

Lectures on Statistical Mechanics

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Lecture 5

1D models and the transfer matrix

In this lecture, we introduce to the statistical physics of one-dimensional systems, starting from two examples, the one-dimensional gas of hard spheres, and the one-dimensional Ising model with next-neighbor interactions. These models are both exactly solvable. In finding the solutions, we will introduce to a fundamental concept called “Transfer Matrix”. We will show that these models exist only in a single phase, so that there cannot be a phase transition. Mathematically, this feature of the exact solution derives from basic properties of the transfer matrix, via two fundamental theorems. A physical argument will also be developed, namely the analysis of domain walls. As often, the well-understood and exactly solved cases are very closely related to hard problems: We will analyze the Ising model with long-range interactions which, understood by Thouless, became the starting point of the now famous Kosterlitz–Thouless theory.

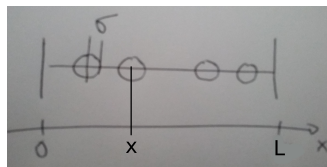


Figure 5.1: $N = 4$ one-dimensional hard spheres of radius σ on an interval of length L , without periodic boundary conditions. Many of the properties of this model can be solved analytically. The position x of each sphere is that of its center, so that $\sigma < x < L - \sigma$.

5.1 One-dimensional hard spheres

The one-dimensional hard-sphere model is exactly solvable (its thermodynamics and some of its structural properties can be obtained in closed form). We will compute the equation of state, check the equivalence of ensembles, introduce to the virial expansion, and compute (what amounts to) correlation functions. In a nutshell, the model can be mapped to non-interacting particles. Nevertheless, the model has non-trivial correlation functions, that are so intricate that not all is known analytically about them.

We consider, as in Fig. 5.1, N one-dimensional spheres in a box of finite volume (length) L without any other interaction than the hard-sphere repulsion. Spheres have velocities and positions, but the momenta can be integrated out in the partition function, so that only the configurational integral remains.

5.1.1 One-dimensional hard spheres - partition function

We can write down the partition function of this system is given by

$$Z_{N,L} = \int_{\sigma}^{L-\sigma} dx_1 \cdots \int_{\sigma}^{L-\sigma} dx_N f(x_1, \dots, x_N), \quad (5.1)$$

where the Boltzmann weight f satisfies

$$f(x_1, \dots, x_N) = f(x_{P_1}, \dots, x_{P_N}) = \begin{cases} 1 & \text{if legal} \\ 0 & \text{otherwise} \end{cases}$$

Note that π is symmetric under permutations. One often multiplies the partition function of eq. (5.1) by a factor $1/N!$, in order to avoid the so-called Gibbs paradox. The question of the presence or not of this factor $1/N!$ is very interesting, and it has nothing to do with quantum mechanics, see [?]. We will just leave it out.

5.1.2 Free energy by mapping to non-interacting particles

The statistical weight f , as discussed, and also the integration domain, are totally symmetric under permutations ($P_1 \dots P_N$) that each indicate a certain ordering of particle indices. We are free to choose one of the $N!$ permutations (for example $x_1 < x_2 < \dots < x_N$), and multiply the integral with this same factor $N!$:

$$Z_{N,L} = N! \int_{\sigma}^{L-\sigma} dx_1 \cdots \int_{\sigma}^{L-\sigma} dx_N f(x_1, \dots, x_N) \Theta(x_1, \dots, x_N) \quad (5.2)$$

(the function Θ is equal to one if $x_1 < x_2 < \dots < x_N$ and zero otherwise). Now, let us change variables $x \rightarrow y$ in the integral as follows:

$$y_1 = x_1 - \sigma, \dots, y_k = x_k - (2k - 1)\sigma, \dots, y_N = x_N - (2N - 1)\sigma,$$

