails, the solution cannot converge to a travelling wave; consider $g(-\alpha/\beta,t)$ from (20), with the value of $C_{r=-\alpha/\beta}$ given by (21):

$$g(-\alpha/\beta,t) = \beta + \left(\int_0^\infty \mathrm{d}z\,h_0(z)\mathrm{e}^{-\frac{\alpha}{\beta}z} - \beta\right)\mathrm{e}^{\left[\left(\frac{\alpha}{\beta}\right)^2+1\right]t + \frac{\alpha}{\beta}\mu_t}. \tag{25}$$

When (24) is not satisfied, $\beta>0$ and the expression in parenthesis is non zero. Then, if the front were to reach a travelling wave, the exponential factor in the right-hand-side of (25) would diverge, in contradiction with the fact that the left-hand-side would converge to $\int \mathrm{d}z\,\omega_v(z)\mathrm{e}^{-(\alpha/\beta)z}$, as can be seen from the definition (16) of g.

Finally in section 4.3, we analyse how the asymptotic decay of h_0 determines that of η_0. In certain regimes, η_0 decays more slowly than h_0 and this shows that (2) does no longer behave as the usual KPP front.

The mapping of section 4.2 gives some insight on the unicity of the solutions to (2). Consider two given boundaries $t\mapsto\mu_t^+$ and $t\mapsto\mu_t^-$ and let h^+ and h^- be the solutions to $\partial_t h^\pm = \partial_x^2 h^\pm + h^\pm$ for $x>\mu_t^\pm$ with $h^\pm(\mu_t^\pm,t)=0$ and $h^\pm(x,0)=h_0(x)$ given. Then, if $\mu_s^+ \geqslant \mu_s^-$ for all $s\leqslant t$ with a strict equality on some interval, it is clear that $\int_{\mu_t^+}^\infty \mathrm{d}x\,h^+(x,t) < \int_{\mu_t^-}^\infty \mathrm{d}x\,h^-(x,t)$. This suggests that the solution to (10) is unique.

(There exists a rigourous proof in some cases, see [19].) But (10) is the same as (2) with ($\alpha = 0, \beta = 1$), and through the mapping of section 4.2, the solution to the general (α, β) case must be unique.

4.1. The case ($\alpha = 1, \beta = 0$)

Let us show how to construct the solution to (2) with parameters ($\alpha = 1, \beta = 0$) and an initial condition $h_0(x)$ defined for $x > 0$. As usual, we assume that $h_0 \in [0, 1]$ decays exponentially fast.

For $n > 1$, we introduce $H_n(x, t)$ as the solution to

$$\partial_t H_n = \partial_x^2 H_n + H_n - H_n^n, \qquad H_n(x, 0) = \begin{cases} h_0(x) & \text{if } x \geqslant 0, \\ 1 & \text{if } x < 0. \end{cases} \tag{26}$$

By standard comparison principle, one has for any x and t,

$$0 \leqslant H_n(x, t) \leqslant H_m(x, t) \leqslant 1 \quad \text{if } n \leqslant m. \tag{27}$$

One concludes that, for fixed x and t, the large n limit of H_n exists, and we define

$$H(x, t) := \lim_{n\to\infty} H_n(x, t). \tag{28}$$

Clearly, $0 \leqslant H(x, t) \leqslant 1$.

Assume for now that h_0 is a decreasing function of x. By standard results on Fisher–KPP equations [6], all of the $H_n(x, t)$ are non-increasing in x, and so is H after taking the limit. Therefore there exists a μ_t such that

$$\begin{cases} H(x, t) = 1 & \text{if } x \leqslant \mu_t \\ H(x, t) < 1 & \text{if } x > \mu_t. \end{cases} \tag{29}$$

The position μ_t above must be finite for all $t > 0$. Indeed, the function H cannot be uniformly equal to 1 since it must be smaller than the solution L to the linearised equation $\partial_t L = \partial_x^2 L + L$ with the same initial condition, and L is smaller than 1 for x large enough. Similarly, H cannot be everywhere smaller than 1: if it were, H would be equal to L, and this is impossible because $L > 1$ for x negative enough.

The couple (μ_t, H) is thus the solution to the system (2) with parameters ($\alpha = 1, \beta = 0$) and initial condition h_0. The condition that h_0 is decreasing is convenient, but by no mean necessary; what really matters to identify H with the solution h to (2) is that (29) holds.

By standard Fisher–KPP theory [6], each of the H_n in (26) converges as $t \to \infty$ to some travelling wave with a velocity v which depends on h_0 but not on n. This v is also the velocity of the front described by the linearised equation $\partial_t L = \partial_x^2 L + L$. The bounds $H_n(x, t) \leqslant H(x, t) \leqslant L(x, t)$ thus yield that asymptotically the front H must also have the velocity v. Although this does not directly yields the convergence towards a travelling wave, it is nevertheless a very strong indication that such a convergence holds. Indeed, establishing the asymptotic velocity is usually the first step of proving the convergence to a travelling wave in many reaction diffusion equations, see for instance the celebrated result of Aronson and Weinberger [23] for the Fisher–KPP equation.

4.2. Mapping the general (α, β) case into the ($\alpha = 1, \beta = 0$) case

We present a procedure to transform the general (α, β) case of (2) with initial condition h_0 into the ($\alpha = 1, \beta = 0$) one. We start by defining

$$\eta_0(x) = \begin{cases} e^{\frac{\alpha}{\beta}x}\left[1 - \frac{1}{\beta}\int_0^x dz\, e^{-\frac{\alpha}{\beta}z} h_0(z)\right] & \text{if } \beta \neq 0, \\ h_0(x)/\alpha & \text{if } \beta = 0. \end{cases} \tag{30}$$

Let $\eta(x,t)$ be the solution to (2) with ($\alpha = 1$, $\beta = 0$) and initial condition $\eta_0(x)$:

$$\partial_t \eta = \partial_x^2 \eta + \eta \text{ if } x > \mu_t, \qquad \eta(\mu_t, t) = 1, \qquad \partial_x \eta(\mu_t, t) = 0, \qquad \eta(x, 0) = \eta_0(x). \tag{31}$$

We are going to check that $h(x,t)$ defined as

$$h(x,t) = \alpha\eta(x,t) - \beta\partial_x\eta(x,t) \tag{32}$$

is the solution of (2) with parameters (α, β) and initial condition h_0. Both the h front and the η front have the same boundary μ_t.

