# Tutorial 4, Advanced MCMC
# 2020/21 ICFP Master (second year)

Werner Krauth
*ENS Paris*

(Dated: 09 February 2022)

1. **Prove the Convergence theorem for irreducible, aperiodic Markov chains**
   ... (please look it up in Section 4.3 of Levin, Peres, Wilmer).

2. **Perfect card shuffling through the top-to-random shuffle**

   In lecture 4 (part 4.1), we discussed a card-shuffling algorithm that produced a perfect sample.

   (a) For a small card deck of only 3 cards, write down the transition matrix of the top-to-random shuffle algorithm.

   (b) Compute the eigenvalues of the top-to-random shuffle for 3 cards, and the inverse gap of the eigenvalue spectrum.

   (c) Prove that the stopping rule that we discussed in the lecture is indeed correct.

   (d) Write a simple MCMC algorithm. Test it for 5 cards, and show (by simulation) that all the 120 possible states of the deck come up with the same probability.

3. **Particle diffusion on the path graph**
   In lecture 4 (part 4.1), we discussed the diffusion of a particle on the path graph $P_5$.

   (a) Compute the transition matrix for this problem.

   (b) Is this transition matrix symmetric? Are all transition matrices of all MCMC algorithms symmetric?

   (c) Determine the eigenvalue spectrum of this transition matrix.

   (d) Describe the eigenvalue spectrum of the entire transition matrix describing the coupling. Explain why its spectral gap must be smaller than that of the original $5 \times 5$ matrix.

4. **Particle diffusion on the path graph and the Ising model (simulation project)**
   In lecture 4 (part 4.1), we discussed the diffusion of a particle on the path graph $P_5$.

(a) Implement this algorithm.

(b) Starting from all possible initial configuration, compute (by simulation) the coupling time.

(c) Generate a histogram of the coupling position, and show that it is not uniformly distributed among the 5 sites.

(d) Implement coupling from the past for this problem, and show (by simulation) it generates a uniform random sample.

(e) Implement coupling from the past for the Ising model.