This transformation simply shrinks the radius of each sphere to zero, and the y_1, \dots, y_N are non-interacting points on an interval of length $L - 2N\sigma$. We obtain the integral

$$Z_{N,L} = N! \int_0^{L-2N\sigma} dy_1 \cdots \int_0^{L-2N\sigma} dy_N \Theta(y_1, \dots, y_N), \quad (5.3)$$

Look at this integral: the bounds for y_1, y_2, \dots are again symmetric, and we can undo the trick that brought us from eq. (5.1) to eq. (5.2) by suppressing the ordering of the variables y and also the factor $N!$. We arrive at the partition function of N non-interacting particles:

$$Z_{N,L} = \int_0^{L-2N\sigma} dy_1 \cdots \int_0^{L-2N\sigma} dy_N = \begin{cases} (L - 2N\sigma)^N & \text{if } L > 2N\sigma \\ 0 & \text{otherwise} \end{cases}. \quad (5.4)$$

We may rewrite the partition function in terms of the density $\rho = 2\sigma N/L$

$$Z_{N,\rho} = \begin{cases} L^N (1 - \rho)^N & \text{if } \rho < 1 \\ 0 & \text{otherwise} \end{cases} \quad (5.5)$$

We see that this is an analytic function for all N and all ρ , and even that the free energy per sphere $-\log Z/N$ is an analytic function, so that this model has no phase transition at finite density.

5.1.3 Partition function, Transfer-matrix derivation

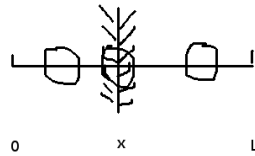


Figure 5.2: Transforming three hard spheres on an interval of length L into two intervals (one of length $x - \sigma$, and one of length $L - x - \sigma$) separated by a wall (represented by the center sphere, at position x). Allowing for all possible x yields the partition function.

FIXME {This is a recursive solution, rather than a transfer-matrix one??} The second derivation of the one-d hard-sphere partition function uses a transfer-matrix strategy. Let us first compute the partition function for $N = 1$ (without invoking eq. (5.4)). Evidently it is given by $Z_{1,L} = L - 2\sigma$ (of course only if $L \geq 2\sigma$). It is zero otherwise. Now we immediately go from one to three spheres (see Fig. 5.2). The center sphere can be transformed into a wall. We find that

$$Z_{3,L} = \int_{3\sigma}^{L-3\sigma} dx Z_{1,x-\sigma} Z_{1,L-x-\sigma} \quad (5.6)$$

$$= \int_{3\sigma}^{L-3\sigma} dx (x - 3\sigma)(L - x - 3\sigma) \quad (5.7)$$

$$= \frac{1}{6}(L - 6\sigma)^3 \quad (5.8)$$

$$= \frac{1}{3!}(L - 2N\sigma)^3 \quad (5.9)$$

The general case is relegated to the homework session of this lecture. It consists in cutting up the partition function for $N + 1$ spheres into a partition function for $N - 1$ spheres in front of a wall (represented by sphere N) and a lone sphere $N + 1$.

5.1.4 Pressure and the equivalence of ensembles

As we have in our possession the partition function $Z_{N,L} = (L - 2N\sigma)^N$, we can compute the pressure:

$$\underline{\beta P} = \frac{\partial \log Z}{\partial L} = N \frac{\partial}{\partial L} [N \log (L - 2N\sigma)] \quad (5.10)$$

$$= \frac{N}{L - 2N\sigma} = \frac{N}{L(1 - \rho)}, \quad (5.11)$$

where ρ is the density (the fraction $2N\sigma/L$ of space occupied by the spheres). The eq. (5.11) is an analytic function, even in the limit $N \rightarrow \infty$, so that there is no phase transition in the one-dimensional hard-sphere model in one dimension. The compressibility

$$\kappa = -\frac{1}{L} \frac{\partial V}{\partial P} \quad (5.12)$$

is always positive (in a finite system as well as in the limit of $L, N \rightarrow \infty$. As discussed in Lecture 4, this condition is a necessary stability condition in an infinite system but it does not hold in a finite system, because of the possible presence of interfaces.

