Complexity barriers for partition functions

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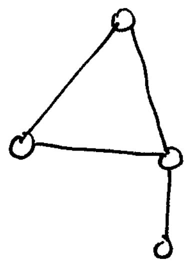
$$G = (V, E)$$

vertices, edges



$e \in E \Rightarrow x_e$ variable associated with edge e

$$\mathbb{x} = (x_e)_{e \in E}$$

Ising partition fction:

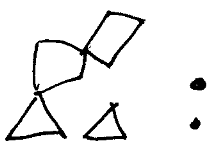
$$\mathcal{E}(G, \mathbb{x}) = \sum_{E' \subseteq E \text{ even}} \prod_{e \in E'} x_e$$

Dimer partition fction:

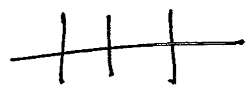
$$P(G, \mathbb{x}) = \sum_{\substack{E' \subseteq E \\ \text{perfect matching}}} \prod_{e \in E'} x_e$$

G bipartite $\Downarrow$ $P(G, \mathbb{x}) = \mathrm{per}(A_G)$

E' even: each deg even



E' perfect match.



substitution: $x_e := e^{-\beta J_e}$

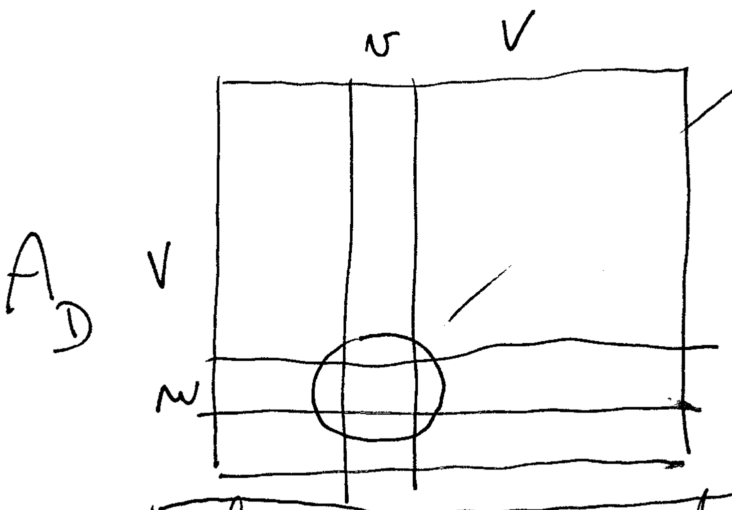
From 60's, basic tools are:

determinants (Pfaffian method)

Kasteleyn, Fisher, ...

$G \to D$ orientation

$\circ\!-\!\!-\!\circ \rightsquigarrow \circ\!\longrightarrow\!\circ$



$$\begin{bmatrix} x_{uv} & \overset{u}{\circ}\!\longrightarrow\!\overset{v}{\circ} \\ -x_{uv} & \underset{v}{\circ}\!\longrightarrow\!\underset{u}{\circ} \end{bmatrix}$$

skew-symmetric

$$\mathrm{Pfaf}\, A_D = (\mathrm{Det}\, A_D)^{1/2}$$

Products over aperiodic closed walks (Ihara-Selberg fction)

discrete

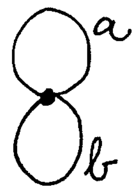
Kac, Ward, Feynman, Sherman...

rotation

$$\prod (1 - \prod_{e \in p} x_e)$$

p aperiodic closed walk

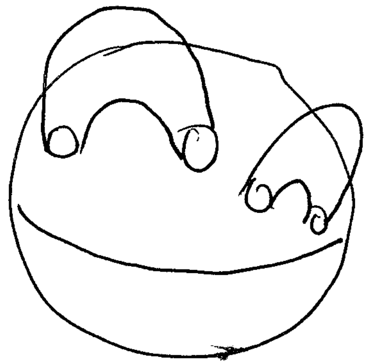
infinite : formal power series



$x_a^n x_b$

Bass' theorem : $\prod_p (1 - \prod x_e)$ is a determinant

Complexity barriers of both tools: genus of G



$g = 2$

<u>Theorem (Loebl, Masbaum, Adv. in Math 2011)</u>

"Minimum number of orientations of G so that $\mathcal{E}(G, x)$ is linear combination of $\overset{\text{Pfaffians}}{\text{determinants}}$ of corresponding matrices is 4^g, g genus of G".