When $\beta = 0$, the claim is trivial by linearity. When $\beta \neq 0$ we need to check that:

- h is solution to $\partial_t h = \partial_x^2 h + h$ for $x > \mu_t$.
  This is obvious from (32) and (31) by linearity.
- $h(\mu_t, t) = \alpha$.
  This is also obvious from (32) and (31).
- $\partial_x h(\mu_t, t) = \beta$.
  This one is less obvious. It works from (32) because one has

$$\partial_x^2 \eta(\mu_t, t) = -1 \qquad \text{for all } t > 0, \tag{33}$$

  as can be seen by taking the time derivative of $1 = \eta(\mu_t, t)$; one gets $0 = \dot{\mu}_t \partial_x \eta(\mu_t, t) + \partial_x^2 \eta(\mu_t, t) + \eta(\mu_t, t) = \partial_x^2 \eta(\mu_t, t) + 1$.
- $h(x, 0) = h_0(x)$.
  It follows from (30) that $\alpha\eta_0 - \beta\eta_0' = h_0$.
- If $\beta \neq 0$, one needs the condition that $\eta_0(x)$ is continuous and such that $\eta_0(0) = 1$.
  The fact that the condition holds is obvious from (30). Here is why it is needed: if the condition did not hold, then around any discontinuity point in η_0 (or around μ_t if $\eta_0(0) \neq 1$), the solution $\eta(x,t)$ would have an arbitrarily large slope at short times even though it would be continuous. This would mean in (32) that $h(x,t)$ would be unbounded around the problematic points in η_0 and, therefore, would not be an acceptable solution to (2).

With the procedure (30)–(32), it is clear that $h(x,t)$ converges to a travelling wave if and only if $\eta(x,t)$ does. The first thing to check is whether the $\eta_0(x)$ in (30) goes to zero exponentially fast as $x \to \infty$. When $\beta \leqslant 0$, this is always the case. When $\beta > 0$, one checks easily that this is the case if and only if the term in square brackets in (30) goes to zero as $x \to \infty$. (In either case, one needs to use the hypothesis that h_0 decays exponentially fast.) We have therefore justified the criterion (24).

In particular, in the case $\alpha = 0$ and $\beta > 0$ and for a h_0 such that (24) holds, the initial condition η_0 (30) is a decreasing function, which is sufficient to ensure that the front reaches a travelling wave.

4.3. Asymptotics of η_0

It is interesting to compare the large x behaviours of η_0 (the initial condition of the ($\alpha = 1$, $\beta = 0$) problem) and of h_0 (the initial condition of the original (α, β) problem). For simplicity, we limit the discussion to initial conditions of the form

$$h_0(x) \sim Ax^{\nu}\mathrm{e}^{-\gamma x} \quad \text{as } x \to \infty, \tag{34}$$

for some values of $A > 0$, $\gamma > 0$ and ν. With this form, one finds in (30)

$$\eta_0(x) \sim \begin{cases} \frac{A}{\alpha+\gamma\beta} x^{\nu}\mathrm{e}^{-\gamma x} & \text{if } \beta > 0 \text{ and assuming (24), or if } \beta < 0 \text{ and } \gamma < -\frac{\alpha}{\beta}, \\ [\text{some constant}]\ \mathrm{e}^{\frac{\alpha}{\beta}x} & \text{if } \beta < 0 \text{ and } \int \mathrm{d}z\, \mathrm{e}^{-\frac{\alpha}{\beta}z} h_0(z) < \infty, \\ \frac{A}{-\beta}(\log x)\mathrm{e}^{\frac{\alpha}{\beta}x} & \text{if } \beta < 0 \text{ and } \gamma = -\frac{\alpha}{\beta} \text{ and } \nu = -1, \\ \frac{A}{-\beta(\nu+1)} x^{\nu+1}\mathrm{e}^{\frac{\alpha}{\beta}x} & \text{if } \beta < 0 \text{ and } \gamma = -\frac{\alpha}{\beta} \text{ and } \nu > -1. \end{cases} \tag{35}$$

From (35), we will obtain in the next section the asymptotic behaviour of μ_t for a front in the so-called 'pushed' regime (if $\alpha + \beta < 0$) by translating it into the 'pulled' problem ($\alpha = 1$, $\beta = 0$) and the initial condition (with different asymptotic) given by (35).

5. The position of the front

In this section, we use the main relation (6) to relate the long time asymptotics of the position μ_t of the front to the initial condition h_0, assuming that the solution converges to a travelling wave.

When $\beta > -\alpha$, we find for initial conditions that decay fast enough that, as $t \to \infty$,

$$\begin{aligned} \mu_t &= 2t - \frac{3}{2}\log t + a + o(1) && \text{iff } \int \mathrm{d}x\, h_0(x) x \mathrm{e}^{x} < \infty, \\ \mu_t &= 2t - \frac{3}{2}\log t + a - 3\frac{\sqrt{\pi}}{\sqrt{t}} + o\Big(\frac{1}{\sqrt{t}}\Big) && \text{if } \int \mathrm{d}x\, h_0(x) x^2 \mathrm{e}^{x} < \infty, \\ \mu_t &= 2t - \frac{3}{2}\log t + a - 3\frac{\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + o\Big(\frac{\log t}{t}\Big) && \text{if } \int \mathrm{d}x\, h_0(x) x^3 \mathrm{e}^{x} < \infty, \end{aligned} \tag{36}$$

where we have no expression of a as a function of h_0, α and β. The asymptotics are the same as for the Fisher–KPP equation or any 'pulled front'. We recover in particular the Bramson term $-\frac{3}{2}\log t$, the Ebert and van Saarloos correction $-3\sqrt{\pi}/\sqrt{t}$ and a new universal term in $(\log t)/t$. Precise necessary conditions for the last two lines of (36) are given in (82) and (90).

If the initial condition decays as

$$h_0(x) \sim Ax^{\nu}\mathrm{e}^{-\gamma x} \quad \text{as } x \to \infty, \tag{37}$$

we also find that

$$\begin{aligned} \mu_t &= \Big(\gamma + \frac{1}{\gamma}\Big)t + \frac{\nu}{\gamma}\log t + a + o(1) && \text{if } \gamma < 1, \\ \mu_t &= 2t - \frac{1-\nu}{2}\log t + a + o(1) && \text{if } \gamma = 1 \text{ and } \nu \in (-2, \infty), \\ \mu_t &= 2t - \frac{3}{2}\log t + \log\log t + a + o(1) && \text{if } \gamma = 1 \text{ and } \nu = -2, \\ \mu_t &= 2t - \frac{3}{2}\log t + a - bt^{1+\frac{\nu}{2}} + o\big(t^{1+\frac{\nu}{2}}\big) && \text{if } \gamma = 1 \text{ and } \nu \in [-3, -2), \\ \mu_t &= 2t - \frac{3}{2}\log t + a - 3\frac{\sqrt{\pi}}{\sqrt{t}} - bt^{1+\frac{\nu}{2}} + o\big(t^{1+\frac{\nu}{2}}\big) && \text{if } \gamma = 1 \text{ and } \nu \in (-4, -3), \\ \mu_t &= 2t - \frac{3}{2}\log t + a - 3\frac{\sqrt{\pi}}{\sqrt{t}} + b\frac{\log t}{t} + o\Big(\frac{\log t}{t}\Big) && \text{if } \gamma = 1 \text{ and } \nu = -4. \end{aligned} \tag{38}$$

For the first three lines, there is a relatively simple expression of a as a function of α, β, A, γ, ν. The constant b can be expressed as a function of a, ν, α, β and A. All these expressions are compatible with what is already known for the Fisher–KPP equation (1).

