Fact sheet: Density of states in the mean-field model.

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The goal of this fact sheet is twofold. We demonstrate that, in practice, there is only one sharp peak in the function c_r that comes up in the mean-field model (see Baxter (1982) eq.(3.1.8), p. 40). Also, we demonstrate numerically the presence of a phase transition in the mean-field model.

The Hamiltonian of the mean-field model (fully connected Ising model) is

$$E(\{\sigma\}) = -\frac{qJ}{N-1}\sum_{(i,j)}\sigma_i\sigma_j - h_{\text{ext}}\sum_{i=1}^{N}\sigma_i$$

where the first sum is over all pairs of lattice sites and the $\sigma_i = \pm 1$ is the Ising spin on site i.

In this fact sheet, we study the property of the mean-field model by looking at its density of states c_r, which is defined as

$$c_r = \binom{N}{r}\exp\left\{-\beta\left[-\frac{qJ}{2(N-1)}\left((N-2r)^2 - N\right) - h_{ext}\,(N-2r)\right]\right\}$$

where r is the number of spins which are down. The partition function of the system could be expressed as

$$Z = \sum_{r=-N}^{N} c_r$$

1 Fluctuation on r

Due to the presence of factor $1/(N-1)$ in the interaction strength, the energy of the system scales as N, i.e. $E \propto N$ for large Ns. And, the entropy of the system scales as N as well, since

$$\begin{aligned}
S &= k \ln \binom{N}{r} \\
&= k \ln \left[\frac{N!}{(N-r)!r!} \right] \\
&\approx k \left(N \ln(N) - N - (N-r)\ln(N-r) + (N-r) - r\ln(r) + r \right) \\
&= k \left[(N-r)\ln(N/(N-r)) + r\ln(N/r) \right] \\
&= kN \left[-\frac{1+m}{2} \ln \frac{1+m}{2} - \frac{1-m}{2} \ln \frac{1-m}{2} \right]
\end{aligned}$$

where $m \sim O(1)$ is the average magnetization for each spin. Thus, $\frac{1+m}{2}\ln\frac{1+m}{2} + \frac{1-m}{2}\ln\frac{1-m}{2}$ should also be $\sim O(1)$. Thus, the free energy $F = E - TS$ must also scale as N. It is possible to define $f = F/N$, which is the free energy per site. f is $O(1)$ with respect to N.

The average value of r is

$$\begin{aligned}
r_0 &= \frac{\sum_r r c_r}{Z} \\
&= -\frac{1}{2\beta} \frac{\partial \ln(Z)}{\partial h_{ext}} \\
&= -\frac{N}{2} \frac{\partial f}{\partial h_{ext}}
\end{aligned}$$

which indicates $r_0 \propto N$. [1] And the variance of r,

$$\langle r^2 \rangle - \langle r \rangle^2 = \frac{N}{4\beta} \frac{\partial^2 f}{\partial h_{\text{ext}}^2}$$

is also $\propto N$. Thus, c_r has a peak at r_0 and the width of the peak, which is characterized by $\sqrt{\langle r^2 \rangle - \langle r \rangle^2}$, is $\propto N^{1/2}$. Thus, the fluctuation of r could be ignored and it is possible to derive the self-consistency relation.

$$m_0 = \tanh\left(qJm_0 + h_{\text{ext}} \right)$$

2 Numerical results

With positive h_{ext} and $T < T_c$, c_rs with different system size are plotted in Fig. 1. There are two peaks in c_r. However, only one of them survives when $N \to \infty$. The width of the peak scales roughly as $N^{1/2}$. The second peak also disappear quickly when increasing N. When there are only 960 sites in the system, the peak is already sharp; and the second peak is almost invisible. Thus, when there are 10^{23} particles in the system, it is safe to ignore the second peak and the width of the peak.

[1] $\frac{\partial f}{\partial h_{\text{ext}}}$ might diverge. However, the size of the system N should not be involved in this divergence. Thus it is still $O(1)$ when compared with N. This is also true for $\frac{\partial^2 f}{\partial h_{\text{ext}}^2}$.

