Tutorial 1, Statistical Mechanics: Concepts and applications
2019/20 ICFP Master (first year)

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Tutorial

(Dated: September 20, 2019)

1. **General properties of the characteristic functions.**

   (a) [EASY] Prove the following properties:

   - $\Phi_\xi(0) = 1$.

     The total probability is normalized to 1.

   - $\Phi_\xi(-t) = \Phi_\xi^*(t)$.

     The probability distribution is real.

   - $|\Phi_\xi(t)| \leq 1$.

     The absolute value of the integral is bounded from above by the integral of the absolute value, which is $\Phi_\xi(0)$.

   - $\Phi_{a\xi+b}(t) = e^{ibt}\Phi_\xi(at)$.

     Under a change of variables $\xi' = f(\xi)$, with $f(\xi)$ monotonous, the probability distribution transforms as follows

$$\pi_{\xi'}(x) = \frac{\pi_\xi(f^{-1}(x)}{|f'(f^{-1}(x))|} \,. \tag{1}$$

     In particular, under a linear transformation we have

$$\pi_{a\xi+b}(x) = |a|^{-1}\pi_\xi((x-b)/a) \tag{2}$$

     and hence

$$\Phi_{a\xi+b}(t) = \int \mathrm{d}x\; e^{ixt}|a|^{-1}\pi_\xi((x-b)/a) = e^{ibt}\Phi_\xi(at) \,. \tag{3}$$

   (b) [EASY] Let ξ_1 and ξ_2 two independent random variables, what is the characteristic function of their sum? What about the sum of n independent random variables?

Since ξ_1 *and* ξ_2 *are independent, so are* $e^{it\xi_1}$ *and* $e^{it\xi_2}$.

$$\Phi_{\xi_1+\xi_2}(t) = \mathbb{E}(e^{it(x_1+x_2)}) = \mathbb{E}(e^{itx_1}e^{itx_2}) = \mathbb{E}(e^{itx_1})\mathbb{E}(e^{itx_2}) = \Phi_{\xi_1}(t)\Phi_{\xi_2}(t)$$

This can be readily generalized to N *variables*

$$\Phi_{\sum_i^N \xi_i}(t) = \prod_{i=1}^{N} \Phi_{\xi_i}(t)\,. \tag{4}$$

(c) [EASY] Name the first two cumulants. What is the variance of the sum of two independent random variables?

The first cumulant κ_1 *is the mean. The second cumulant is the variance. The cumulants are proportional to the coefficients of the series expansion of the logarithm of the characteristic function. Since the logarithm of a product is the sum of the logarithms, the* n*-th cumulant of the sum of two independent random variables is the sum of the* n*-th cumulant of the random variables. In particular, this applies to* $n = 2$, i.e. *to the variance.*

2. **Sum of random variables with uniform distribution.**

(a) [EASY] Compute the characteristic function of the sum of n independent random variables ξ_j with uniform distribution $\pi_{\xi_j}(x) = \frac{1}{2a}\theta_H(x+a)\theta_H(a-x)$, where $\theta_H(x+a)$ is the Heaviside theta function.

The characteristic function of π_{ξ_j} *is simply given by* $\frac{\sin(at)}{at}$*, therefore the characteristic function of the sum of* n *uniform variables is* $(\frac{\sin(at)}{at})^n$.

(b) [EASY-MEDIUM] Show that the characteristic function of $\xi^{(n)} = \sum_{j=1}^{n} \xi_j$ can be written in the following form:

$$\Phi_{\xi^{(n)}} = \frac{1}{(2ia)^n} t^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{i(n-2k)at} \tag{5}$$

Hint 1: Express the sin functions using complex exponentials ($\sin x = \frac{e^{ix}-e^{-ix}}{2i}$) and use the binomial theorem $(a+b)^j = \sum_{k=0}^{j} \binom{j}{k} a^j b^{j-k}$.

$$\left(\frac{\sin(at)}{at}\right)^n = \frac{t^{-n}}{(2ia)^n}(e^{iat} - e^{-iat})^n = \frac{t^{-n}}{(2ia)^n}\sum_{k=0}^{n}\binom{n}{k}[e^{iat}]^{n-k}[-e^{-iat}]^k = \frac{t^{-n}}{(2ia)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{i(n-2k)at} \tag{6}$$

(c) [HARD] Compute the inverse Fourier transform of the characteristic function and show that the distribution of $\xi^{(n)}$ can be written as?

