Tutorial 11, Statistical Mechanics: Concepts and applications 2016/17 ICFP Master (first year)

Maurizio Fagotti, Olga Petrova, Werner Krauth
Tutorial exercises

I. WORKSHEET: REAL SPACE RENORMALIZATION GROUP OF PERCOLATION

Source: P. J. Reynolds, H. E. Stanley and W. Klein, J. of Phys. C, **10**, *L167.*
D. Stauffer, A. Aharony, Introduction to Percolation Theory*, 2nd rev. ed., Taylor & Francis, 2003.*

1. **Percolation in one dimension**

The model: Consider a $1d$ chain, whose sites are either occupied, or free. The probability of each site being occupied is independent of the other sites' occupancy, and is equal to p. Neighboring occupied sites form *clusters*. The behavior of these clusters as a function of p is the central subject of percolation theory. Similar models in higher dimensions have proven useful across many different domains, e.g. transport in porous media, epidemic spreads, etc.

For an infinite size system, there is a critical value of p, called *percolation threshold* p_c, such that for $p < p_c$ the system does not have any clusters that span the entire system length, and for $p > p_c$ it does.

(a) Define n_s to be the *cluster number* – number of clusters containing s sites, per lattice site. Find it in terms of p and s.

(b) What is S, the average size of a finite cluster?

(c) Calculate the correlation function $g(r)$ – the probability that a site, that is a distance r away from an occupied site, belongs to the same cluster. Rewrite it in terms of the *correlation length* $\xi = -\frac{1}{\ln p}$.

(d) What is the percolation threshold in this problem? What happens to S and ξ at p_c?

The fact that the divergence of ξ at the percolation threshold can be described by a simple power law, is general. Additionally, the power ν in

$$\xi \propto |p - p_c|^{-\nu}$$

depends only on the dimension of the system.

(e) Calculate $S(L, p)$ for a finite segment of L sites, with [periodic boundary conditions. Discuss how S scales with the system size, and the limits $L \ll \xi$ and $L \gg \xi$.

Now we will use this exactly solvable model to demonstrate how *real-space renormalization* works. The method uses the fact that, when averaged over different realizations of the system, the picture that we get looks similar on different lengthscales between the original lattice spacing and ξ. The idea is to replace a cell of sites by a single super-site, whose linear dimension is $b < \xi$. In the process, some information will be lost, but nevertheless this scheme has been successfully used to study the behavior of systems near a phase transition. The super-site is said to be occupied if there is a cluster of the original sites that spans the length of the cell. Away from the critical point, the probability (also known as concentration) p' of the super-sites will be different from p. Since it is ξ that limits the validity of our assumption of similarity, we need it to be the same in the original and renormalized lattices. The new lattice will have a new lattice constant b, and ξ' will be measured in the new units. It follows that

$$b|p' - p_c|^{-\nu} = |p - p_c|^{-\nu}$$

and

$$\frac{1}{\nu} = \frac{\ln\left[(p' - p_c)/(p - p_c)\right]}{\ln b} = \frac{\ln \frac{dp'}{dp}|_{p=p_c}}{\ln b}.$$

The second expression allows us to estimate the critical exponent ν via the renormalization group method.

(f) Group the sites in our one-dimensional problem into cells of b. Find p'.

(g) Values of p that stay constant under renormalization are called *fixed points*. What are the fixed points in this problem?

(h) Calculate ν and compare it to the earlier result.