Our calculation, so far, has been in the NVT ensemble, where of course the role of the volume is played by the length L (we are in one dimensions), and where the temperature is irrelevant because we deal with hard spheres (π equals 0 or 1 at all temperatures). But this gives an equation of state, where the control parameter is the volume (in other words L), and we put it on the x -axis. We can compute the behavior in other ensembles, for example at constant chemical potential and pressure, but let us check, for the time being, the constant pressure (NPT) ensemble (see [5, chap??]). In constant pressure, but variable volume L , the partition function is:

$$Z_{NPT} = \int dL e^{-\beta PL} Z_{N,L} \quad (5.13)$$

$$= \int_{2N\sigma}^{\infty} dL e^{-\beta PL} \underbrace{(L - 2N\sigma)^N}_{\text{set } L' = L - 2N\sigma} \quad (5.14)$$

$$= \int_0^{\infty} dL' e^{-\beta P(L' + 2N\sigma)} L'^N = \quad (5.15)$$

In the NPT ensemble, the volume L is an observable, and we can compute its expectation value:

$$\langle L - 2N\sigma \rangle = \dots = \frac{N + 1}{\beta P}, \quad (5.16)$$

so that we obtain

$$P = \frac{N + 1}{\langle L - 2N\sigma \rangle}, \quad (5.17)$$

which is almost the same as eq. (5.11). Task: For added symmetry, formulate eqs (5.17) and (5.11) such that both contain a mean value.

5.1.5 The virial expansion

An ideal gas satisfies $PV = NRT$ or, in our variables of the one-dimensional hard-sphere gas, it satisfies $\beta PL/N = 1$. This is nothing but the limit of eq. (5.11) in the small-density limit $\rho \rightarrow 0$. The corrections to the ideal-gas behavior for physical systems in higher dimensions, and in particular for hard-sphere models, have fascinated generations of physicists, starting with Boltzmann, in 1874. In the one-dimensional hard-sphere gas, we can of course compute:

$$\beta PL/N = \frac{1}{1-\rho} = 1 + \rho + \rho^2 \dots \quad (5.18)$$

$$= 1 + B \frac{1}{L} + C \frac{1}{L^2} \dots \quad (5.19)$$

(the above equation is interpreted as a function of the volume L at fixed N). The coefficient $B = 2N\sigma$ is the second “virial” exponent. As mentioned in the lecture, Boltzmann computed the fourth virial coefficient for three-dimensional hard spheres, and it took half a century to check that his calculation was correct. It was believed that the virial expansion of eq. (5.19) allowed one to access the behavior of real systems, but this was not really true. In the one-dimensional hard-sphere gas, we can

1. compute all the virial coefficients,
2. prove that the virial expansion actually converges for all $\rho < 1$.

This is much more than we can usually achieve.

5.1.6 The probability to be at position x

Computing [5] the probability $\pi(x)$ to be at x , for N one-dimensional hard spheres on an interval of length L , is represented by the statistical weight of having a sphere at x and then k more spheres to the left of x and $N - 1 - k$ spheres to its right. As, initially, we have to choose one sphere out of N to put it at x and then k spheres out of the remaining $N - 1$, $\pi(x)$ is given by the sum of the statistical weights of putting k disks to the left (in the remaining left interval of length $x - \sigma$) and the $N - 1$ remaining disks to the right (length of interval $L - x - \sigma$):

$$\pi(x) = \sum_{k=0}^{N-1} \frac{1}{Z_{N,L}} \underbrace{\binom{N-1}{k} Z_{k,x-\sigma} Z_{N-1-k,L-x-\sigma}}_{\pi_k(x)} \quad (5.20)$$

The function $\pi(x)$ is easy to write down, and also to evaluate numerically (this is done in a program on a web site. You must to make sure that both cases in eq. (5.4) are implemented, that is, that $Z_{N,L} = 0$ if $L < 2N\sigma$. (see Fig. 5.3). For $\rho < 1/2$, there is a central region, where the density $\pi(x)$ is strictly independent of x . You can check this out numerically, but to prove it is really complicated [?].