$$\pi_{\xi^{(n)}} = \frac{1}{(n-1)!(2a)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \max\Big((n-2k)a - x, 0\Big)^{n-1} . \tag{7}$$

Hint 1: Move the sum outside of the integral of the inverse Fourier transform. *Warning*: the resulting integrals are divergent, but the divergencies have to simplify, so don't worry too much! The finite part of the integrals can be extracted using the *Cauchy principal value*, usually denoted by P.V., which, in the case of a singularity at zero, reads as

$$\text{P.V.} \int_{-\infty}^{\infty} f(t) = \lim_{\epsilon \to 0^+} \Big[\int_{-\infty}^{-\epsilon} f(t) + \int_{\epsilon}^{\infty} f(t) \Big] . \tag{8}$$

Hint 2: Compute the (finite part of the) integrals by integrating by parts $n-1$ times (note that the original product of sin functions has a zero of order n at $t = 0$).

Hint 3: P.V. $\int_{-\infty}^{\infty} \mathrm{d}t t^{-1} e^{itb} = i\pi \mathrm{sgn}(b)$.

Hint 4: $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (x+k)^j = 0$ for any x and integer $j = 1, \ldots, n-1$.

We must compute

$$\pi_{\xi^{(n)}}(x) = \frac{1}{2\pi} \int \mathrm{d}t e^{-ixt} \frac{1}{(2ia)^n} t^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{i(n-2k)at} . \tag{9}$$

First, we move the sum outside of the integral and take the principal value

$$\pi_{\xi^{(n)}}(x) = \frac{1}{2\pi} \sum_{k=0}^{n} \text{P.V.} \int \mathrm{d}t \frac{1}{(2ia)^n} t^{-n} \binom{n}{k} (-1)^k e^{i((n-2k)a-x)t} . \tag{10}$$

Since $(\sin(at))^n$ *has a zero of order* n *at* $t = 0$, *the boundary parts which come from the integration by parts (taking the integral of* t^{-n} *and the derivative of the rest) and which could have given contribution from* $t = 0$ *are in fact zero for* $n-1$ *consecutive integration by parts. Thus we find*

$$\pi_{\xi^{(n)}}(x) = \frac{1}{2\pi} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{((n-2k)a - x)^{n-1}}{(2a)^n (n-1)!} \text{P.V.} \int \mathrm{d}t\, t^{-1} e^{i((n-2k)a-x)t} . \tag{11}$$

The integral can be easily evaluated and gives

$$\begin{aligned} \pi_{\xi^{(n)}}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{((n-2k)a - x)^{n-1}}{2(2a)^n (n-1)!} \mathrm{sgn}((n-2k)a - x) = \\ \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{((n-2k)a - x)^{n-1}}{2(2a)^n (n-1)!} [2\theta_H((n-2k)a - x) - 1] = \\ \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{((n-2k)a - x)^{n-1}}{(2a)^n (n-1)!} \theta_H((n-2k)a - x) , \end{aligned} \tag{12}$$

where in the last step we used the identity in Hint 4 to keep only the term multiplied by the step function. The proof is concluded noting that $x\theta_H(x) = \max(x, 0)$.

(d) [MEDIUM] Verify the validity of the central limit theorem for the sum of variables with uniform distribution (you can work with the characteristic function).

Hint 1: $\log \frac{\sin t}{t} = \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{2n(2n)!}(2t)^{2n}$, where the coefficients B_n are known as "Bernoulli numbers", $B_0 = 1$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, et cetera.

The characteristic function of $\tilde{\xi} = \frac{\xi^{(n)}}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \xi_j$ *is given by*

$$\phi_{\tilde{\xi}}(t) = \phi_{\xi^{(j)}}(t/\sqrt{n}) = \left(\sqrt{n}\frac{\sin(at/\sqrt{n})}{at}\right)^n = \exp\left(n \log\left(\sqrt{n}\frac{\sin(at/\sqrt{n})}{at}\right)\right) =$$
$$\exp\left(\sum_{j=1}^{\infty} n^{1-j} \frac{(-1)^j B_{2j}}{2j(2j)!}(2at)^{2j}\right) = \exp\left(-\frac{(at)^2}{6} + O(1/n)\right). \quad (13)$$

In the limit $n \to \infty$ *this approaches the characteristic function of a Gaussian with mean zero and variance* $a^2/3$.