The results (36) and (38) concern 'pulled' front equation ($\beta > -\alpha$). For pushed and critical fronts, we could also use the main relation (6) to derive the asymptotic position of the front. It is however simpler to translate the front into a pulled front, as explained in section 4.3, and use (35) to obtain the results. We find that

- When $\beta < -\alpha$ ('pushed' front equation), setting $\gamma_* = \alpha/(-\beta)$ and $v_* = \gamma_* + \gamma_*^{-1}$,

$$\begin{aligned}
\mu_t &= \Big(\gamma + \frac{1}{\gamma}\Big)t + \frac{\nu}{\gamma}\log t + a + o(1) && \text{for (37) if } \gamma < \gamma_*,\\
\mu_t &= v_* t + \frac{\nu+1}{\gamma_*}\log t + a + o(1) && \text{for (37) if } \gamma = \gamma_* \text{ and } \nu \in (-1,\infty),\\
\mu_t &= v_* t + \frac{1}{\gamma_*}\log\log t + a + o(1) && \text{for (37) if } \gamma = \gamma_* \text{ and } \nu = -1,\\
\mu_t &= v_* t + a + o(1) && \text{if } \int \mathrm{d}x\, h_0(x)\mathrm{e}^{\gamma_* x} < \infty.
\end{aligned} \tag{39}$$

The constant a depends on α, β and the whole function h_0 in the last line. For the other three lines, a can be expressed as a function of α, β, A, γ, ν. To illustrate how (39) is obtained, consider the second line (i.e. $h_0 \sim Ax^\nu \mathrm{e}^{-\gamma_* x}$ with $\nu > -1$): the last line of (35) tells us that this corresponds to a pulled front with initial condition $\eta_0 \sim A' x^{\nu+1}\mathrm{e}^{-\gamma_* x}$, and the first line of (38) with ν replaced to $\nu + 1$ gives the answer. For the third line of (39), the initial condition would be $\eta_0 \sim A'(\log x)\mathrm{e}^{-\gamma_* x}$, which is not in (38), but which could be computed easily using the methods explained in this section.

- When $\beta = -\alpha$ (critical front equation), we get

$$\begin{aligned}
\mu_t &= \Big(\gamma + \frac{1}{\gamma}\Big)t + \frac{\nu}{\gamma}\log t + a + o(1) && \text{for (37) if } \gamma < 1,\\
\mu_t &= 2t + \frac{\nu}{2}\log t + a + o(1) && \text{for (37) if } \gamma = 1 \text{ and } \nu \in (-1,\infty),\\
\mu_t &= 2t - \frac{1}{2}\log t + \log\log t + a + o(1) && \text{for (37) if } \gamma = 1 \text{ and } \nu = -1,\\
\mu_t &= 2t - \frac{1}{2}\log t + a + o(1) && \text{if } \int \mathrm{d}x\, h_0(x)\mathrm{e}^{x} < \infty.
\end{aligned} \tag{40}$$

Notice how Bramson's $\frac{3}{2}\log t$ correction is replaced here by a $\frac{1}{2}\log t$ correction. As in the pushed case, the constant a depends on α, β and the whole function h_0 in the last line. For the other three lines, a can be expressed as a function of α, β, A, γ, ν.

We now turn to the derivation of (36) and (38). From now on in this section, we assume that $\beta > -\alpha$ ('pulled' case) and that the front converges to a travelling wave with velocity v:

$$h(\mu_t + z, t) \to \omega_v(z), \qquad \frac{\mu_t}{t} \to v, \qquad v = \lambda + \frac{1}{\lambda} \text{ with } \lambda \leqslant 1. \tag{41}$$

We introduce

$$\Psi_1(r) = \int_0^\infty \mathrm{d}z\, h_0(z)\mathrm{e}^{rz}, \qquad \Psi_2(r) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{r\mu_t - (1+r^2)t}, \qquad \gamma = \sup\{r; \Psi_1(r) < \infty\}. \tag{42}$$

(γ is the exponential decay rate of h_0. By hypothesis on h_0, one has $\gamma > 0$.)

$\Psi_1(r)$ is finite for $r < \gamma$. $\Psi_2(r)$ is finite for $r < \lambda$ or $r > \lambda^{-1}$, where λ is defined in (41). Our main relation (6) states that

$$\Psi_1(r) = -\frac{\alpha}{r} + \left(\beta + \frac{\alpha}{r}\right)\Psi_2(r) \qquad \text{for } r < \min(\lambda, \gamma). \tag{43}$$

The main idea to obtain the asymptotics of μ_t is to express that both sides of (43) have the same first singularity in r. Matching the position of this singularity determines the velocity of the front, see section 5.1, while matching the nature of the singularity determines the sublinear term, as explained in the subsequent sections.

5.1. Velocity selection

The basic idea to find the final velocity of the front is that the two functions $\Psi_1(r)$ and $\Psi_2(r)$ become singular at the same value of r. Given γ (a property of initial condition h_0, see (42)), we want to compute the velocity v or, equivalently, λ, see (41). Recall that $\lambda \ll 1$.

It can be checked that

- $\Psi_1(r)$ is analytic for any $r < \gamma$ but becomes singular at $r = \gamma$ when γ is finite,
- $\Psi_2(r)$ is analytic for any $r < \lambda$. When $\lambda < 1$, it becomes singular at $r = \lambda$. (For $\lambda = 1$, we will see that, depending on μ_t, $\Psi_2(r)$ can be either singular or analytic at $r = \lambda = 1$.)

Then, (43) implies that

$$\lambda = \begin{cases} \gamma & \text{if } \gamma \leqslant 1 \quad (\text{both } \Psi_1 \text{ and } \Psi_2 \text{ are singular at } r = \lambda = \gamma), \\ 1 & \text{if } \gamma > 1 \quad (\text{both } \Psi_1 \text{ and } \Psi_2 \text{ are analytic at } r = 1). \end{cases} \tag{44}$$

The velocity is then $v = 2$ if $\gamma \geqslant 1$ and $v = \gamma + \gamma^{-1}$ if $\gamma \leqslant 1$.

Remark: in the 'pushed case', the prefactor $(\beta + \frac{\alpha}{r})$ of Ψ_2 in (43) vanishes at $r = \gamma_* = \alpha/(-\beta)$ and, when $\gamma > \gamma_*$, Ψ_1 is analytic at $r = \gamma_*$ while Ψ_2 has a single pole.

