Resonant Deloc. on the Complete Graph

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Based on:
M.A. - M. Shamis - S. Warzel: “Partial delocalization on the complete graph” (2014)
Random Schrödinger operators - the question of spectral characteristics

Single quantum particle on regular graph $G$ (e.g. $\mathbb{Z}^d$)

$$H(\omega) := -\Delta + \lambda V(x; \omega)$$

on $\ell^2(G)$

(Anderson ’58, Mott - Twose ’61,...)

- discrete Laplacian: $(\Delta \psi)(x) := \sum_{\text{dist}(x,y)=1} \psi(y) - n(x)\psi(x)$
- Disorder parameter: $\lambda > 0$
- $V(x; \cdot)$, $x \in G$, i.i.d. rand. var., e.g. abs. cont distr. $\mathbb{P}(V(0) \in d\nu)$

Of particular interest: Localization and delocalization under disorder
Random Schrödinger operators - the question of spectral characteristics

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(“steelpan”, Trinidad and Tobago)
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Of particular interest: Localization and delocalization under disorder

Currently, delocalization remains less understood.

Possible mechanisms:
- continuity (?) (trees: [K’96, ASiW’06])
- quantum diffusion (?) [EY’00]
- resonant delocalization
Eigenfunction hybridization (tunneling amplitude vs. energy gaps)

Reminder from QM 101: Two-level system

\[ H = \begin{pmatrix} E_1 & \tau \\ \tau^* & E_2 \end{pmatrix} \]

Energy gap: \( \Delta E := E_1 - E_2 \)

Tunneling amplitude: \( \tau \).

- Case \( |\Delta E| \gg |\tau| \): Localization
  \[ \psi_1 \approx (1, 0), \quad \psi_2 \approx (0, 1). \]

- Case \( |\Delta E| \ll |\tau| \): Hybridized eigenfunctions
  \[ \psi_1 \approx \frac{1}{\sqrt{2}} (1, 1), \quad \psi_2 \approx \frac{1}{\sqrt{2}} (1, -1). \]

Heuristic explanation of the abs. cont. spectrum on tree graphs: (A-W ‘11)

Tunnelling amp. for states with energy \( E \) at distances \( R \):
\[ e^{-L_\lambda(E)R} \text{ (typ.)} \]

Since the volume grows exponentially fast as \( K^R \), extended states will form in spectral regimes with \( L_\lambda(E) < \log K \).


EPL 96: 37004 (2011)

[The implications include a surprising correction of the standard picture of the phase diagram: absence of a mobility edge for the Anderson Hamiltonian on tree graphs at weak disorder (Aiz-Warzel, EPL 2011).]
Quasimodes & their tunnelling amplitude

Definition:

1. A **quasi-mode** (qm) with discrepancy $d$ for a self-adjoint operator $H$ is a pair $(E, \psi)$ s.t.
   \[ \| (H - E) \psi \| \leq d \| \psi \|. \]

2. The pairwise **tunnelling amplitude**, among orthogonal qm’s of energy close to $E$ may be defined as $\tau_{jk}(E)$ in
   \[ P_{jk}(H - E)^{-1} P_{jk} = \begin{bmatrix} e_j + \sigma_{jj}(E) & \tau_{jk}(E) \\ \tau_{kj}(E) & e_k + \sigma_{kk}(E) \end{bmatrix}^{-1}. \]
   (the “Schur complement” representation).

Seems reasonable to expect:

If the typical **gap size** for quasi-modes is $\Delta(E)$, the condition for **resonant delocalization** at energies $E + \Theta(\Delta E)$ is:

\[ \Delta(E) \leq |\tau_{jk}(E)|. \]

**Question:**
how does that work in case of many co-resonating modes?
Example: Schrödinger operator on the complete graph (of $M$ sites)

$$H_M = -|\varphi_0\rangle\langle\varphi_0| + \kappa_M V$$

with:

- $|\varphi_0\rangle = (1, 1, \ldots, 1)/\sqrt{M}$,
- $V_1, V_2, \ldots V_M$ iid standard Gaussian rv's, i.e.

$$\varrho(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2},$$

- $\kappa_M := \lambda/\sqrt{2 \log M}$.

