Advanced topics in Markov-chain Monte Carlo

Lecture 4:
Perfect sampling in Markov-chain Monte Carlo
Part 1/3: Introduction

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References

- C. Chanal *Thesis* University Paris VI (2010)
- J. G. Propp, D. B. Wilson **Exact sampling with coupled Markov chains and applications to statistical mechanics** http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.27.1022
- W. Krauth "**Statistical Mechanics: Algorithms and Computations**" (Oxford University Press, 2006)

MCMC convergence theorem 1/3

MCMC results are difficult to render rigorous because of:

- Compiler bugs.
- Program bugs (unittests, software engineering, proven correctness).
- Sampling uncertainty ($\#$ of samples $\ll \infty$).
- Mixing (Convergence) problems (sampling $\pi^{\{t\}}$, not π).

MCMC convergence theorem 2/3

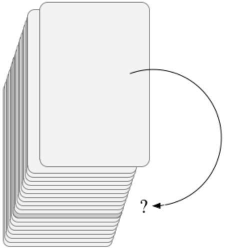
- Irreducible aperiodic (finite) Markov chains convergence exponentially:

$$\max_{x \in \Omega} ||P^t(x,.) - \pi||_{\text{TV}} < C\alpha^t$$

  with $C > 0$ and $\alpha \in (0, 1)$.
- Aperiodicity: $\exists r : P(x,.)^r > 0$
- Proof = TD.
- Exponential convergence $\implies$ time scale.
- The values of C and of α are generally unknown.

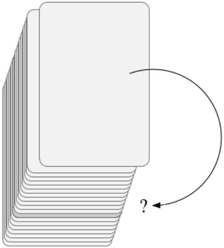
- Convergence is asking for too much, because the ergodic theorem only requires irreducibility.

Shuffling of cards 1/2



- $\Omega = \{\text{Permutations of } \{1, \ldots, N\}\}$
- $\pi^{t=0} = \delta((1, , \ldots, N))$
- Top-to-random shuffle
- Top (bottom) card to random
- Mixing time finite, but uniform distribution for $t \to \infty$ only

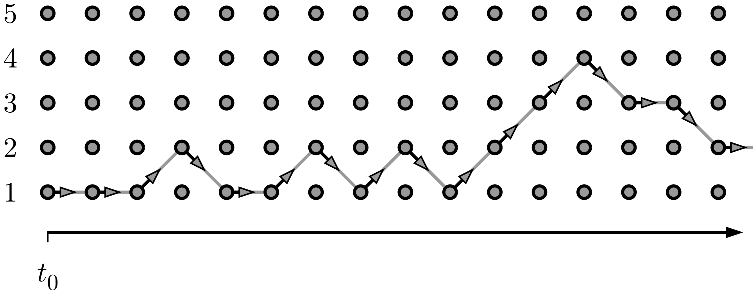
Shuffling of cards 2/2



- Mark the bottom card
- “Top-to-random” shuffles
- Top (bottom) card to random
- Stop! (Perfect sample)

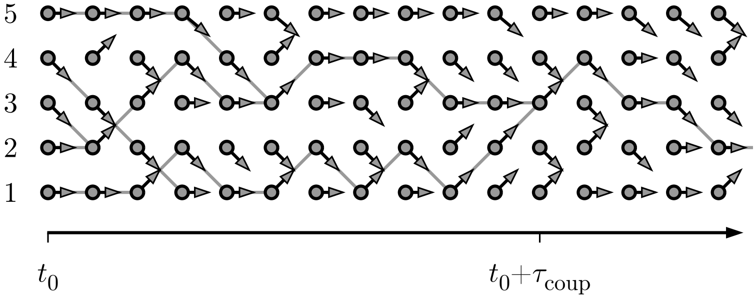
Example of a stopping rule in MCMC

Markov chain (traditional view)



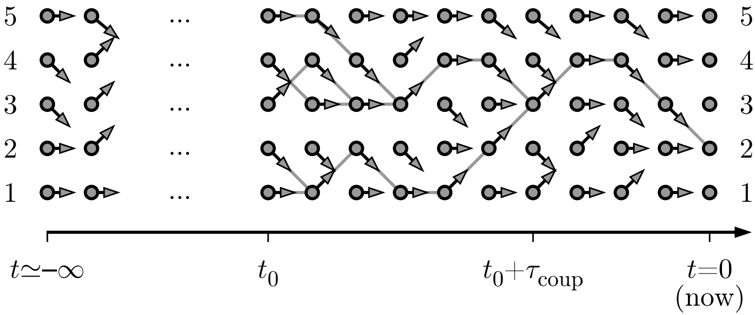
- Configuration c_t, move δ_t.
- Set $t_0 = 0$.

Markov chain (random maps), coupling



- Each configuration has its move at each time step.
- Coupling (Doeblin, 1930s).

Coupling from the past



- Starting an MCMC simulation at $t = -\infty$
- Propp & Wilson (1997)