5.2. The singularities in Ψ_1 and Ψ_2

5.2.1. *The singularity in* Ψ_1. When the initial condition h_0 is of the form (37), one has

$$\Psi_1(\gamma - \epsilon) = \int_0^\infty \mathrm{d}z \left(h_0(z)\mathrm{e}^{\gamma z}\right) \mathrm{e}^{-\epsilon z}, \qquad \text{with } h_0(z)\mathrm{e}^{\gamma z} \sim A z^\nu, \tag{45}$$

and it is clear that $\Psi_1(\gamma - \epsilon)$ is singular at $\epsilon = 0$: for $\nu \geqslant -1$, one has $\Psi_1(\gamma) = \infty$; for $\nu \in [-2, -1)$, then $\Psi_1(\gamma)$ is finite but $\Psi_1'(\gamma)$ is infinite, etc. In fact, still assuming (37), one can show that

$$\mathrm{FST}_\epsilon\left[\Psi_1(\gamma - \epsilon)\right] = \begin{bmatrix} \text{First singular term} \\ \text{in an } \epsilon \text{ expansion} \\ \text{of } \Psi_1(\gamma - \epsilon) \end{bmatrix} = A \begin{cases} \Gamma(\nu + 1)\epsilon^{-\nu - 1} & \text{if } \nu \notin \{-1, -2, -3, \ldots\}, \\ \frac{(-)^\nu \epsilon^{-\nu-1} \log \epsilon}{(-\nu - 1)!} & \text{if } \nu \in \{-1, -2, -3, \ldots\}. \end{cases} \tag{46}$$

For example, when $\nu = -2.9$, we would write $\mathrm{FST}_\epsilon[\Psi_1(\gamma - \epsilon)] = A\Gamma(-1.9)\epsilon^{1.9}$, meaning that $\Psi_1(\gamma - \epsilon) = \Psi_1(\gamma) - \Psi_1'(\gamma)\epsilon + A\Gamma(-1.9)\epsilon^{1.9} + o(\epsilon^{1.9})$. In general, the small ϵ expansion of $\Psi_1(\gamma - \epsilon)$ starts like some polynomial in ϵ and, then, the first singular term is given by (46).

One can understand (46) by comparing Ψ_1 with the following functions of $\epsilon > 0$, which can be written as an analytic function plus one singular term:

$$\int_1^\infty \mathrm{d}z\, z^\nu \mathrm{e}^{-\epsilon z} = \big[\text{analytic function of } \epsilon\big] + \begin{cases} \Gamma(\nu+1)\epsilon^{-\nu-1} & \text{if } \nu \notin \{-1,-2,-3,\ldots\}, \\ \frac{(-)^\nu \epsilon^{-\nu-1}\log\epsilon}{(-\nu-1)!} & \text{if } \nu \in \{-1,-2,-3,\ldots\}. \end{cases} \tag{47}$$

For $\nu > -1$, obtaining (47) is easy: the analytic function is simply $-\int_0^1 \mathrm{d}z\, z^\nu \mathrm{e}^{-\epsilon z}$. For $\nu = -1$, a similar argument holds after an integration by parts of $1/z$. For $\nu < -1$, one simply needs to integrate (47) with $\nu \geqslant -1$ over ϵ as many times as needed. Note that one could change the lower bound of the integral in the left hand side of (47) to any positive value without changing the right hand side, as the nature of the singularity is governed by the large z regime.

5.2.2. The singularity in Ψ_2. We now turn to writing the singularity in Ψ_2. We define $\delta_t = o(t)$ as the sublinear correction in the position:

$$\mu_t = vt + \delta_t. \tag{48}$$

We need to consider two cases.

- If $\gamma < 1$, then $\lambda = \gamma$, $v = \gamma + \gamma^{-1}$, and $\Psi_2(r)$ is singular at $r = \gamma$. Using (48) in (42) we obtain

$$\Psi_2(\gamma - \epsilon) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon(\gamma^{-1}-\gamma+\epsilon)t + (\gamma-\epsilon)\delta_t} \approx \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon(\gamma^{-1}-\gamma)t + \gamma\delta_t} \quad \text{for } \epsilon > 0. \tag{49}$$

  It is then clear that, by taking δ_t logarithmic in t for large t, one recovers the same kind of integrals as in (47) with ϵ replaced by $(\gamma^{-1} - \gamma)\epsilon$, and one will be able to easily match the singularities with (46). This is done in detail in section 5.3.
- If $\gamma \geqslant 1$, then $\lambda = 1$ and $v = 2$. The function $\Psi_2(r)$ is singular at $r = 1$ if $\gamma = 1$ and unexpectedly analytic at $r = 1$ if $\gamma > 1$. With the form (48), one gets

$$\Psi_2(1 - \epsilon) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon^2 t + (1-\epsilon)\delta_t} \quad \text{for } \epsilon > 0. \tag{50}$$

  Again, by choosing δ_t logarithmic in t, one recovers the same kind of integrals as in (47) but with ϵ replaced by ϵ^2. As shown below, this difference is important. The matching of singularities with Ψ_1 is explained in section 5.4.

5.2.3. Other useful identities. We need in sections 5.3 and 5.4 some generalisations of (47) which we now enumerate. By taking the derivative of (47) with respect to ν when $\nu \notin \{-1,-2,-3,\ldots\}$, one gets

$$\int_1^\infty \mathrm{d}z\, (\log z) z^\nu \mathrm{e}^{-\epsilon z} = \big[\text{analytic function of } \epsilon\big] + \Big[\Gamma(\nu+1)(-\log\epsilon) + \Gamma'(\nu+1)\Big]\epsilon^{-\nu-1}. \tag{51}$$

Let us write more explicitly the ones we actually use:

$$\int_1^\infty \mathrm{d}z\, (\log z) z^{-\frac{3}{2}} \mathrm{e}^{-\epsilon z} = \big[\text{analytic function of } \epsilon\big] + 2\sqrt{\pi\epsilon}\big[\log\epsilon + \mathrm{Cst}\big], \tag{52}$$

$$\int_1^\infty \mathrm{d}z\, (\log z) z^{-\frac{5}{2}} \mathrm{e}^{-\epsilon z} = \big[\text{analytic function of } \epsilon\big] - \frac{4}{3}\sqrt{\pi}\epsilon^{\frac{3}{2}}\big[\log\epsilon + \mathrm{Cst}\big]. \tag{53}$$

(The constants in the two lines are different.) By taking the derivative of (51) with respect to ν, an extra $\log z$ term appears in the integral and one obtains an expression for $\int_1^\infty \mathrm{d}z(\log z)^2 z^\nu \mathrm{e}^{-\epsilon z}$. We only need the case $\nu = -\frac{3}{2}$, which is given by

$$\int_1^\infty \mathrm{d}z\,(\log z)^2 z^{-\frac{3}{2}}\mathrm{e}^{-\epsilon z} = \big[\text{analytic function of } \epsilon\big] - 2\sqrt{\pi\epsilon}\big[\log^2\epsilon - (4 - 4\log 2 - 2\gamma_E)\log\epsilon + \mathrm{Cst}\big] \quad (54)$$

Taking $\nu = -2 - u$ with $u > 0$ in (47) leads, after a small u expansion, to

$$\int_1^\infty \mathrm{d}z\,\frac{\log z}{z^2}\mathrm{e}^{-\epsilon z} = 1 - \epsilon\left(\frac{\log^2\epsilon}{2} - (1-\gamma_E)\log\epsilon + \mathrm{Cst}\right) + \mathcal{O}(\epsilon^2). \quad (55)$$

5.3. Sublinear terms in the position when $\nu > 2$

We assume that h_0 is of the form (37): $h_0(x) \sim Ax^\nu \mathrm{e}^{-\gamma x}$. The first singular term of Ψ_1 is given by (46) and $\Psi_2(\gamma - \epsilon)$ is given by (49), where $\delta_t = \mu_t - vt$. It is easy to see that the singularities in Ψ_2 and Ψ_1 match if

$$\delta_t = \frac{\nu}{\gamma}\log t + a + o(1) \qquad \text{as } t \to \infty. \quad (56)$$

