Tutorial 1, Advanced MCMC
2021/22 ICFP Master (second year)

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1. ***A priori* probabilities and filters**

   (a) In the lecture, we discussed *a priori* probabilities $\mathcal{A}$ and filters $\mathcal{P}$. Discuss those for the 3×3 pebble game.

   (b) We discussed that a finite Markov chain takes place on a graph. Draw this graph for the 3×3 pebble game.

2. **Irreducibility and uniqueness of stationary distribution**

   In the lecture, we discussed that an irreducible transition matrix has a unique stationary distribution (but that convergence is not guaranteed). We now consider the matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

   (a) Show that P is a transition matrix of a Markov chain.

   (b) Draw the graph of P.

   (c) Is P irreducible?

   (d) Does P have a unique stationary distribution?

   (e) Is there a contradition with what was said in the lecture?? (look up “Unique essential communicating class” in Levin & Peres).

3. **Different categories of transition matrices**

   In the lecture, we discussed the different categories of transition matrices. We now consider the matrix:

$$P = \begin{pmatrix} 5/12 & 5/12 & 1/6 \\ 1/4 & 1/4 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \tag{2}$$

(a) Is P the transition matrix of a Markov chain?

(b) If yes, is P an irreducible transition matrix?

(c) If yes, is P an aperiodic transition matrix?

(d) Is P reversible or non-reversible?

(e) What is its category?

4. **Total variation distance and mixing time**

In the lecture, we discussed the total variation distance (TVD)

(a) There was a "tiny" theorem to be proven... Prove it!

(b) Compute the TVD for the 3×3 pebble game, starting from the upper right corner, for the first few steps (if you have no computer) or for all t (if you have a computer).

(c) Compute the quantity $d(t)$ (with the maximum taken over initial *configurations*), and the mixing time as a function of the parameter ϵ (with a computer).

5. **Conductance on the path graph**

The path graph $P_n = (\Omega, V)$ is the one-dimensional lattice without periodic boundary conditions, with $\Omega = \{1, \ldots, n\}$, and $V = \{(1,2), \ldots, (n-1, n)\}$. In week 2, we will consider a number of different MCMC algorithms, variants of the Metropolis algorithm, where one proposes a move from i to $j = i \pm 1$ with probability $\frac{1}{2}$, and accepts it with probability $\min(1, \pi_j/\pi_i)$. It will be interesting to know the conductance of the Metropolis algorithm for different choices of π.

(a) What is the conductance for the Metropolis algorithm for the constant distribution $\pi_i = \text{const}$? (NB: the distribution must be normalized)

(b) What is the conductance for the Metropolis algorithm for the square-wave distribution $\pi_{2k-1} = \frac{2}{3n}$, $\pi_{2k} = \frac{4}{3n}$ $\forall k \in \{1, \ldots, n/2\}$ (for even n)?

(c) What is the conductance for the Metropolis algorithm for the V-shaped distribution: $\pi_i = \text{const}|\frac{n+1}{2} - i|$ $\forall i \in \{1, \ldots, n\}$?

NB: Next week, we will connect these conductances to the mixing and correlation times. If you have time, you can compute the TVD (by simulation) as a function of time (start from configuration $i = 1$).