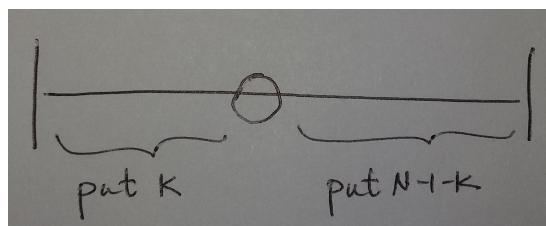


Figure 5.3: Probability to be at x in the one-dimensional hard-sphere gas. It is computed by placing a first sphere at position x and by asking about the statistical weight of all possible configurations of k spheres to the left and $N - 1 - k$ spheres to the right.

5.1.7 Hard spheres on a ring of length L , correlation functions

One might think that in the problem of one-dimensional hard spheres, all we did was to consider boundary effects close to a wall. After all, in the center of the system, the density is constant. This reasoning is wrong: The probability to be at a point x is closely related to the correlation function at distance x (that is, the probability to have two disks a distance x apart).

5.2 One-dimensional Ising model

5.2.1 Partition function, Transfer-matrix derivation $h = 0$

We consider the Ising model in one dimensions (Ising chain), for the moment without a magnetic field. The hamiltonian (the energy) of the system is given by

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}. \quad (5.21)$$

Open boundary conditions

Let us first compute the partition function for two sites ($N = 2$) without periodic boundary conditions. It is given by the following four configurations and the partition function is the sum of their statistical weights:

$$Z_2 = \sum \begin{array}{l} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} \begin{array}{l} e^{\beta J} \\ e^{-\beta J} \\ e^{-\beta J} \\ e^{\beta J} \end{array} = 2 \left(e^{\beta J} + e^{-\beta J} \right) = 4 \cosh(\beta J). \quad (5.22)$$

In a typical “transfer-matrix” approach, we can now move from the partition func-

tion with $N - 1$ spins to the partition function with N spins and find:

$$Z_N = \sum \begin{array}{l} \boxed{\dots \uparrow} \downarrow = \frac{1}{2} Z_{N-1} \exp(-\beta J) \\ \boxed{\dots \uparrow} \uparrow = \frac{1}{2} Z_{N-1} \exp(\beta J) \\ \boxed{\dots \downarrow} \downarrow = \frac{1}{2} Z_{N-1} \exp(\beta J) \\ \boxed{\dots \downarrow} \uparrow = \frac{1}{2} Z_{N-1} \exp(-\beta J) \end{array} = 2 \left(e^{\beta J} + e^{-\beta J} \right) = 4 \cosh(\beta J) \quad (5.23)$$

where each of the boxes contains all the configurations of $N - 1$ spins with the spin $N - 1$ oriented as indicated. We notice that the basic symmetry between up and down spins requires that the partition function of a system of $N - 1$ spins with the final spin in up position is simply $\frac{1}{2} Z_{N-1}$. We find

$$Z_N = Z_{N-1} [2 \cosh(\beta J)] = Z_2 [2 \cosh(\beta J)]^{N-2} = 2 [2 \cosh(\beta J)]^{N-1} \quad (5.24)$$

so that one has

$$F = -kT \log Z_N = \frac{1}{\beta} [\log 2 + (N - 1) \log [2 \cosh(\beta J)]] \quad (5.25)$$

Clearly this is an analytic function at all temperatures and there is no phase transition.

