Tutorial 5, Statistical Mechanics: Concepts and applications
2016/17 ICFP Master (first year)

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Tutorial exercises

I. WORKSHEET: ISING MODEL IN $D \geq 2$ – THE PEIERLS ARGUMENT

[Source: R. Peierls, M. Born, Proceedings of the Cambridge Philosophical Society, **32***, 477 (1936)*
C. Bonati, Eur. J. Phys. **35***, 035002 (2014)]*

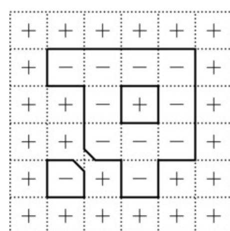
1. **Peierls argument for the Ising model in 2D**

Reminder: Peierls contours are drawn via the following procedure:

(a) draw a unit square on each site i with $\sigma_i = -1$

(b) cancel the edges that appear twice

An example of the Peierls contour for a given spin configuration is shown below:



The model: Consider a classical Ising ferromagnet, defined for spins $\sigma \in \{+1, -1\}$:

$$E = -J \sum_{(i,j)} \sigma_i \sigma_j - h \sum_i \sigma_i \ , \tag{1}$$

where J is assumed to be positive, and we set the applied magnetic field h to zero. We define the average magnetization per lattice site as

$$m = \frac{1}{N} \sum_i \sigma_i = \frac{N_+ - N_-}{N} = 1 - 2\frac{N_-}{N} \tag{2}$$

where N is the total number of spins and $N_\pm$ is the number of ± 1 spins. In $D \geq 2$, this system undergoes a phase transition at the critical temperature T_c. In the paramagnetic phase ($T > T_c$), the average magnetization in thermodynamic limit $\langle m \rangle$ vanishes, whereas in the ferromagnetic phase ($T < T_c$) it does not. This non-vanishing value of $\langle m \rangle$ is generally ill-defined, so we use one of the possible ways to overcome this issue by imposing the following boundary condition:

$$\sigma_{i_b} = +1.$$

In this set of exercises, we will use the Peierls argument to show that $\langle N_- \rangle / N < 1/2 - \epsilon$ (for every N) in ferromagnetic phase, from which it follows that $\langle m \rangle > 0$.

(a) Assume a finite $\sqrt{N} \times \sqrt{N}$ lattice. Label an arbitrary Peierls contour by γ_L^i, where L is the length of the contour (e.g., $L = 4$ for a single closed square). Provide an upper bound for N_- in a given spin configuration in terms of the areas inside applicable Peierls contours.

Hint: You may find it useful to define a variable $X(\gamma_L^i)$ that is equal to 1 if γ_L^i occurs in the given spin configuration and zero otherwise.

(b) Give an upper bound $A(L)$ on the area inside a contour $A(\gamma_L^i)$ as a function of only the contour's length L.

(c) The quantity we are interested in is the thermal average $\langle N_- \rangle$. Find an upper bound $X(L)$ on $\langle X(\gamma_L^i) \rangle$.

(d) Derive an upper bound on the number of closed paths of length L.

(e) Use the quantities you calculated to write down an expression for $\langle N_- \rangle$, which will be proportional to a sum over contour lengths L. Simplify it. The final result should be of the form

$$\langle N_- \rangle \leq N f_2(x)$$

where $x = 9e^{-4J\beta}$ and $f_2(x)$ is a continuous function of x.

Explain the physical meaning of this result.

2. **Peierls argument for the Ising model in 3D**

The model: Consider a three dimensional cubic lattice with dimensions $N^{1/3} \times N^{1/3} \times N^{1/3}$. The Peierls contours of the two dimensional case now become surfaces, but their construction proceeds along the same lines as in the two dimensional case.

(a) Label an arbitrary Peierls surface by γ_S^i, where S is the surface area measured in units of elementary squares. Provide an upper bound for N_- in a given spin configuration in terms of the volumes inside applicable Peierls surfaces.

(b) Give an upper bound $V(S)$ on the volume inside a surface $V(\gamma_S^i)$ as a function of only the surface area S.

(c) Find an upper bound $X(S)$ on $\langle X(\gamma_S^i)\rangle$.

(d) Derive an upper bound on the number of closed surfaces of area S.

(e) Use the quantities you calculated to write down an expression for $\langle N_-\rangle$, which will be proportional to a sum over surface areas S. The final result should be of the form

$$\langle N_-\rangle \leq N f_3(x)$$

where $x = 9e^{-4J\beta}$ and $f_3(x)$ is a continuous function of x.

(f) Use the same reasoning to arrive at a similar result for the general $D \geq 3$ case.

(g) Why cannot the Peierls argument be applied to the one dimensional Ising model?