Advanced topics in Markov-chain Monte Carlo

Lecture 4:
Perfect sampling in Markov-chain Monte Carlo
Part 2/3: Coupling from the past / Ising model

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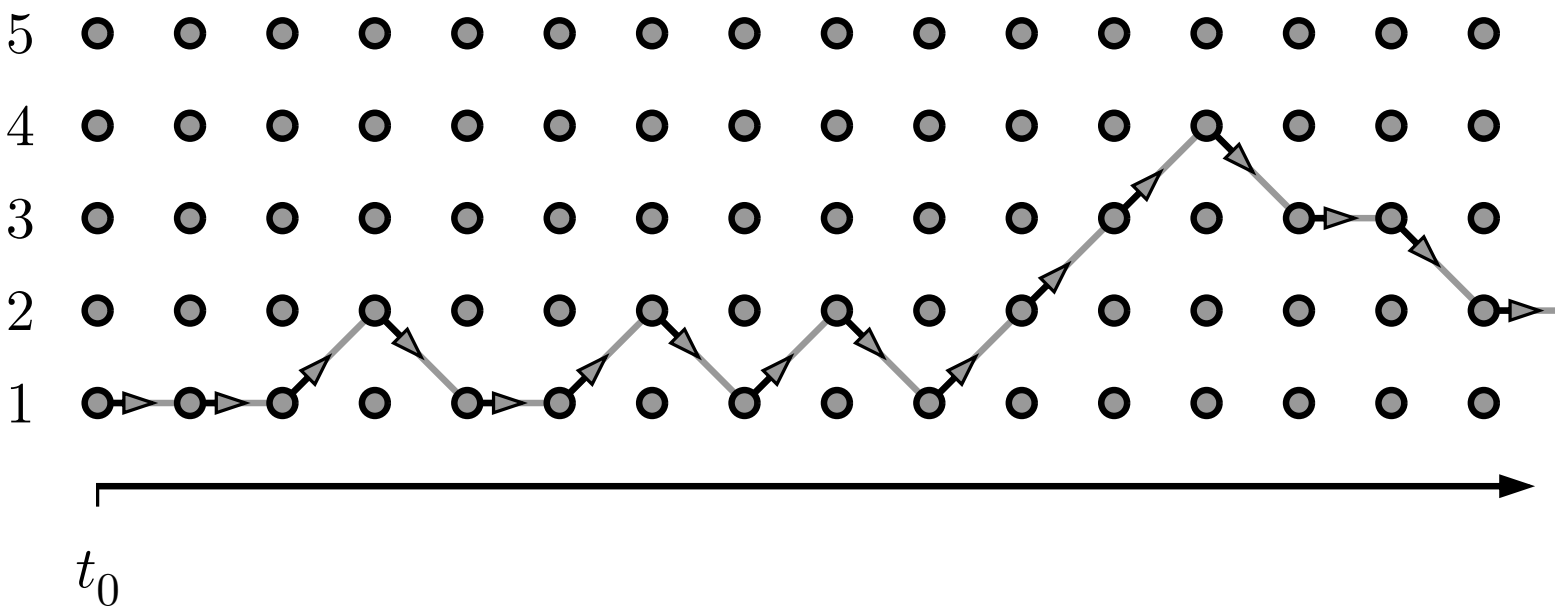
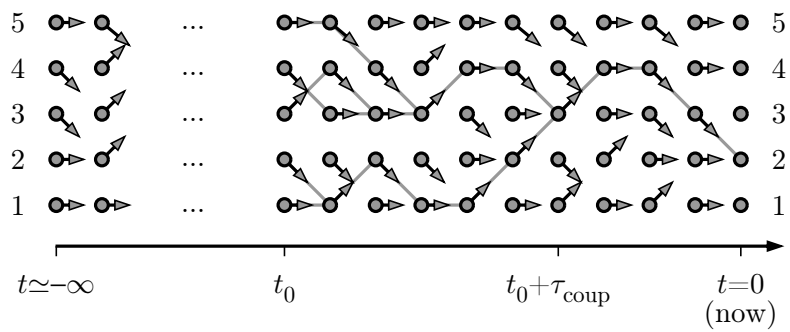
Markov chain (traditional view)



- Configuration c_t, move δ_t.
- Set $t_0 = 0$.
- Transition matrix easy to write down (TD)