3. **Stable distributions. Definition**: A non-degenerate distribution π_ξ is a stable distribution if it satisfies: let ξ_1 and ξ_2 be independent copies of a random variable ξ (they have the same distribution π_ξ). Then π_ξ is said to be stable if for any constants $a > 0$ and $b > 0$ the random variable $a\xi_1 + b\xi_2$ has the distribution $\pi_{c\xi+d}$ for some constants $c > 0$ and d.

(a) [MEDIUM] Prove that the Gaussian $\pi_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is a stable distribution.

Remember the propriety of the generating functions

$$\Phi_{a\xi+b}(t) = e^{ibt}\Phi_\xi(at) \quad (14)$$

The generating function $\phi^{(G)}(t)$ *of a centered* ($\mu = 0$) *Gaussian distribution is given by*

$$\Phi^{(G)}(t) = e^{-\frac{\sigma^2}{2}t^2} \quad (15)$$

We consider the characteristic function of the sum of the two random variables $a\xi + b\xi$, *which we know to be given by the product of the characteristic functions of the single (Gaussian) distributions*

$$\Phi_{a\xi+b\xi}(t) = \Phi^{(G)}_{a\xi}(t)\Phi^{(G)}_{b\xi}(t) = e^{-\frac{\sigma^2}{2}t^2(a^2+b^2)} = \Phi^{(G)}_{\sqrt{a^2+b^2}\xi}(t) \quad (16)$$

We have then shown that the distribution of the sum of the two Gaussian random variables $a\xi + b\xi$ is a Gaussian distribution of the variable $c\xi$ with $c = \sqrt{a^2+b^2}$. Therefore the Gaussian distribution is stable.

(b) [EASY] Consider a characteristic function of the form

$$\Phi_\xi(t) = \exp\bigl(it\mu - (c_0 + ic_1 f_\alpha(t))|t|^\alpha)\bigr)\,, \tag{17}$$

with $1 \leq \alpha < 2$. Show that $f_\alpha(t) = \mathrm{sgn}(t)$, for $\alpha \neq 1$, and $f_1(t) = \mathrm{sgn}(t)\log|t|$ produce stable distributions. These are also known as *Lévy distributions*, after Paul Lévy, the first mathematician who studied them.

As before we consider the conbination of two random variables with a Lévi distribution has the characteristic function

$$\Phi_{a\xi_1+b\xi_2}(t) = \exp\bigl(it(a+b)\mu - (c_0 + ic_1 f_\alpha(t))|(a^\alpha + b^\alpha)^{1/\alpha} t|^\alpha\bigr)\,.) \tag{18}$$

If $\alpha \neq 1$ this is mapped into the same distribution by the transformation $t \to (a^\alpha + b^\alpha)^{-1/\alpha} t$ and $\mu \to (a^\alpha + b^\alpha)^{1/\alpha}(a+b)^{-1}\mu$. For $\alpha = 1$ the transformation is $t \to (a+b)^{-1} t$ and $\mu \to \mu - \frac{c_1}{\alpha}\log(a+b)$.

(c) [EASY] Find a distinctive feature of the Lévy distributions.

The second cumulant

$$\kappa_2 = (-i)^2 \partial_t^2 \log \Phi(t)\Big|_{t=0} \sim sign(t)|t|^{\alpha-2} + \ldots\Big|_{t=0} = \infty \tag{19}$$

as $\alpha < 2$, it diverges.

(d) [EASY] Assumes $\alpha \neq 1$ and show that, in order to be $\Phi_\xi(t)$ the Fourier transform of a probability distribution, the coefficient c_1 can not be arbitrarily large; determine its maximal value.

Hint 1: One can show (MEDIUM-HARD) that the inverse Fourier transform of (17) has the tails

$$\pi_\xi(x) \xrightarrow{|x|\gg 1} \frac{\Gamma(1+\alpha)}{2\pi|x|^{1+\alpha}}\Bigl(c_0 \sin\frac{\pi\alpha}{2} - c_1 \mathrm{sgn}(x)\cos\frac{\pi\alpha}{2}\Bigr)\,. \tag{20}$$

The probability distribution must be positive or equal to zero, therefore the coefficients of the tails of the Lévi distributions must be positive. Since

$$\pi_\xi(x) \xrightarrow{|x|\gg 1} \frac{\Gamma(1+\alpha)}{2\pi|x|^{1+\alpha}}\Bigl(c_0 \sin\frac{\pi\alpha}{2} - c_1 \mathrm{sgn}(x)\cos\frac{\pi\alpha}{2}\Bigr) \tag{21}$$

we find

$$|c_1| < c_0 \left| \tan \frac{\pi\alpha}{2} \right| \tag{22}$$