Remarks:

- Choice of $\kappa_M$ motivated by: $\max\{V_1, \ldots, V_M\}^{\text{inProb}} = \sqrt{2 \log M} + o(1)$.
- The spectrum of $H$ for $M \to \infty$:

$$\sigma(H_M) \longrightarrow [-\lambda, \lambda] \cup \{-1, 0\} \quad \text{(on the ‘macroscopic scale’)}.$$

- Eigenvalues interlace with the values of $K_M V$
- Studied earlier by Bogachev and Molchanov (‘89), and Ossipov (‘13) - both works focused on localization.
Two phase transitions for $H_M = -|\varphi_0\rangle\langle\varphi_0| + \frac{\lambda}{\sqrt{2\log M}} V$

Quasi-modes: $|\varphi_0\rangle$ (extended), and $|\delta_j\rangle$, $j = 1, \ldots, M$ (localized).

1. A transition at the spectral edge ($1^{st}$-order), at $\lambda = 1$:

\[ \begin{align*}
\lambda < 1 & : \quad E_0 = -1 + o(1), \quad \Psi_0 \approx \varphi_0 \quad \text{(the ground state is extended)} \\
\lambda > 1 & : \quad E_0 = -\lambda + o(1), \quad \Psi_0 \approx \delta_{\text{argmin}(V)} \quad \text{(the ground state is localized except for ‘avoided crossings’)}
\end{align*} \]

(Similar first order trans. in QREM and ... were studied [num. & rep.] by Jörg, Krzakala, Kurchan, Maggs ’08, Jörg, Krzakala, Semerjian, Zamponi ’10, ... More on the subject in the talks of Leticia Cugliandolo and Simone Warzel)

2. Emergence of a band of semi-delocalized states: of main interest here

at energies near $E = -1$, for $\lambda > \sqrt{2}$. A similar band near $E = 0$ is found for all $\lambda > 0$. 
Proposition
The eigenvalues of $H_M$ intertwine with the values of $\kappa V$.
The spectrum of $H_M$ consists of the collection of energies $E$ for which

$$F_M(E) := \frac{1}{M} \sum_{x=1}^{M} \frac{1}{\kappa_M V(x) - E} = 1,$$

and the corresponding eigenfunctions are given by:

$$\psi_E(x) = \frac{\text{Const.}}{\kappa_M V(x) - E}.$$

Proof: “rank one” perturbation theory $\implies$ for any $z \in \mathbb{C}\setminus\mathbb{R}$:

$$\frac{1}{H_M - z} = \frac{1}{\kappa_M V - z} + [1 - F_M(z)]^{-1} \frac{1}{\kappa_M V - z} |\varphi_0\rangle\langle\varphi_0| \frac{1}{\kappa_M V - z},$$

In particular, $\langle\varphi_0, (H_M - z)^{-1} \varphi_0\rangle = (F_M(z)^{-1} - 1)^{-1}$. The spectrum and eigenfunctions are given by the poles and residues of this “resolvent”.
The scaling limit

**Zooming onto scaling windows** centered at a sequence of energies $\mathcal{E}_M$ with:

$$\lim_{M \to \infty} \mathcal{E}_M = \mathcal{E} \in [-\lambda, \lambda], \quad \text{and} \quad |\mathcal{E}_M - \mathcal{E}| \leq C/\ln M,$$

denote

\[
\begin{align*}
\nu_{n,M} &:= \frac{E_{n,M} - \mathcal{E}_M}{\Delta_M(\mathcal{E}_M)}, \\
\omega_{n,M} &:= \frac{\kappa_M V_j - \mathcal{E}_M}{\Delta_M(\mathcal{E}_M)}.
\end{align*}
\]

rescaled eigenvalues          rescaled potential values

Questions of interest:

1. the nature of the limiting point process of the rescaled eigenvalues (including: extent of level repulsion (?), and relation to rescaled potential values)

2. the nature of the corresponding eigenfunctions (**extended** versus **localized**, and possible meaning of these terms).
Results (informal summary)

**Theorem 1** [Bands of partial delocalization (A., Shamis, Warzel)]

If either

- $\mathcal{E} = 0, \lambda > 0$; or
- $\mathcal{E} = -1$, and $\lambda > \sqrt{2}$,

and additionally the limit exists:

$$\lim_{M \to \infty} M \Delta_M(\mathcal{E}) \left(1 - \kappa_M^{-1} \mathcal{Q}(\mathcal{E}_M / \kappa_M)\right) =: \alpha$$

then:

I. the eigenvalues within the scaling window are *delocalized in $\ell^1$ sense*, localized in $\ell^2$ sense.

II. the rescaled eigenvalue point process converges in distribution to the Šeba point process at level $\alpha$ [defined below].

**Theorem 2** [A non-resonant delocalized state for $\lambda < \sqrt{2}$]

For $\lambda < \sqrt{2}$, there is a sequence of energies satisfying $\lim_{M \to \infty} \mathcal{E}_M = -1$ such that within the scaling windows centered at $\mathcal{E}_M$:

1. There exists one eigenvalue for which the corresponding eigenfunction $\psi_E$ is $\ell^2$-delocalized [...]
2. All other eigenfunctions in the scaling window are $\ell^2$-localized [...]

Elsewhere localization (Theorem 3 — not displayed here).
Key elements of the proof

▶ Rank-one perturbation arguments yield the characteristic equation:

Eigenvalues: \[ \frac{1}{M} \sum_n \frac{1}{\kappa_M V_n - E} = 1 \] (*)

Eigenvectors: \( \psi_{j,E} = \frac{1}{\kappa_M V_j - E} \) up to normalization

▶ To study the scaling limit we distinguish between the head contribution in (*), \( S_M,\omega(u) \), and the tail sum, transforming (*) into:

\[
S_M,\omega(u) = M\Delta_M(\mathcal{E}) - T_M,\omega(u) \quad := \quad -R_M,\omega(u)
\]

with

\[
T_M,\omega(u) = \sum_n \frac{1[|\omega_n| \geq \ln M]}{\omega_M,n - u}
\]

▶ Prove & apply some general results concerning limits of random Pick functions (aka Herglotz - Nevanlinna functions).

In particular: the scaling limit of a function such as \( R_M,\omega(u) \) is either:

i. constant \( \Rightarrow \) Šeba process & semi-delocalization,

ii. singular \((+\infty)\) or \((-\infty)\) \( \Rightarrow \) localization, or

iii. singular with transition \( \Rightarrow \) localization + single deloc. state \( (\mathcal{E} = -1, \lambda < \sqrt{2}) \)
Putting it all together (with details in appended slides)

1. Proofs of Theorems 1 - 3 (the spectral characteristics of \( H_{M,\omega} \))
   
   Recall: 
   
   Eigenvalues: \[
   \frac{1}{M} \sum_n \frac{1}{\kappa M V_n - E} = 1 \quad (*)
   \]
   
   Eigenvectors: \[
   \psi_{j,E} = \frac{1}{\kappa M V_j - E} \quad \text{up to normalization}
   \]
   
   distinguishing head \( S_{M,\omega}(u) \) versus tail contributions, rewrite (*) as:
   
   \[
   S_{M,\omega}(u) = M\Delta_M(\varepsilon) - T_{M,\omega}(u)
   \]
   
   with \( S_{M,\omega}(u) = \sum_n \frac{1[|\omega_n| \leq \ln M]}{\omega_{M,n} - u} \) and \( T_{M,\omega}(u) = \sum_n \frac{1[|\omega_n| \geq \ln M]}{\omega_{M,n} - u} \),
   
   apply the general results on such functions.