Indeed, comparing (49) to (47) leads to

$$\mathrm{FST}_\epsilon\big[\Psi_2(\gamma-\epsilon)\big] = \begin{cases} \mathrm{e}^{\gamma a}\Gamma(\nu+1)\big[(\gamma^{-1}-\gamma)\epsilon\big]^{-\nu-1} & \text{if } \nu \notin \{-1,-2,-3,\ldots\}, \\ \mathrm{e}^{\gamma a}\frac{(-)^\nu\big[(\gamma^{-1}-\gamma)\epsilon\big]^{-\nu-1}\log\epsilon}{(-\nu-1)!} & \text{if } \nu \in \{-1,-2,-3,\ldots\}. \end{cases} \quad (57)$$

The relation (43) yields that

$$\mathrm{FST}_\epsilon\big[\Psi_1(\gamma-\epsilon)\big] = (\beta + \frac{\alpha}{\gamma})\mathrm{FST}_\epsilon\big[\Psi_2(\gamma-\epsilon)\big]. \quad (58)$$

Thus, one must choose a such that

$$A = \left(\beta + \frac{\alpha}{\gamma}\right)\mathrm{e}^{\gamma a}(\gamma^{-1}-\gamma)^{-\nu-1}. \quad (59)$$

Finally,

$$\mu_t = vt + \frac{\nu}{\gamma}\log t + \underbrace{\frac{1}{\gamma}\log\frac{A\gamma(\gamma^{-1}-\gamma)^{\nu+1}}{\alpha+\beta\gamma}}_{a} + o(1). \quad (60)$$

5.4. Sublinear terms in the position when $v = 2$

The case $v = 2$ corresponds to $\lambda = 1$. Either $\gamma > 1$, and both Ψ_1 and Ψ_2 are analytic around $r = 1$, or $\gamma = 1$ and they have matching singularities. $\Psi_2(1-\epsilon)$ is given by (50) where $\delta_t = \mu_t - 2t$. If one chooses δ_t logarithmic in t

$$\delta_t = \xi\log t + a + o(1), \quad (61)$$

then, using (47) with ϵ replaced by ϵ^2, one gets from (50)

$$\mathrm{FST}_{\epsilon^2}\big[\Psi_2(1-\epsilon)\big] = \begin{cases} \mathrm{e}^a\Gamma(\xi+1)(\epsilon^2)^{-\xi-1} & \text{if } \xi \notin \{-1,-2,-3,\ldots\}, \\ \mathrm{e}^a\frac{(-)^\xi(\epsilon^2)^{-\xi-1}\log(\epsilon^2)}{(-\xi-1)!} & \text{if } \xi \in \{-1,-2,-3,\ldots\}. \end{cases} \tag{62}$$

Observe that this is the first singular term for an expansion in powers of ϵ^2. In an expansion in powers of ϵ, the term above is not singular for $\xi \in \{-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},\ldots\}$, and one concludes that, in an expansion in powers of ϵ,

$$\mathrm{FST}_{\epsilon}\big[\Psi_2(1-\epsilon)\big] = \begin{cases} \mathrm{e}^a\Gamma(\xi+1)\epsilon^{-2\xi-2} & \text{if } \xi \notin \{-1,-\frac{3}{2},-2,-\frac{5}{2},-3,-\frac{7}{2},\ldots\}, \\ (-)^\xi 2\mathrm{e}^a\frac{\epsilon^{-2\xi-2}\log\epsilon}{(-\xi-1)!} & \text{if } \xi \in \{-1,-2,-3,\ldots\}, \\ o\big(\epsilon^{-2\xi-2}\big) & \text{if } \xi \in \{-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},\ldots\}. \end{cases} \tag{63}$$

Let us assume that h_0 is of the form (37) with $\gamma = 1$: $h_0(x) \sim Ax^\nu\mathrm{e}^{-x}$. The first singular term in ϵ must be, from (46), either in $\epsilon^{-\nu-1}$ or in $\epsilon^{-\nu-1}\log\epsilon$. For example, if $\nu = -1.8$, then the singularity is $\epsilon^{0.8}$ and (63) leads to $\xi = -1.4$. On the other hand, if $\nu = -2.2$, the singularity is $\epsilon^{1.2}$ and (63) gives two possible solutions: either $\xi = -1.6$ or $\xi = -\frac{3}{2}$. Thus, we see that the case $\nu \geqslant -2$ (where there is no ambiguity) is simpler than the case $\nu < -2$ (where one needs to determine the correct solution amongst several possibilities).

5.4.1. The case $\gamma = 1$ and $\nu \geqslant -2$. When $\nu > -2$, one can match unambiguously (46) and (63) and one finds that $\xi = (\nu-1)/2$. Putting aside the case $\nu = -1$ for now, one finds

$$A\Gamma(\nu+1) = (\alpha+\beta)\mathrm{e}^a\Gamma\left(\frac{\nu+1}{2}\right) \qquad \text{for } \nu > -2 \text{ and } \nu \neq -1, \tag{64}$$

and finally

$$\mu_t = 2t + \frac{\nu-1}{2}\log t + \underbrace{\log\frac{A\Gamma(\nu+1)}{(\alpha+\beta)\Gamma[(\nu+1)/2]}}_{a} + o(1) \qquad \text{for } \nu > -2 \text{ and } \nu \neq -1. \tag{65}$$

For $\nu = -1$, the singularity in (46) is of order $\log\epsilon$ and a simple matching gives

$$\mu_t = 2t - \log t + \underbrace{\log\frac{2A}{\alpha+\beta}}_{a} + o(1) \qquad \text{for } \nu = -1. \tag{66}$$

The case $\nu = -2$ is slightly more problematic because the first singular term in Ψ_1 is $A\epsilon\log\epsilon$ which cannot be obtained from (63). The correction to the position is not of the form (61), but one can check that in fact $\delta_t = -\frac{3}{2}\log t + \log\log t + a + o(1)$ matches the singularity and, finally, using (52),

$$\mu_t = 2t - \frac{3}{2}\log t + \log\log t + \underbrace{\log\frac{A}{(\alpha+\beta)4\sqrt{\pi}}}_{a} + o(1) \qquad \text{for } \nu = -2. \tag{67}$$

5.4.2. The leading term when $\gamma = 1$ and $\nu < -2$. When $\nu < -2$, the first singular term in Ψ_1 is small compared to ϵ, see (46), and there are several ways of matching such a singular term from (63). For instance, if $\nu = -2.2$, the singular term is $\epsilon^{1.2}$ and, from (63), one has either $\xi = (\nu - 1)/2 = -1.6$ or $\xi = -3/2$. If $\nu = -4.2$, one could either have $\xi = -2.6$ or $\xi = -3/2$ or $\xi = -5/2$, etc.