We used in eq. (5.23) that the partition function $Z_{N-1}^\downarrow = \boxed{\dots \downarrow}$ was the same as $Z_{N-1}^\uparrow = \boxed{\dots \uparrow}$. More generally, we have that

$$Z_N^\uparrow = Z_{N-1}^\uparrow \times \uparrow\uparrow + Z_{N-1}^\downarrow \times \downarrow\uparrow \quad (5.26)$$

$$Z_N^\downarrow = Z_{N-1}^\uparrow \times \uparrow\downarrow + Z_{N-1}^\downarrow \times \downarrow\downarrow, \quad (5.27)$$

where we remember that Z_N^\uparrow is the partition function of the Ising model with the restriction that the final spin (spin N) is "up". We write eq. (5.27) as

$$\begin{bmatrix} Z_N^\uparrow \\ Z_N^\downarrow \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{bmatrix}}_{\text{Transfer Matrix } T} \begin{bmatrix} Z_{N-1}^\uparrow \\ Z_{N-1}^\downarrow \end{bmatrix} \quad (5.28)$$

where the 2×2 matrix here is called the transfer matrix. Clearly we have:

$$\begin{bmatrix} Z_N^\uparrow \\ Z_N^\downarrow \end{bmatrix} = T^{N-1} \begin{bmatrix} Z_1^\uparrow \\ Z_1^\downarrow \end{bmatrix} \quad (5.29)$$

With $Z_1^\uparrow = Z_1^\downarrow = 1$, you easily check the value of eq. (??) for Z_2 .

Periodic boundary conditions

The Ising chain of N spins with periodic boundary conditions is the same as an Ising chain on $N + 1$ spins with open boundary conditions and two additional conditions:

1. If spin 1 is \uparrow , then spin $N + 1$ is \uparrow .
2. If spin 1 is \downarrow , then spin $N + 1$ is \downarrow .

(Next week, you'll need full understanding of the above sentence in a much more complicated setting, so don't hesitate to spend 20 minutes having the above sink in: With the two conditions, we have N independent spins, and spin $N + 1$ is simply glued onto spin 1, making the two to be one and the same.) We can impose these two condition separately by considering only Z_{N+1}^\uparrow for the case where $[Z_1^\uparrow, Z_1^\downarrow] = [1, 0]$ and by considering only Z_{N+1}^\downarrow for the case where $[Z_1^\uparrow, Z_1^\downarrow] = [0, 1]$. We find that the partition function is given by

$$Z_N^{\text{period}} = (T^N)(1,1) + (T^N)(2,2) = \text{Tr} (T^N) \quad (5.30)$$

where Tr stands for the trace of the matrix, the sum of its diagonal elements. T^N is the N th power of the matrix T . In eq. (5.30), one first takes the N th power of the matrix, then sums the diagonal elements.

5.2.2 Ising partition function (finite field)

If we consider the short-range Ising model in a magnetic field, then the hamiltonian is given by

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} - h \sum_{i=1}^N \sigma_i \quad (5.31)$$

and we may write the magnetic-field dependent term as $\frac{1}{2}h(\sigma_i + \sigma_{i+1})$, although that is not a big deal. In any case, the transfer matrix is

$$\begin{bmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J+h)} \end{bmatrix} \quad (5.32)$$

Example: For a two-site Ising chain with periodic boundary conditions, square the matrix and take the trace, and then check that this corresponds to the naive sum over the four terms. This is a useful exercise for next week. The outcome is

$$\text{Tr} Z_2 = e^{2\beta(J+h)} + e^{-2\beta J} e^{2\beta(J-h)} e^{-2\beta J}. \quad (5.33)$$

Also, using for a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2} \left(a + d \pm \sqrt{a^2 + d^2 + 4bc - 2ad} \right) \quad (5.34)$$

and using that the partition function with periodic boundary conditions is $Z_N = \lambda_1^N + \lambda_2^N \dots$ you easily compute the free energy. One can also compute $m = -1/N \partial F / \partial h = \dots$ and one easily finds, with "un coup de Mathematica",

$$m = \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}} \quad (5.35)$$

so that one obtains $m = 0$ for $h \rightarrow 0$ for all temperatures T .