2. The heuristic criterion for resonant delocalization “checks out”
   
   yields the correct answer.

3. The localization criteria require some discussion (\( \ell^2 \) versus \( \ell^1 \)).

4. Comment on operators with many mixing modes
   
   (crossover to random matrix asymptotics)
Thank you for your attention

Alternatively - some further details are given below
Random Pick functions, and some facts about their limits

Pick class functions(*): functions $F : \mathbb{C}_+ \mapsto \mathbb{C}_+$ which are:

i) analytic in $\mathbb{C}_+$, and ii) satisfy $\text{Im } F(x + iy) \geq 0$ for $y > 0$.

Such functions have the Herglotz representation:

$$F(z) = a_F z + b_F + \int \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) \mu_F(dx)$$

$P(a, b)$ - the subclass of Pick functions which are analytic in $(a, b) \subset \mathbb{R}$.

Random Pick functions:

$\mu_F(dx)$ a random measure, e.g. point process, $(a_F, b_F)$ may also be random.

The charact. eq. $S_{M, \omega}(u) = -R_{M, \omega}(u)$ relates two rather different examples:

1. $S_{M, \omega}(u)$: its spectral measure $\mu_S$ converges to a Poisson process
2. $R_{M, \omega}(u)$: is in $P(-L_M, L_M)$ for $L_M = \ln M \to \infty$
The “oscillatory part”

Prop 1: For any Pick function $S_\omega(x)$ which is stationary and ergodic under shifts, and of purely singular spectral measure, the value of $S_\omega(x)$ has the general Cauchy distribution ($\overset{D}{=} aY + b$; $Y$ Cauchy RV)

(See A.-Warzel ‘13, may have been know to Methuselah.)

Among the interesting examples:

1. **(periodic)** the function $S_\theta(u) = \cot(u + \theta)$
2. **(random, no level repulsion)** the Poisson-Stieltjes function $S_\omega(u)$
3. **(random, with level repulsion)** the Wigner matrix resolvent

$$S(u) = \langle 0 | \frac{\Delta_N(\epsilon)}{H_{\omega,N} - (\epsilon + u\Delta_N(\epsilon))} | 0 \rangle$$
Lemma: Let $F(z)$ be a function in $P(-L, L)$. Then $\forall \ W < L/3$ and $u, u_0, u_1 \in [-W, W]$,

$$\left| \frac{F(u) - F(u_0)}{u - u_0} - \frac{F(u_1) - F(u_0)}{u_1 - u_0} \right| \leq 2 \frac{W}{L} \frac{F(u_1) - F(u_0)}{u_1 - u_0}$$

Prop. 2: (A-S-W) Functions $F_M \in P(-L_M, L_M)$ with $L_M \to \infty$
can only have one of the following 3 limits

i. $F(z) = az + b$,

ii. singular: $(+\infty)$ or $(-\infty)$,

iii. singular with transition

and for (i) & (ii) convergence at two points suffices
The Šeba process

Let $\omega$ be the Poisson process of constant intensity 1. The corresponding Stieltjes-Poisson random function

$$S_\omega(u) := \lim_{w \to \infty} \sum_n \frac{1[|\omega_n| \leq w]}{\omega_n - u}$$

(lim exists a.s.)

For specified $\alpha \in [-\infty, \infty]$, denote by $\{u_n, \omega_n(\alpha)\}$ the solutions of:

$$S_\omega(u) = \alpha$$

Definition
We refer to the intertwined point process $\{\{u_n, \omega_n\}\}$ as the Šeba point processes at level $\alpha$.

Remarks:
- Limiting cases $\alpha = \pm \infty$: Poisson process
- Intermediate statistics with some level repulsion

Šeba 1990, Albeverio-Šeba 1991
Turn back to Page 12.