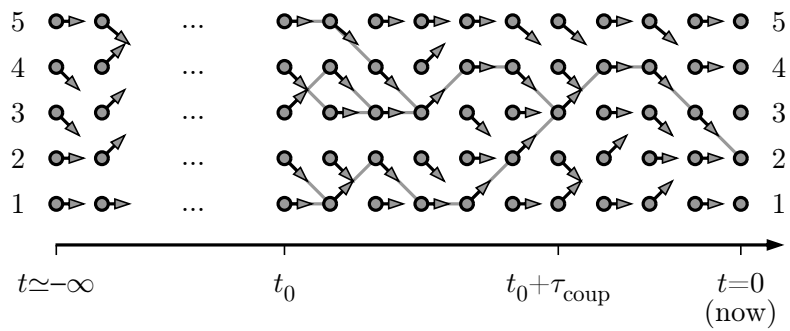
Markov chain coupling



$$p(|\circ\circ\bullet\circ\bullet\rangle \to |\circ\bullet\circ\bullet\circ\rangle) = 1/9$$
$$p(|\circ\circ\bullet\circ\bullet\rangle \to |\circ\circ\bullet\circ\bullet\rangle) = 2/9$$
$$p(|\circ\circ\bullet\circ\bullet\rangle \to |\circ\circ\circ\bullet\circ\rangle) = 1/9$$
$$p(|\circ\circ\bullet\bullet\bullet\rangle \to |\circ\circ\circ\bullet\circ\rangle) = 1/27,$$

- Each configuration has its move at each time step.
- Coupling (Doeblin, 1930s).

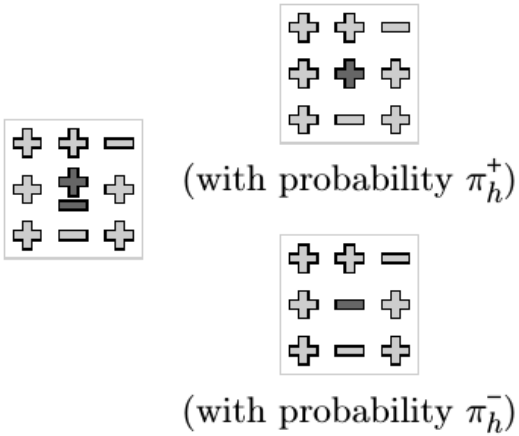
Coupling forward



$$P = \begin{pmatrix} P^{1,1} & P^{2,1} & \dots & \dots \\ 0 & P^{2,2} & T^{3,2} & \dots \\ 0 & 0 & P^{3,3} & \dots \\ \dots & & & \\ 0 & 0 & & P^{N,N} \end{pmatrix}$$

- $P^{k,l}$: Go from k particles to l particles in one step.
- The back-coupling transition matrix has the same eigenvalues as the forward-coupling transition matrix.
- interpretation

Ising model - heat bath

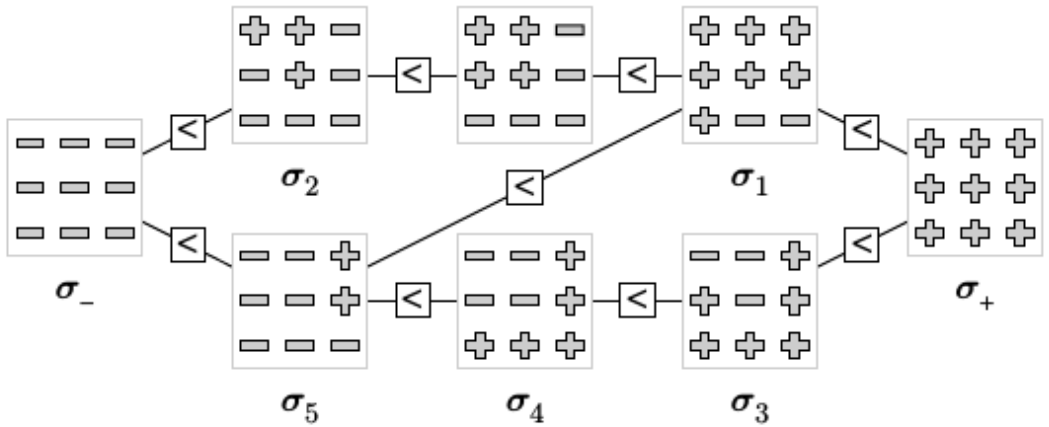


$$\pi_h^+ = \frac{e^{-\beta E^+}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{-2\beta h}},$$

$$\pi_h^- = \frac{e^{-\beta E^-}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{+2\beta h}}.$$

- Roughly equivalent to Metropolis algorithm.

