Mathematica notebook for the Kac-Ward solution of the two-dimensional Ising model without periodic boundary conditions. Here we present the matrix for the 2x2 model and check that it yields the correct solution.

```
β = 1.2
1.2

ν = Tanh[β ]
0.833655

α = Exp[I Pi / 4] ν
0.589483 + 0.589483 ⅈ

α' = Conjugate[α]
0.589483 - 0.589483 ⅈ

null = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

one  = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

right = {{ν, α, 0, α'}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
{{0.833655, 0.589483 + 0.589483 ⅈ, 0, 0.589483 - 0.589483 ⅈ},
 {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

up = {{0, 0, 0, 0}, {α', ν, α, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
{{0, 0, 0, 0}, {0.589483 - 0.589483 ⅈ, 0.833655, 0.589483 + 0.589483 ⅈ, 0},
 {0, 0, 0, 0}, {0, 0, 0, 0}}

left = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, α', ν , α}, {0, 0, 0, 0}}
{{0, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0.589483 - 0.589483 ⅈ, 0.833655, 0.589483 + 0.589483 ⅈ}, {0, 0, 0, 0}}

down = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {α, 0, α', ν}}
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
 {0.589483 + 0.589483 ⅈ, 0, 0.589483 - 0.589483 ⅈ, 0.833655}}
```

```
U2x2 = ArrayFlatten[ {{one, right, up, null},
   {left, one, null , up}, {down, null, one, right}, {null, down, left, one}}]
```

```
{{1, 0, 0, 0, 0.833655, 0.589483 + 0.589483 i, 0, 0.589483 - 0.589483 i, 0, 0,
  0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0.589483 - 0.589483 i, 0.833655,
  0.589483 + 0.589483 i, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0,
  0, 0, 0, 0, 0.589483 - 0.589483 i, 0.833655, 0.589483 + 0.589483 i, 0},
 {0, 0.589483 - 0.589483 i, 0.833655, 0.589483 + 0.589483 i, 0, 0, 1, 0,
  0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0.833655, 0.589483 + 0.589483 i, 0,
  0.589483 - 0.589483 i}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
 {0.589483 + 0.589483 i, 0, 0.589483 - 0.589483 i, 0.833655, 0, 0, 0, 0,
  0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
  0.589483 - 0.589483 i, 0.833655, 0.589483 + 0.589483 i, 0, 0, 1, 0}, {0, 0, 0, 0,
  0.589483 + 0.589483 i, 0, 0.589483 - 0.589483 i, 0.833655, 0, 0, 0, 0, 0, 0, 0, 1}}
```

```
Det[U2x2]
```

$2.19928 + 1.19473 \times 10^{-17}\, i$

Comparison with the analytic solution (high-temperature solution with just two terms.

```
(1 + Tanh[β]^4) ^2
```

2.19928