Tutorial 9, Statistical Mechanics: Concepts and applications
2016/17 ICFP Master (first year)

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Tutorial exercises

I. WORKSHEET: PHYSICS IN INFINITE DIMENSIONS

Source: R. J. Baxter, Exactly Solved Models in Statistical Mechanics*, Dover Publications (2008).*

1. **Ising model on the Bethe lattice**

Reminder: Consider the graph constructed as follows: start from a central point 0 and add q points all connected to 0. Call the set of these q points the "first shell". Now create further shells by taking a point in shell r and connecting $q-1$ new points to it. Do this for all points in shell r and call the set of all the new points "shell $r+1$". Proceeding iteratively in this way, construct shells $2, 3, \dots, n$. This gives a graph like that shown in Fig. 1. This is called *Cayley tree.* If the local properties of sites deep within the graph are considered (*i.e.* infinitely far away from the boundary in the limit $n \to \infty$), the boundary sites can be ignored (despite their number being comparable with the rest), and the graph is called *Bethe lattice.*

The model: Consider the Ising model on a Bethe lattice at inverse temperature β. The energy reads as

$$E(\{\sigma\}) = -J(\sum_{(i,j)} \sigma_i \sigma_j + h \sum_i \sigma_i)\,, \qquad (1)$$

and the first sum is over all the bonds of the Bethe lattice.

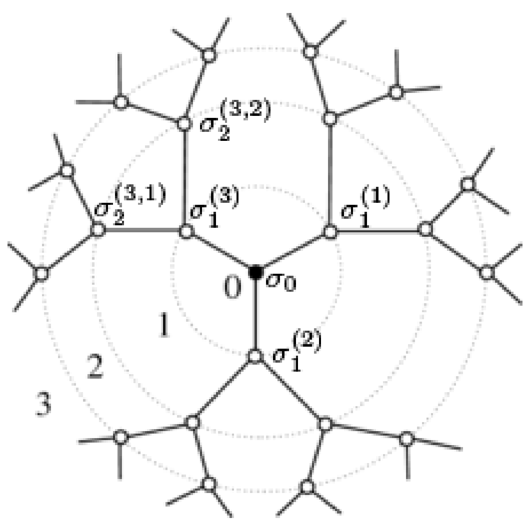

FIG. 1. A Cayley tree (with $q = 3$ and $n = 4$), divided at the central site 0 into three sub-trees.

(a) We define the dimensionality of a lattice as $d = \lim_{n\to\infty} \frac{\log c_n}{\log n}$, where $c_n = 1 + m_1 + \cdots + m_n$ with m_1 the number of neighbors per site, m_2 the number of next-nearest neighbors, and so on. Show that, for some regular lattices you know, this definition is consistent with our intuition. Which is the dimensionality of the Bethe lattice?

(b) Rewrite the probability of a configuration in the form

$$P(\{\sigma\}) = \frac{1}{Z} e^{\beta J h \sigma_0} \prod_{j=1}^{q} Q_n(\sigma_0|\{\sigma\}^{(j)})\,, \tag{2}$$

where $\{\sigma\}^j$ are all the spins in the j-th subtree (we are labeling the spins as $\sigma_i^{(j,k,\dots)}$, where $(j,k,\dots)$ identifies the specific branch and i is a redundant index equal to the number of the shell - Fig. 1).

(c) Show that $Q_n(\sigma_0|\{\sigma\}^j)$ satisfies the recurrence relation

$$Q_n(\sigma_0|\{\sigma\}^{(j)}) = e^{\beta J \sigma_0 \sigma_1^{(j)} + \beta J h \sigma_1^{(j)}} \prod_{k=1}^{q-1} Q_{n-1}(\sigma_1^{(j)}|\{\sigma\}^{(j,k)})\,, \tag{3}$$

where $\{\sigma\}^{j,k}$ are all the spins in k-th brunch of the j-th subtree.

(d) Define $g_n(\sigma_0) = \sum_{\{\sigma\}^{(j)}} Q_n(\sigma_0|\{\sigma\}^j)$ and write a recurrence relation for $x_n = \frac{g_n(-)}{g_n(+)}$. Show that, by consistency, $g_0(\sigma_0)$ must be set equal to 1.

(e) Express the local magnetization $\langle\sigma_0\rangle$ as a function of x_n.

(f) What happens in the thermodynamic limit $n \to \infty$? Does the model exhibit ferromagnetism?

Hint 1: The recurrence relation for x_n has the form $x_n = y(x_{n-1})$. This can be thought of graphically by simultaneously plotting $y = y(x)$ and $y = x$ (even if you do not know the exact form of the function, you can still identify the possible scenarios, at least qualitatively.) Let P_{n-1} be the point $(x_{n-1}, y(x_{n-1}))$ in the (x, y) plane. To construct P_n, draw a horizontal line through P_{n-1} to intercept the line $y = x$ at a point Q_n. Now, draw a vertical line through Q_n. Its intercept with $y = y(x)$ is the point P_n.

Hint 2: Reinterpret the equation $x = f(x)$ as an equation that defines h as a function of x. Determine the interval where x lies, and identify the range of temperatures where the equation $x = f(x)$ has only one solution (take the derivative with respect to x!). How many solutions does the equation have otherwise?

(g) Show that there is a phase transition ($q > 2$) and compute the critical temperature.

(h) Expand around the critical temperature and show that, in the limit of small h and temperature close to the critical one, the magnetization $\langle\sigma_0\rangle$ satisfies

$$\beta J h = \langle\sigma_0\rangle^3 \, b\Big(\frac{T - T_c}{T_c \langle\sigma_0\rangle^2}\Big)\,, \tag{4}$$

where

$$b(x) = \frac{1}{2}(q-2)x \log\frac{q}{q-2} + \frac{(q-1)(q-2)}{3q^2}\,. \tag{5}$$

Which are the values of the critical exponents β and δ?

Hint: You must assume that x is close to 1.