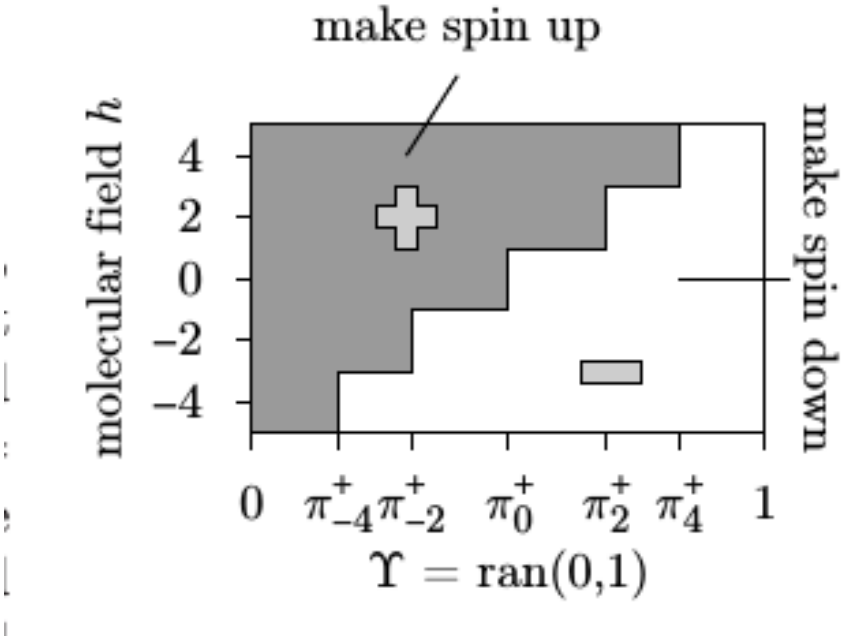
Ising model - half order



$$\pi_h^+ = \frac{e^{-\beta E^+}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{-2\beta h}},$$

$$\pi_h^- = \frac{e^{-\beta E^-}}{e^{-\beta E^+} + e^{-\beta E^-}} = \frac{1}{1 + e^{+2\beta h}}.$$

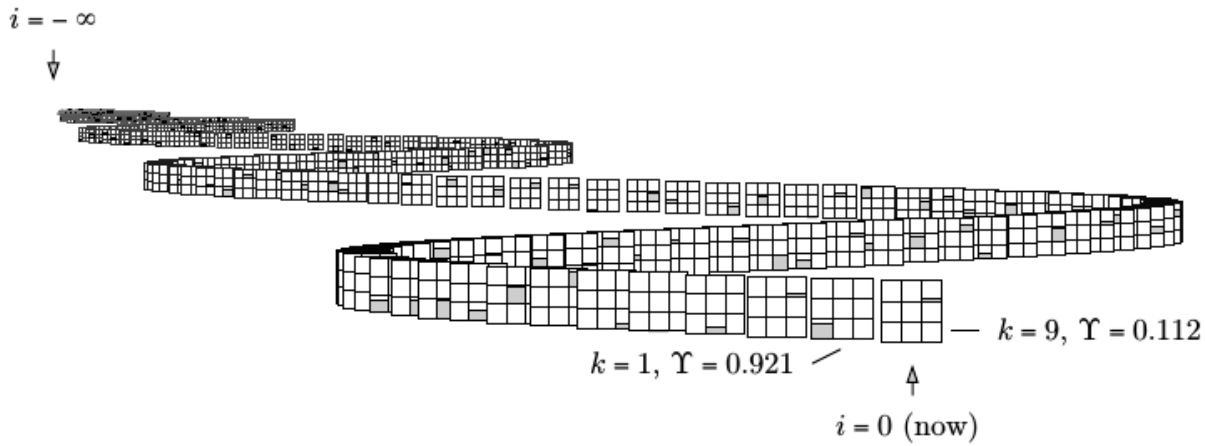
Ising model - half order



$$\pi_h^+ = \frac{\mathrm{e}^{-\beta E^+}}{\mathrm{e}^{-\beta E^+} + \mathrm{e}^{-\beta E^-}} = \frac{1}{1 + \mathrm{e}^{-2\beta h}},$$

$$\pi_h^- = \frac{\mathrm{e}^{-\beta E^-}}{\mathrm{e}^{-\beta E^+} + \mathrm{e}^{-\beta E^-}} = \frac{1}{1 + \mathrm{e}^{+2\beta h}}.$$

Ising model - coupling from the past



- Ising-model simulation that has run since time $i = -\infty$.
- Allows us to produce perfect samples.