5.2.3 Spin correlations

The two-point function $\langle s_i s_{i+\ell} \rangle$ can be computed within the transfer-matrix formalism with the insertion of $(-\sigma^z)$'s:

$$\begin{aligned} \langle s_i s_{i+\ell} \rangle &= \frac{\text{tr}[T^i \sigma^z T^\ell \sigma^z T^{N-i-\ell}]}{Z} \rightarrow \lambda_+^{-\ell} \langle \lambda_+ | \sigma^z T^\ell \sigma^z | \lambda_+ \rangle \\ &= \langle \lambda_+ | \sigma^z | \lambda_+ \rangle^2 + \left(\frac{\lambda_+}{\lambda_-} \right)^{-\ell} |\langle \lambda_+ | \sigma^z | \lambda_- \rangle|^2 \end{aligned} \quad (5.36)$$

The first term is nothing but the magnetization per unit length, so the connected correlation is given by

$$\langle s_i s_{i+\ell} \rangle - \langle s_i \rangle \langle s_{i+\ell} \rangle = \left(\frac{\lambda_+}{\lambda_-} \right)^{-\ell} |\langle \lambda_+ | \sigma^z | \lambda_- \rangle|^2 \quad (5.37)$$

and the correlation length is

$$\xi = \left(\log \frac{\lambda_+}{\lambda_-} \right)^{-1}. \quad (5.38)$$

We note that the correlation length depends on the subleading eigenvalue of the transfer matrix. Again, this is quite general.

More explicitly we find

$$\begin{aligned} \xi &= \left(\log \frac{\cosh(\beta J h) + \sqrt{\sinh^2(\beta J h) + e^{-4\beta J}}}{\cosh(\beta J h) - \sqrt{\sinh^2(\beta J h) + e^{-4\beta J}}} \right)^{-1} \\ i &= \left(2 \operatorname{arctanh} \frac{\sqrt{\sinh^2(\beta J h) + e^{-4\beta J}}}{\cosh(\beta J h)} \right)^{-1}. \end{aligned} \quad (5.39)$$

5.3 Absence of phase transition in one-dimensional statistical systems

We discuss two reasons why in one-dimensional statistical-physics systems a phase transition is often absent. One of the reasons is mathematical: The transfer matrix is irreducible, so that its dominant eigenvalue is non-degenerate (there is only a single one of them). Furthermore, one can prove that if the matrix elements are analytic functions, then so must be the largest eigenvalue. The second reason is more qualitative, often wrong, but essential to be known. It is related to domain walls.

5.3.1 Frobenius–Perron Theorem

The result obtained in Section 5.2.1 is in fact rather general, indeed one can show that models with *finite-dimensional* transfer matrices can have phase transitions only if there are *forbidden* (infinite energy) configurations. This is a consequence of two theorems. The first is known as Perron-Frobenius theorem:

Theorem 5.3.1 *Let A be an irreducible matrix with non-negative elements; the maximum eigenvalue is positive and non-degenerate.*

We remind that a matrix M is reducible if and only if it can be placed into block upper-triangular form by simultaneous row/column permutations. Clearly, a matrix with strictly positive elements is automatically irreducible.

$$P^t M P = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}, \quad (5.40)$$

where P is a permutation matrix and X and Z are square matrices.

