Mathematica notebook for the Kac-Ward solution of the two-dimensional Ising model without periodic boundary conditions. Here we present the matrix for the 4x4 model and check that it yields the correct solution.

```
In[1]:= ν = Tanh[β ]

Out[1]= Tanh[β]

In[2]:= α = Exp[I Pi / 4] ν
```

Out[2]= $e^{\frac{i\pi}{4}}$ Tanh[β]

```
In[12]:= α' = Exp[-I Pi / 4] ν
```

Out[12]= $e^{-\frac{i\pi}{4}}$ Tanh[β]

```
In[4]:= null = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[4]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[5]:= one  = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

Out[5]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

In[14]:= right = {{ν, α, 0, α'}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

Out[14]= {{Tanh[β], $e^{\frac{i\pi}{4}}$ Tanh[β], 0, $e^{-\frac{i\pi}{4}}$ Tanh[β]}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
In[15]:= up = {{0, 0, 0, 0}, {α', ν, α, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

Out[15]= {{0, 0, 0, 0}, {$e^{-\frac{i\pi}{4}}$ Tanh[β], Tanh[β], $e^{\frac{i\pi}{4}}$ Tanh[β], 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
In[16]:= left = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, α', ν , α}, {0, 0, 0, 0}}
```

Out[16]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, $e^{-\frac{i\pi}{4}}$ Tanh[β], Tanh[β], $e^{\frac{i\pi}{4}}$ Tanh[β]}, {0, 0, 0, 0}}

```
In[17]:= down = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {α, 0, α', ν}}
```

Out[17]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {$e^{\frac{i\pi}{4}}$ Tanh[β], 0, $e^{-\frac{i\pi}{4}}$ Tanh[β], Tanh[β]}}

```
In[23]:= U4x4 =
  ArrayFlatten[ {{one, right, null, null, up, null, null, null, null, null, null, null,
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     null, null, null, null, null, null, null, null}, {null, left, one, right,
     null, null, up, null, null, null, null, null, null, null, null, null},
    {null, null, left, one, null, null, null, up, null, null, null, null,
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     left, one, right, null, null, up, null, null, null, null, null, null},
    {null, null, down, null, null, left, one, right, null, null, up, null,
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     down, null, null, null, one, right, null, null, up, null, null, null},
    {null, null, null, null, null, down, null, null, left, one, right, null,
     null, up, null, null}, {null, null, null, null, null, null, down, null,
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    {null, null, null, null, null, null, null, null, down, null, null,
     null, one, right, null, null}, {null, null, null, null, null,
     null, null, null, null, down, null, null, left, one, right, null},
    {null, null, null, null, null, null, null, null, null, null, down,
     null, null, left, one, right}, {null, null, null, null, null,
     null, null, null, null, null, null, down, null, null, left, one}}]
```

```
Out[23]= {{1, 0, 0, 0, Tanh[β], e^(iπ/4) Tanh[β], 0, e^(-iπ/4) Tanh[β], 0, 0, 0, 0, 0,
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   e^(iπ/4) Tanh[β], 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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```

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0, 0, 0, 0, 0, 0, 0, e^(-iπ/4) Tanh[β], Tanh[β], e^(iπ/4) Tanh[β], 0, 0, 1, 0},
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e^(-iπ/4) Tanh[β], Tanh[β], 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}}
```

In[30]:= **Simplify[Det[U4x4]]**

Out[30]= $\left(1 + 9\,\text{Tanh}[\beta]^{4} + 12\,\text{Tanh}[\beta]^{6} + 50\,\text{Tanh}[\beta]^{8} + 92\,\text{Tanh}[\beta]^{10} + 158\,\text{Tanh}[\beta]^{12} + 116\,\text{Tanh}[\beta]^{14} + 69\,\text{Tanh}[\beta]^{16} + 4\,\text{Tanh}[\beta]^{18} + \text{Tanh}[\beta]^{20}\right)^{2}$

The coefficients here are identical to the coefficients given by high temperature expension.