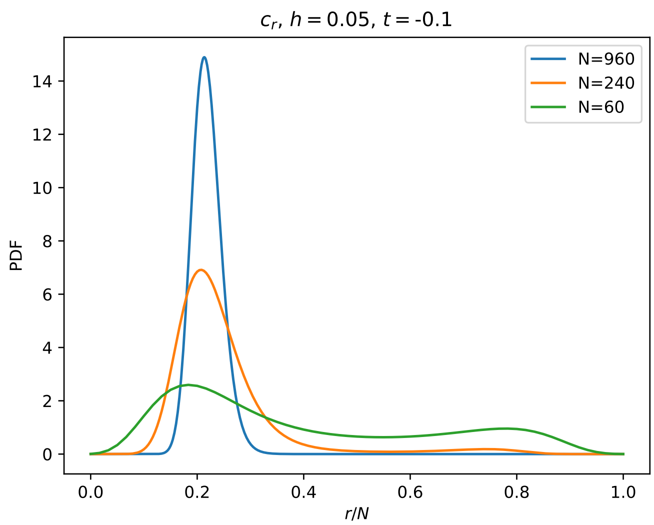

Figure 1: c_r as a function of r/N with different N. In order to compare these functions, c_r is normalized as probability distribution function, instead of probability, which c_r is supposed to be. When $N = 60$, the second peak is obverse, and the peaks are blunt. However, when $N = 960$, the lower peak is no longer visible; and the remaining peak is much sharper compared with the $N = 60$ case. It could be shown that the width of this peak scales as $N^{1/2}$. When $N \gg 1$, c_r should behave like a delta function; and the assumption, made when deriving the self-consistence relation, should be correct.

With fixed system size and $T < T_c$, c_r with various external magnetic fields is shown in Fig. 2. When $h_{\text{ext}} > 0$, the system prefers $m_0 > 0$ (small r_0); meanwhile, when $h_{\text{ext}} < 0$, the system prefers $m_0 < 0$ (large r_0). When $h_{\text{ext}} = 0$, the probability of having a positive m_0 and a negative m_0 are identical. This means that, when there are a large number of identical systems, a half of them will have $m_0 > 0$, while the other half have $m_0 < 0$. However, for each of these systems, most the spins will point uniformly upwards or downwards, instead of pointing upwards or downwards in a mixed manner. This is referred to as spontaneous symmetry breaking.

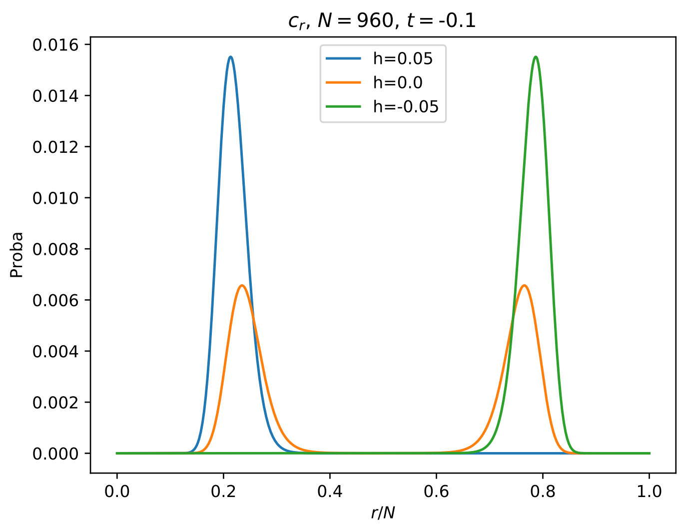

Figure 2: c_r as a function of r/N with different h. When $h_{\text{ext}} > 0$, the system prefers $m_0 > 0$ (small r_0); meanwhile, when $h_{\text{ext}} < 0$, the system prefers $m_0 < 0$ (large r_0). When $h_{\text{ext}} = 0$, the probability of having a positive m_0 and a negative m_0 are identical.

With fixed system size and negative h_{ext}, c_r with various temperatures is shown in Fig. 3. When $T < T_c$, there are two peaks. The small peak almost vanishes. When $T > T_c$, there is only one peak at the center of the plot. Thus, when $T > T_c$, $m_0 = 0$ while when $T < T_c$, $m_0 \neq 0$. This indicates the presence of phase transition.

References

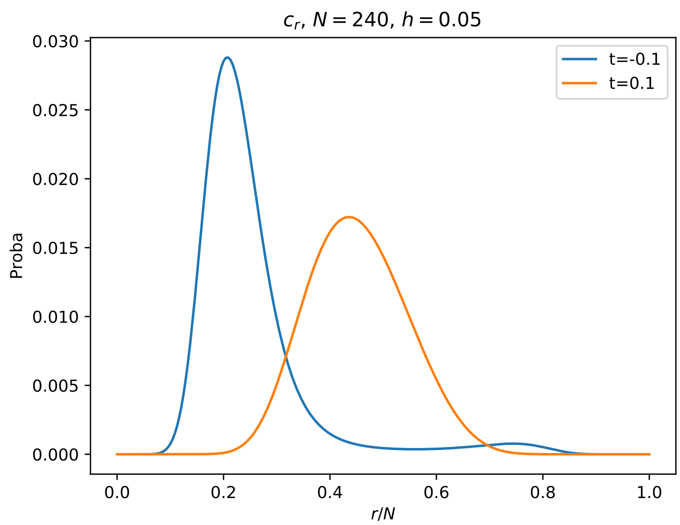

Figure 3: When $T < T_c$, there are two peaks. The small peak almost vanishes. When $T > T_c$, there is only one peak at the center of the plot. Thus, when $T > T_c$, $m_0 = 0$ while when $T < T_c$, $m_0 \neq 0$