To resolve this difficulty, we now argue that there exists a constant C such that, for any initial condition h_0, one has

$$\delta_t + (3/2)\log t \geqslant C \qquad \text{for } t \text{ large enough.} \tag{68}$$

This will imply that $\xi \geqslant -3/2$, always, and thus that we have $\xi = -3/2$ when $\nu < -2$. In fact, more generally, $\xi = -3/2$ for any initial condition such that $\mathrm{FST}_\epsilon[\Psi_1(1-\epsilon)] = o(\epsilon)$ or, equivalently, such that $\int \mathrm{d}x\, h_0(x) x \mathrm{e}^x < \infty$:

$$\mu_t = 2t - \frac{3}{2}\log t + a + o(1) \qquad \text{if and only if} \int \mathrm{d}x\, h_0(x) x \mathrm{e}^x < \infty, \tag{69}$$

for some constant a.

We now turn to showing (68). We only need to consider the case ($\alpha = 1, \beta = 0$) because of the mapping explained in section 4.2. With the construction of a solution to the ($\alpha = 1, \beta = 0$) explained in section 4.1, it is clear that there is a comparison principle: for two ordered initial conditions $h_0(x) \leqslant \tilde{h}_0(x)$, one must have $\mu_t \leqslant \tilde{\mu}_t$ at all times where (μ_t resp. $\tilde{\mu}_t$) is the position of the solution with initial condition h_0 (resp. $\tilde{h}_0$). This implies that it is sufficient to show (68) for the initial condition $h_0 = 0$.

When $h_0 = 0$, the main relation (43) reduces to $\Psi_2(1-\epsilon) = 1$ for $\epsilon > 0$. Writing as usual $\mu_t = 2t + \delta_t$, we expand (50) up to order $o(\epsilon)$.

$$\Psi_2(1-\epsilon) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon^2 t + (1-\epsilon)\delta_t} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\delta_t} \mathrm{e}^{-\epsilon^2 t} - \epsilon \int_0^\infty \mathrm{d}t\, \delta_t \mathrm{e}^{\delta_t} + o(\epsilon). \tag{70}$$

The behaviour of the first term in the right-hand-side depends on $\exp(\delta_t)$. From (47):

- If $\exp(\delta_t) \sim \mathrm{e}^a t^{-3/2}$ then $\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\delta_t} \mathrm{e}^{-\epsilon^2 t} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\delta_t} - \epsilon 2\mathrm{e}^a \sqrt{\pi} + o(\epsilon)$.
- If $\exp(\delta_t) = o(t^{-3/2})$, for instance if $\delta_t = \xi \log t + a + o(1)$ with $\xi < -3/2$, then one finds $\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\delta_t} \mathrm{e}^{-\epsilon^2 t} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\delta_t} + o(\epsilon)$.
- Finally, if $\exp(\delta_t) \gg t^{-3/2}$, for instance if $\delta_t = \xi \log t + a + o(1)$ with $\xi > -3/2$, then $\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\delta_t} \mathrm{e}^{-\epsilon^2 t} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\delta_t} + R(\epsilon)$ with $R(\epsilon) \gg \epsilon$.

The initial condition $h_0 = 0$ is of course below the travelling wave, and the position of the front started from the travelling wave is exactly $2t$. By using again the comparison principle, we conclude that the position of the front with $h_0 = 0$ is such that $\delta_t \leqslant 0$ at all times. Thus, $\int_0^\infty \mathrm{d}t\, \delta_t \mathrm{e}^{\delta_t} < 0$ and the only way that $\Psi_2(1-\epsilon) = 1$ is that

$$\exp(\delta_t) \sim \mathrm{e}^a t^{-3/2} \quad \text{with} \quad 2\mathrm{e}^a \sqrt{\pi} = -\int_0^\infty \mathrm{d}t\, \delta_t \mathrm{e}^{\delta_t} \tag{71}$$

This concludes the argument. We were not able to find a simpler expression for a.

5.4.3. Vanishing corrections when $\gamma = 1$ and $\nu < -2$. We still consider the case where $\int \mathrm{d}x\, h_0(x) x \mathrm{e}^x < \infty$ or, when considering only initial conditions of the form $h_0(x) \sim A x^\nu \mathrm{e}^{-x}$, the case $\nu < -2$. From the previous argument, we must have

$$\delta_t = -\frac{3}{2}\log t + a + q_t \qquad \text{with } q_t = o(1). \tag{72}$$

In (50),

$$\Psi_2(1-\epsilon) = \mathrm{e}^{a(1-\epsilon)}\int_1^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{3/2}}\mathrm{e}^{\epsilon\frac{3}{2}\log t + q_t - \epsilon q_t} + f(\epsilon), \tag{73}$$

where $f(\epsilon)$, which represents the integral from 0 to 1, has no singularity in a small ϵ expansion.

The singularity of $\Psi_2(1-\epsilon)$ at $\epsilon = 0$ is dominated by the large t decay. The second exponential in (73) can be expanded; the leading term in this expansion gives $\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}/t^{3/2}$, which is not singular in ϵ. The next term is either

$$\mathrm{FST}_\epsilon\left[\int_1^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{3/2}}\epsilon\frac{3}{2}\log t\right] = 6\sqrt{\pi}\epsilon^2\log\epsilon, \tag{74}$$

see (52), or

$$\mathrm{FST}_\epsilon\left[\int_1^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{3/2}}q_t\right], \tag{75}$$

or the sum of the two if (75) is also of order $\epsilon^2\log\epsilon$, or a higher order term is the sum cancels.

From (47), the quantity (75) is of order $\epsilon^2\log\epsilon$ when q_t decays as $1/\sqrt{t}$:

$$\mathrm{FST}_\epsilon\left[\int_1^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{3/2}}\frac{1}{\sqrt{t}}\right] = 2\epsilon^2\log\epsilon. \tag{76}$$

We are now ready to match the singularities of Ψ_1 and Ψ_2. Recall that for an initial condition $h_0(x) \sim Ax^\nu\mathrm{e}^{-x}$, the first singular term in Ψ_1 is given by (46) and is of order $\epsilon^{-\nu-1}$. We consider three cases.

- When $\nu \in (-3,-2)$, the singularity in Ψ_1 is between ϵ and ϵ^2 and can only be matched by the q_t term (75). One must choose $q_t \sim -bt^{1+\nu/2}$ which leads to

$$A\Gamma(\nu+1) = -(\alpha+\beta)\mathrm{e}^a b\Gamma\Big(\frac{\nu+1}{2}\Big), \tag{77}$$

and finally

$$\mu_t = 2t - \frac{3}{2}\log t + a - \underbrace{\frac{-A\mathrm{e}^{-a}\Gamma(\nu+1)}{\Gamma\big[(\nu+1)/2\big](\alpha+\beta)}}_{b} t^{1+\frac{\nu}{2}} + o(t^{1+\frac{\nu}{2}}) \qquad \text{for } \nu\in(-3,-2). \tag{78}$$

- When $\nu = -3$, the singular term in Ψ_1 is $-(A/2)\epsilon^2\log\epsilon$. One needs to take q_t of the form $q_t \sim -bt^{-1/2}$ and both (74) and (76) contribute. Matching singularities gives

$$-\frac{A}{2} = (\alpha+\beta)\mathrm{e}^a\Big[6\sqrt{\pi} - 2b\Big], \tag{79}$$

and finally

$$\mu_t = 2t - \frac{3}{2}\log t + a - \underbrace{\left[\frac{Ae^{-a}}{4(\alpha+\beta)} + 3\sqrt{\pi}\right]}_{b} t^{-1/2} + o(t^{-1/2}) \qquad \text{for } \nu = -3. \tag{80}$$

