Tutorial 6, Statistical Mechanics: Concepts and applications
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Maurizio Fagotti, Olga Petrova, Werner Krauth
Tutorial exercises

I. WORKSHEET: THERMODYNAMIC QUANTITIES AND CORRELATION FUNCTIONS IN SERIES EXPANSIONS

1. **Ising model in 2D in magnetic field: high temperature expansion**

The model: Consider the high temperature expansion of the classical Ising model, defined on the square lattice with energy

$$E(\{\sigma\}) = -J \sum_{(i,j)} \sigma_i \sigma_j - H \sum_i \sigma_i \ , \tag{1}$$

where (i, j) signifies nearest neighbor pairs and J is assumed to be positive.

(a) Write down the partition function Z in the form that contains products of variables $v = \tanh \beta J$ and $y = \tanh \beta H$. The high temperature expansion derived in the steps to follow will be a series of powers of v and y.

(b) Show that the terms in Z can be represented by graphs on the square lattice where each graph with l bonds and m odd vertices contributes a factor $2^N v^l y^m$ (here an odd vertex refers to there being an odd number of bonds connected to the vertex).

(c) Write down Z up to the fourth order each in v and y.

(d) Take the logarithm of Z to find the free energy F and expand for small v and y. What can you say about the contributions coming from disconnected graphs in the Z expansion?

(e) Compute the high temperature series for the zero-field susceptibility.

2. **Ising model in 2D: connected correlations functions**

Reminder: A connected correlation function is the joint cumulant of some random variables. For example, indicating the classical spins of a given statistical model by $\sigma_i \in \{-1, 1\}$, we have

$$\langle \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_n}\rangle_c = \frac{\partial^n}{\partial J_1\cdots\partial J_n}\log\langle e^{\sum_i J_i\sigma_{j_i}}\rangle\Big|_{J_1=\cdots=J_n=0}\,, \tag{2}$$

where the subscript c on the left hand side is used to indicate the connected correlation, while on the right hand side the brackets stand for the mean value.

The model: Consider a classical Ising model in a square lattice with energy

$$E(\{\sigma\}) = -J\sum_{(i,j)}\sigma_i\sigma_j - h\sum_i \sigma_i\ , \tag{3}$$

where (i,j) means that σ_j is adjacent to σ_i and periodic boundary conditions are imposed at the edges. As well known, for $J > 0$, the model is ferromagnetic below the critical temperature and paramagnetic at high temperature. Let J be positive.

(a) Set $h = 0$ and consider the connected two-point function $\langle\sigma_1\sigma_{1+n}\rangle_c$ in the paramagnetic phase (let us assume that the spins σ_1 and σ_{1+n} belong to the same line and have distance n). Perform a high-temperature expansion and list all the diagrams that give a contribution up to $O(\tanh^{n+4}(\beta J))$.

(b) Write down the correlator up to $O(\tanh^{n+2}(\beta J))$.

(c) Identify the contribution given by the first diagram $O(\tanh^{n+4}(\beta J))$ consisting of two disconnected parts. Can there be contributions proportional to the volume?

(d) Can you draw some physical conclusion in the limit of large distance?

HINT: You can assume $n \ll \tanh^{-2}(\beta J)$.

(e) Assume again $h = 0^+$ and consider the (connected) one-point function $\langle\sigma_1\rangle$ in the ferromagnetic phase. Perform a low-temperature expansion and compute the correlator at $O(e^{-16\beta J})$.

HINT: If (and only if) you are able to explain the reason, you can avoid computing all the terms proportional to N^j, with $j > 0$.

(f) Compute the connected two-point function $\langle\sigma_1\sigma_{1+n}\rangle_c$ up to $O(e^{-12\beta J})$ (or, if you are quick enough, $O(e^{-16\beta J})$). What happens in the limit of large distance?