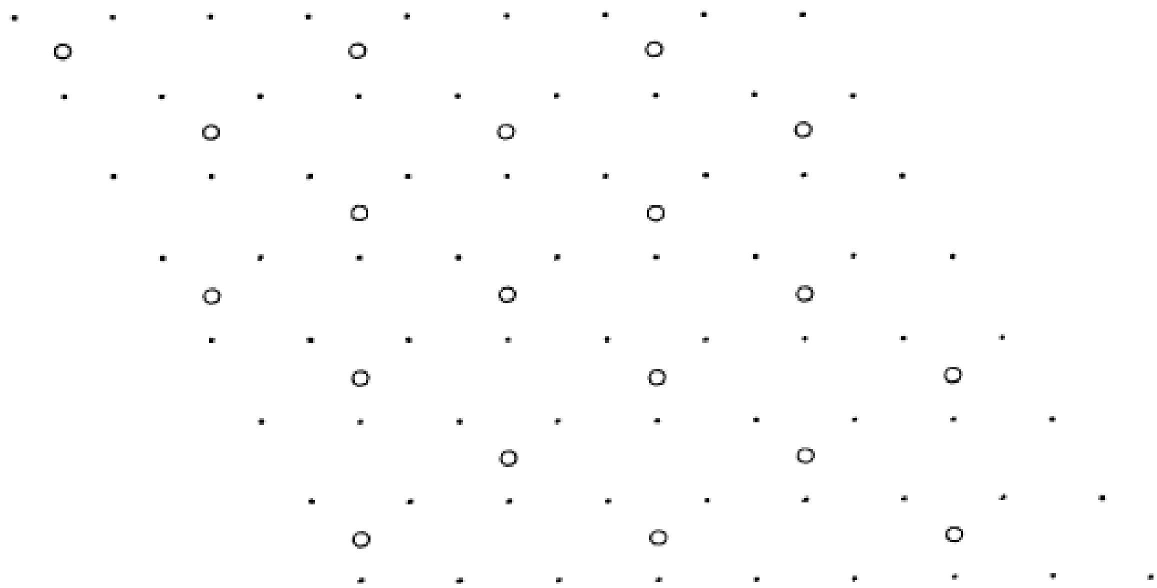

FIG. 1. Renormalization of a triangular lattice. The unfilled circles denote the super-sites, each representing three original sites. The super-sites live on a triangular lattice as well. Credit: *D. Stauffer, A. Aharony,* Introduction to Percolation Theory*, 2nd rev. ed., Taylor & Francis, 2003.*

2. **Site percolation on the triangular lattice**

   The model: Consider a percolation model defined on the triangular lattice. The probability of each site being occupied is p. Associate one super-site with each triangle. The super-site is occupied if a spanning cluster exists. We shall use the renormalization group approach.

   (a) Find p' as a function of p.

   (b) What are the fixed points in this problem? What is p_c?

   (c) Estimate ν and compare it to the known result for two dimensions, 4/3.

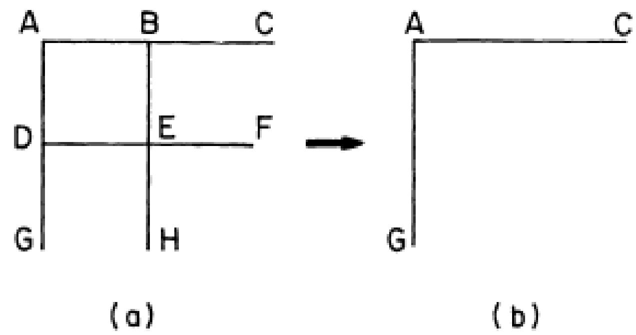

FIG. 2. Renormalization for bond percolation on the square lattice. (a) Original 2×2 cell of eight bonds. (b) Renormalized cell, with two bonds. Credit: *D. Stauffer, A. Aharony,* Introduction to Percolation Theory*, 2nd rev. ed., Taylor & Francis, 2003.*

3. **Bond percolation on the square lattice**

The model: Another type of percolation that can be defined is *bond percolation*. Here all lattice sites are connected by bonds, which can be either open (with probability p), or closed. A cluster is a group of sites connected by open bonds.

In the renormalization scheme, we are replacing each 2×2 cell of eight bonds [Fig. 2(a)] with a supercell of two bonds [Fig. 2(b)]. The two new bonds represent the connectivities in the horizontal and vertical directions: e.g.m the renormalized AC bond is open if one can go from A or from D to C or F in the original system.

(a) What is the scale factor b for this RG transformation?

(b) Find p' as a function of p.

Hint: Consider the different configurations of the AB, BC, BE, DE, and EF original bonds that determine the connectivity p' of the renormalized AC bond.

(c) What are the trivial fixed points in this problem?

(d) Consider the behavior of p' around $p = 1/2$. Can you find p_c?

(e) Estimate ν and compare it to the known result for two dimensions, 4/3.