- When $\nu < -3$, there cannot exist a $\epsilon^2 \log \epsilon$ singularity in Ψ_2 and the terms (74) and (75) must cancel. This leads to

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + o(t^{-1/2}) \qquad \text{for } \nu < -3. \tag{81}$$

For general initial condition (not necessarily such that $h_0 \sim Ax^\nu e^{-x}$), a necessary and sufficient condition to have the expansion (81) is simply that the first singular term in $\Psi_1(1-\epsilon)$ is smaller than $\epsilon^2 \log \epsilon$:

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + o(t^{-1/2}) \qquad \text{if and only if } \mathrm{FST}_\epsilon[\Psi_1(1-\epsilon)] = o(\epsilon^2 \log \epsilon). \tag{82}$$

In particular, the condition in (36), which is equivalent to $\mathrm{FST}_\epsilon[\Psi_1(1-\epsilon)] = o(\epsilon^2)$, is sufficient.

5.4.4. Second order vanishing corrections when $\gamma = 1$ and $\nu < -3$. We only consider the case where (82) holds; when considering initial conditions of the form $h_0(x) \sim Ax^\nu e^{-x}$, this corresponds to $\nu < -3$. We write

$$\delta_t = -\frac{3}{2}\log t + a - 3\frac{\sqrt{\pi}}{\sqrt{t}} + s_t \qquad \text{with } s_t = o(t^{-1/2}), \tag{83}$$

and, as in (73),

$$\Psi_2(1-\epsilon) = e^{a(1-\epsilon)} \int_1^\infty dt\, \frac{e^{-\epsilon^2 t}}{t^{3/2}} e^{\epsilon\frac{3}{2}\log t - 3\sqrt{\pi/t} + s_t + \epsilon\left(3\sqrt{\pi/t} - s_t\right)} + f(\epsilon). \tag{84}$$

We expand the last exponential in (84), keeping all the terms that lead to singularities larger than ϵ^3. The terms with $\epsilon\frac{3}{2}\log t$ and $-3\sqrt{\pi/t}$ have cancelling singularities, so there remains

$$\mathrm{FST}_\epsilon\left[\int_1^\infty dt\, \frac{e^{-\epsilon^2 t}}{t^{3/2}} s_t\right], \tag{85}$$

which is not yet known, and

$$\mathrm{FST}_\epsilon\left[\int_1^\infty dt\, \frac{e^{-\epsilon^2 t}}{t^{3/2}}\left[\frac{1}{2}\left(\epsilon\frac{3}{2}\log t - \frac{3\sqrt{\pi}}{\sqrt{t}}\right)^2 + \epsilon\frac{3\sqrt{\pi}}{\sqrt{t}}\right]\right] = (15 - 18\log 2)\sqrt{\pi}\epsilon^3 \log \epsilon, \tag{86}$$

which we computed using (47), (54) and (55).

The argument is then the same as before and, for an initial condition $h_0(x) \sim Ax^\nu e^{-x}$, one needs to consider three cases.

- When $\nu \in (-4, -3)$, the singularity in Ψ_1 is between ϵ^2 and ϵ^3 and must be matched by (85) because (86) is too small. This leads to taking $s_t \sim -bt^{1+\nu/2}$ and finally

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} - \underbrace{\frac{-A\mathrm{e}^{-a}\Gamma(\nu+1)}{\Gamma\big[(\nu+1)/2\big](\alpha+\beta)}}_{b} t^{1+\frac{\nu}{2}} + o(t^{1+\frac{\nu}{2}}) \qquad \text{for } \nu \in (-4,-3). \tag{87}$$

- When $\nu = -4$, the singularity in Ψ_1 is $A\epsilon^3(\log\epsilon)/6$. The term (86) contributes, as well as (85) with $s_t \sim b(\log t)/t$ using (53):

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + \underbrace{\left(\frac{9}{8}(5 - 6\log 2) - \frac{A\mathrm{e}^{-a}}{16\sqrt{\pi}(\alpha+\beta)}\right)}_{b} \frac{\log t}{t} + o\Big(\frac{\log t}{t}\Big) \quad \text{for } \nu = -4. \tag{88}$$

- When $\nu < -4$, there cannot exist a $\epsilon^3\log\epsilon$ singularity in Ψ_2 and the terms (85) and (86) must cancel. This leads to s_t of order $(\log t)/t$ and

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + o\Big(\frac{\log t}{t}\Big) \quad \text{for } \nu < -4. \tag{89}$$

For general initial condition (not necessarily such that $h_0 \sim Ax^\nu, \mathrm{e}^{-x}$), a necessary and sufficient condition to have the expansion (89) is simply that the first singular term in $\Psi_1(1-\epsilon)$ is smaller than $\epsilon^3\log\epsilon$:

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + o\Big(\frac{\log t}{t}\Big) \quad \text{iff } \mathrm{FST}_\epsilon[\Psi_1(1-\epsilon)] = o(\epsilon^3\log\epsilon). \tag{90}$$

In particular, the condition in (36), which is equivalent to $\mathrm{FST}_\epsilon[\Psi_1(1-\epsilon)] = o(\epsilon^3)$, is sufficient.

6. What happens when the front does not go to a travelling wave

In section 4, we established that the solution to (2) cannot converge to a travelling wave unless condition (24) is satisfied. In this section, we investigate briefly what happens when (24) is not met.

Let $h(x,t)$ be the solution to (2) with initial condition h_0. (As usual, we assume that $h_0 \geqslant 0$ decays exponentially fast at infinity.) To simplify the discussion, we only consider the case ($\alpha = 0$, $\beta = 1$). As in (30), let us introduce $\eta_0(x)$ by

$$\eta_0(x) = 1 - \int_0^x \mathrm{d}z\, h_0(z), \tag{91}$$

and recall from (32) that $h(x,t)$ can be written as

$$h(x,t) = -\partial_x \eta(x,t), \tag{92}$$

where $\eta(x,t)$ is the solution to (2) with parameters ($\alpha = 1$, $\beta = 0$) and initial condition η_0.

The total mass in the system at time t is given by $g(0,t)$ and its evolution is simply given by (25):

$$g(0,t) = \int_0^\infty \mathrm{d}z\, h(\mu_t + z, t) = 1 + \Big(\int_0^\infty \mathrm{d}z\, h_0(z) - 1\Big)\mathrm{e}^t. \tag{93}$$

There are three cases to consider in order to understand the evolution of h:

If $\int_0^\infty \mathrm{d}z\, h_0(z) = 1$.

Then (24) holds. The function $\eta_0(x) = \int_x^\infty \mathrm{d}z\, h_0(z)$ is a decreasing function decaying exponentially fast at infinity. The front $\eta(x,t)$ converges to a travelling wave, and so does $h(x,t)$. Notice that the mass $g(0,t)$ is equal to 1 at all time t.

