Tutorial 3, Advanced MCMC
2021/22 ICFP Master (second year)

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1. **Direct sampling for hard spheres**

   In lecture 3 (part 3.1), we discussed a direct-sampling algorithm for four disks in a box (In week 4, we will treat a few far more successful direct-sampling algorithms for the same problem).

   (a) Prove (formally) that the *tabula-rasa* algorithm presented in the lecture samples the uniform distribution in $[0,1]^8 \setminus \cup_{i<j} Q_{ij}$.

   (b) Prove that the alternative to the *tabula-rasa* algorithm, where you only resample the "offending" last disk, is incorrect.

2. **Non-reversible hard-sphere algorithms in $1d$**

   In lecture 3 (part 3.2), we discussed a number of reversible and non-reversible Markov chains for hard spheres in one dimension, and among them the sequential MCMC (moving disk 1 then 2, then 3... , then N, then 1...)

   (a) Precisely define the lifting that is realized in that case.

   (b) Is the sequential MCMC a reversible or a non-reversible lifting?

   (c) Prove again (as we did in the lecture) that the sequential hard-sphere MCMC satisfies global balance.

   (d) Prove that for an N-particle system, and for a single-particle-move algorithm that satisfies detailed balance, the particle-lifted sequential algorithm satisfies global balance.

3. **Severely lifted non-reversible hard-sphere algorithms in $1d$**

   In lecture 3 (part 3.2), we discussed the forward Metropolis algorithm for a one-dimensional hard-sphere system.

   (a) Precisely define the lifting that is realized in that case.

(b) Prove again (as we did in the lecture) that the forward Metropolis algorithm satisfies global balance.

(c) Do the same for the sequential forward algorithm. Prove that this variant of the algorithm violates the global-balance condition (sequential forward: move forward, but sequentially 1 then 2, . . .).

4. **Mixing times of $1d$ hard-sphere MCMC**

At the end of lecture 3 (part 3.2), we discussed mixing times for non-reversible MCMC algorithms. We did not use the TVD to track the convergence but found an alternative in the "half-system-distance" distribution.

(a) Program one or two of the non-reversible MCMC algorithms and convince yourself that the half-system distance converges to the same limit (for $t \to \infty$) in all cases.

(b) Show that the mixing is much faster for the forward Metropolis algorithm than for the reversible Metropolis algorithm.