Tutorial 10, Statistical Mechanics: Concepts and applications
2016/17 ICFP Master (first year)

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Tutorial exercises

I. WORKSHEET: PHYSICS IN INFINITE DIMENSIONS II

Source: R. J. Baxter, Exactly Solved Models in Statistical Mechanics*, Dover Publications (2008).*

1. **Spherical model**

Reminder: The *n-vector model* is a generalization of the Ising model, where the Hamiltonian reads as

$$H = -J \sum_{(i,j)} \vec{s}_i \cdot \vec{s}_j \,. \tag{1}$$

Here $\vec{s}_i$ are n-component, unit length, classical spins. In particular, the classical Ising model corresponds to $n = 1$, while the (classical) XY model to $n = 2$. Despite its simple form, it is still generally unsolvable. The search for a model somehow tractable in more than two dimensions has lead to the invention of the so-called "spherical model", where *continuous* classical spin variables σ_i interact between nearest neighbor and are subjected to an overall constraint

$$\sum_i \sigma_i^2 = N \,, \tag{2}$$

where N is the number of the sites. In practice, the Hamiltonian is given by

$$H = -J \sum_{(i,j)} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{3}$$

with the constraint (2). This constraint is unphysical, in that it implies an equal coupling, or interaction, between all spins, no matter how far apart on the lattice they are. Fortunately, this model was shown to be equivalent to the *n-vector* model in the limit $n \to \infty$ [M. Kac and C. J. Thompson, Physica Norvegica **5**, 163 (1971)].

The model: We consider the spherical model in a d-dimensional hyper-cubic lattice with periodic boundary conditions. By performing gaussian integrations, one can show that the partition function can be written as

$$Z_N = \beta J \Big(\frac{\pi}{\beta J}\Big)^{\frac{N}{2}} \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}z}{2\pi i} e^{N\phi(z)} \,, \tag{4}$$

where $c > 0$ (being the integrand analytic for $\mathrm{Re}(z) > 0$, the exact value of c does not matter) and

$$\phi(z) = \beta J(z+d) + \frac{\beta h^2}{4Jz} - \frac{1}{2} g(z) \,, \tag{5}$$

with

$$g(z) = \iiint\limits_{[0,2\pi]^d} \frac{\mathrm{d}^d\omega}{(2\pi)^d} \log\Big(z + d - \sum_{i=1}^{d} \cos(\omega_i)\Big) \,. \tag{6}$$

(a) Show that one can choose c in such a way that $\phi(z)$ has a maximum at $z = z_0$, with $\mathrm{Im}[z_0] = 0$.

HINT: Use that $\phi(z)$ is an analytic function of z for $\mathrm{Re}(z) > 0$.

(b) The asymptotic behavior of the integral for $N \to \infty$ can be obtained using the method of steepest descent. In practice, this gives $\lim_{N\to\infty} \frac{1}{N} \log \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}z}{2\pi i} e^{N\phi(z)} = \phi(z_0)$. Express the free energy as a(n implicit) function of β and h.

HINT: Just exhibit a system of equations whose solution gives the desired result.

(c) Write an implicit equation for the magnetization as a function of β and h (the equation of state).

(d) It is not difficult to show that $g'(z)$ can be written as

$$g'(z) = \int_0^\infty \mathrm{d}t e^{-t(z+d)} [J_0(it)]^d \,, \tag{7}$$

where J_0 is the Bessel function of the first kind. This is a special function which, for large t, behaves as

$$J_0(it) \sim \frac{e^t}{\sqrt{2\pi t}} (1 + O(t^{-1})) \,. \tag{8}$$

Analyze the behavior of $g'(z)$ in the limit of small z.

(e) Write the magnetization as a function of h and z_0, and show that there is a phase transition for $d > 2$. Which is the critical temperature?

HINT: Use the equation that determines z_0, *i.e.* $\phi'(z_0) = 0$, in the limit of small h.

(f) The critical exponent β characterizes the behavior of the magnetization as the temperature gets closer to the critical temperature in the ferromagnetic phase for $h = 0$

$$M_0 = \left(\frac{T_c - T}{T_c}\right)^\beta \,. \tag{9}$$

Compute β.

(g) The critical exponent α can be defined as follows:

$$u_+(T) - u_-(T) \sim \left(\frac{T - T_c}{T_c}\right)^{1-\alpha} \,, \tag{10}$$

where $u_\pm(T)$ is the internal energy for a temperature T slightly larger/smaller than the critical one. Compute α.

(h) Derive (4).

HINT 1: Use the identity

$$\delta(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{2\pi} e^{isx} \,. \tag{11}$$

HINT 2: It is convenient to work in the Fourier space.

HINT 3: In the thermodynamic limit, $\sum_{n=1}^{L} f(2\pi n/L) \approx N \int_0^{2\pi} \frac{\mathrm{d}\omega}{2\pi} f(\omega)$.