If $\int_0^\infty \mathrm{d}z\, h_0(z) < 1$.

Then (24) does not hold. The mass $g(0,t)$ reaches zero at some finite time t_c. As $h(x,t)$ is non-negative, this implies that $h(\mu_t + z, t) \to 0$ for all $z > 0$ as $t \to t_c$; the front disappears in finite time, and the solution to (2) does not exist for $t > t_c$. This is easy to understand by considering $\eta(x,t)$; the initial condition $\eta_0(x)$ is a decreasing function bounded away from zero: $\eta_0(\infty) = 1 - \int \mathrm{d}z\, h_0(z) > 0$. At all times, $\eta(x,t)$ is a decreasing function of x with $\eta(\infty,t) = \big(1 - \int \mathrm{d}z\, h_0(z)\big)\mathrm{e}^t = 1 - g(0,t) \to 1$ as $t \to t_c$. Thus, $\eta(x,t) \to 1$ uniformly while μ_t diverges.

If $\int_0^\infty \mathrm{d}z\, h_0(z) > 1$.

Then (24) does not hold. The mass $g(0,t)$ diverges exponentially. We argue below that the position μ_t of the boundary runs to the left with velocity 2 and that it converges to the pseudo-travelling wave $z\mathrm{e}^z$:

$$h(\mu_t + z, t) \to z\mathrm{e}^z \qquad \text{with} \qquad -\mu_t = 2t - \frac{3}{2}\log t + a + 3\frac{\sqrt{\pi}}{\sqrt{t}} + o(t^{-\frac{1}{2}}). \tag{94}$$

(In section 2, we insisted that a proper travelling wave ω_v must be positive and satisfy $\omega_v(\infty) = 0$, so in that sense $z\mathrm{e}^z$ is not really a travelling wave.) We wrote $-\mu_t$ rather than μ_t in (94) to have the usual signs for the velocity and logarithmic correction. Notice that the $1/\sqrt{t}$ correction has the same coefficient with an opposite sign as the Ebert and van Saarloos correction for the position of the Fisher–KPP front.

An intuitive way to visualize these three cases is the following: as $\alpha = 0$, the boundary at μ_t is an absorbing boundary. If it were not moving ($\mu_t = 0\ \forall t$), the front h would grow and so would the slope at μ_t. In order to fix $\beta = \partial_x h(\mu_t, t) = 1$, we must prevent this growth. There are two strategies: either μ_t moves to the right to prevent the front from growing, or μ_t moves to the left to 'escape' the ever-growing front and to find a region where the slope is not yet large. The first two cases above correspond to the first strategy, while the third case corresponds to the second strategy.

It would be sufficient to assume that $h(\mu_t + z, t)$ has a large time limit to derive (94) using the techniques developed in the present paper. To make this section short and simple, we now make the stronger assumption that $h(\mu_t + z, t) \to z\mathrm{e}^z$ and $-\mu_t = 2t + o(t)$, and we explain briefly how the sub-linear terms in μ_t can be obtained. The derivation of section 3 still holds, except that $g(r,t)$ has a long time limit only for $r < -1$. We conclude that

$$\Psi_1(r) = \Psi_2(r) \text{ for } r < -1, \quad \text{with } \Psi_1(r) = \int_0^\infty \mathrm{d}z\, h_0(z)\mathrm{e}^{rz},\ \Psi_2(r) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{r\mu_t - (r^2+1)t}. \tag{95}$$

(This is the same as (42) and (43) with ($\alpha = 0$, $\beta = 1$), but with the extra condition $r < -1$.)

Following section 5.4, we write $\mu_t = -(2t + \delta_t)$ and $r = -1 - \epsilon$ and obtain

$$\Psi_2(-1-\epsilon) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon^2 t + (1+\epsilon)\delta_t}. \tag{96}$$

The right hand side is similar to (50), but with the opposite sign for the term $\epsilon\delta_t$. As in section 5.4, the question is how to choose δ_t in such a way that (96) has no singularity as $\epsilon \to 0^+$. By following the same line of argument as in section 5.4, one finds that $\delta_t = -\frac{3}{2}\log t + a + 3\sqrt{\pi/t} + o(1/\sqrt{t})$, which is the same result as in section 5.4 except for the sign of the $1/\sqrt{t}$ correction. The difference comes directly from the sign difference of the $\epsilon\delta_t$ term between (50) and (96).

Remark that an expression similar to (94), with its unexpected sign in front of the $1/\sqrt{t}$ correction, has already been obtained in [21]. In that paper, the authors study the typical density of particles in a branching Brownian motion. The *expected* density of particles ρ satisfies the linear equation $\partial_t\rho = \partial_x^2\rho + \rho$, but ρ does not represent well the typical density of particles because of the effect of rare paths leading to many particles. In [21] it was proposed to rather consider $\psi(x,t)$ defined as the expected density of particles who never went further than some $\bar{X}_t$ from their starting point. The equation followed by ψ is then $\partial_t\psi = \partial_x^2\psi + \psi$ for $|x| < \bar{X}_t$ and $\psi(\pm\bar{X}_t, t) = 0$. In this view, $\bar{X}_t$ had to be determined in a self consistent way by requiring that the density ψ at a distance of order 1 from $\pm\bar{X}_t$ is of order 1. This equation is very similar to our problem (2) with $(\alpha = 0, \beta = 1)$ and, in fact, it was found [21] that $\psi(\bar{X}_t - z, t) \propto z\mathrm{e}^z$ and that $\bar{X}_t$ has the same asymptotics as $-\mu_t$ in (94).

7. Conclusion

In this work we have studied the long time asymptotics of the solutions of (2). When the solution converges to a travelling wave solution, we have obtained precise expressions (36), (38), (39) and (40) for the position of the front. In the pulled case our linear problem (2) reproduces the known expected asymptotics for Fisher–KPP like equations, including Bramson's logarithmic shift [6] and the power law correction predicted by Ebert and van Saarloos [2], see (36). Our analysis allowed us to even predict a further logarithmic correction, see last line of (36), and this raises the question of the existence and of the universality of a whole series of correction terms for travelling wave equations in the Fisher–KPP class with fast enough decaying initial conditions. For our linear problem (2), we could also analyse the pushed case, see (39) and (40).

Surprisingly all the cases could be analysed from a single compact equation (6). This equation relates the position μ_t at time t of the front to the initial condition $h_0(x)$, and the large time asymptotics of μ_t can be obtained by matching the first singularity of both sides of (6).

As an illustration of our results, by choosing $(\alpha = 1, \beta = 0)$ and $h_0(x) = 0$ we find (6) that the position μ_t is implicitly given by the following integral equation:

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{r\mu_t - (1+r^2)t} = 1, \quad \text{for all } r < 1. \tag{97}$$

This leads to the asymptotics predicted by (36):

$$\mu_t = 2t - \frac{3}{2}\log t + a - 3\frac{\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + \cdots \tag{98}$$

It would certainly be interesting to develop a more direct approach to equations of the type (97) to extract asymptotics such as (98).

A question which remains is to formulate the precise conditions that the initial condition h_0 should satisfy for the solution to converge to a travelling wave and to analyse the general case when it does not.

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