The second useful theorem is a well-known result in complex analysis. It can be expressed as follows:

Theorem 5.3.2 *If $T(\beta)$ is a complex matrix with elements analytic functions of β , the eigenvalues are (branches) of analytic functions of β with algebraic singularities only localized at the points at which eigenvalues split or coalesce [].*

In statistical physics, the elements of the transfer matrix are generally exponentials, therefore, if any configuration is allowed, all the elements of a finite transfer matrix are nonzero and the matrix is in turn irreducible. By the Frobenius–Perron theorem, the leading eigenvalue is non-degenerate and, from Theorem 5.3.2, turns out to be an analytic function of β . This explains why the simplest one-dimensional (classical) models do not exhibit phase transitions. In this argument it is of importance that the transfer matrix in a one-dimensional system does not itself depend on N . This changes in more than one dimension, or for long-ranged interactions.

5.3.2 Domain walls in the 1D Ising model (local interactions)

Domain walls are excitations where the system of Ising spins is separated into one part with all up spins, followed by a part with all down spins. However, it is easy to see that the energy of a domain wall is J , but the entropy of a domain wall (entropy: Logarithm of the number of microstates), the log of the number of possibilities, which is L/a , where L is the system size and $a = 1$ the lattice parameter. We find that the free energy is

$$\Delta F \sim -kT \log \left(\frac{L}{a} \right) \quad (5.41)$$

which is negative. It is therefore favorable to add a domain wall, or two domain walls, and destroy the ferromagnetic ground state. We should be warned that domain wall (or spin wave) arguments are in general easy, and very often they turn out to be wrong.

5.3.3 Domain walls in the 1D Ising model with $1/r^2$ interactions

Notable work on the one-dimensional Ising model with $1/r^2$ interaction (all spins interacting, but with an energy decreasing with the square of the distance) is due to

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Fisher et al (1972)[12]. There was also the influential mathematical proof by Dyson[13], which showed that for interactions decaying slower than $1/r^2$, there had to be a phase transition, however this did not clear up the situation of the $1/r^2$ interaction.

Of particular interest is the two-page 1969 article by Thouless[14], which uses spin waves to explain that something unexpected must happen for the $1/r^2$ Ising model. This work cleared the way for the establishment of Kosterlitz–Thouless theory. Indeed, the one-dimensional Ising model with a $1/r^2$ interaction undergoes such a transition. There is also important work by Kosterlitz [15].

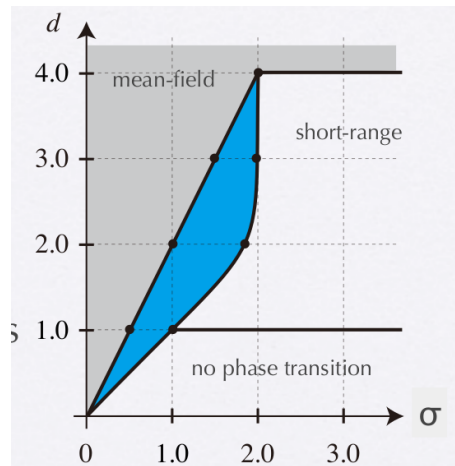


Figure 5.4: Phase diagram of long-range Ising models in D dimensions with interaction $1/r^{D+\sigma}$ (Illustration from a talk by Synge Todo, Univ of Tokyo)

This is an example of where the domain wall argument is not easy, and (as much research has shown) not wrong. It goes back to Thouless[14]. Consider a hamiltonian with interactions

$$H = - \sum_{i,j} \frac{\sigma_i \sigma_j}{(i-j)^2} \quad (5.42)$$

The cost of a domain wall at position x is

$$E = J \int_0^{x-a/2} \int_{x+a/2}^L \frac{dx_1 dx_2}{(x_1 - x_2)^2} \quad (5.43)$$

Integrating this energy twice, one may see that the energy of a domain wall on a system of length L with lattice parameter a is $\log L/a$. (This precise calculation will be the object of Homework 06). Both the energy of a domain wall and its entropy now scale like $\log L$. At low temperature, it becomes unfavorable to put a domain wall, while at high temperature, domain walls are favored. This result of Thouless[14] has been confirmed by much further research. See Fig. 5.4 for the general situation of long-range Ising models in D dimensions.

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