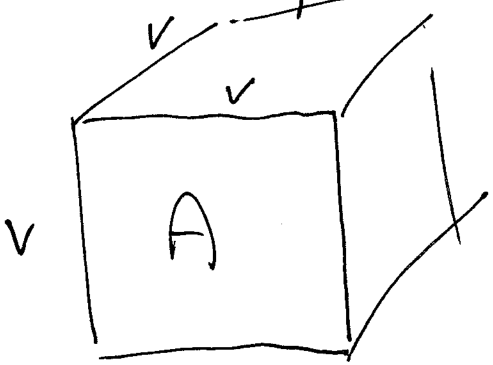
- enumeration characterisation of genus
- exponential lower bound in very restricted computation model

- still open for $P(G, x)$: 4^g not true
- determinantal complexity: is ~~det~~ per exponentially harder than det ?

How to deal with the genus barrier: study a "new" structure of the problems

* Permanent has only combinatorial structure
* ? Polyhedral structure ?
* A suggestion: replace det by hyperdeterminants

3D matrix



$$\mathrm{Per}\, A = \sum_{\pi_1 \pi_2} \prod_{i \in V} A_{i\,\pi_1(i)\,\pi_2(i)}$$

$$\mathrm{Det}\, A = \sum_{\pi_1, \pi_2} \cancel{\prod_{i\in V}} \operatorname{sign}\pi_1 \operatorname{sign}\pi_2 \prod_{i \in V} A_{i\,\pi_1(i)\,\pi_2(i)}$$

The same for 4D, 5D, ...

3D matrix A is Kastelyn if there is signing A' of A so that $\mathrm{Per}\, A = \mathrm{Det}\, A'$.

* 2D Kastelyn matrices rare : basically correspond to planar graphs

* Theorem (Göbl, Rytíř 2013)
- 3D Kastelyn matrices are not rare.
- G cubic lattice $\Rightarrow$ there is 3D Kastelyn matrix A_G : $P(G, x) = \det A_G$. Moreover A_G has "3D structure".

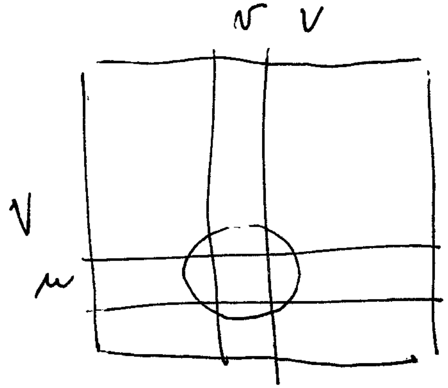
BUT : computing hyperdet is #P complete (as hard as per) few formulas

? Which advantage hyperdet has over per ?

Discrete Ihara-Selberg function

$G = (V, E)$, D orientation of G,

$\tilde{A}_D$:



x_e if $\underset{u}{\circ} \xrightarrow{e} \underset{v}{\circ}$

0 otherwise

$$\circledast \quad \det(I - \tilde{A}_D) = \sum_{\mathcal{C} = \{c_1 \dots c_\ell\} \text{ disjoint dicycles}} (-1)^\ell \cdot \prod_{e \in \mathcal{C}} x_e$$

Theorem (Bass, Foata, Zeilberger ...)

$$\det(I - \tilde{A}_D) = \prod_{p \text{ aperiodic closed walk}} \left(1 - \prod_{e \in p} x_e\right)$$

Theorem (Feynman, Sherman)

$$\mathcal{E}^2(G, x) = \prod_{\substack{p \text{ reduced} \\ \text{ap. closed}}} \left[1 - (-1)^{\mathrm{rot}\, p} \prod_{e \in p} x_e\right]$$

planar

$$\| $$

$$\det(\mathrm{rot})$$

90's :

J. Distler, 3D Ising as a string theory

critical Ising : discrete analogue of some conformal field theory [Alvarez-Gaumé, Moore, Vafa] ? $g \geq 2$?

4D Bass' theorem

Loebl 2014

2D $\det(I - \tilde{A}_D)$

incidence matrix of a digraph

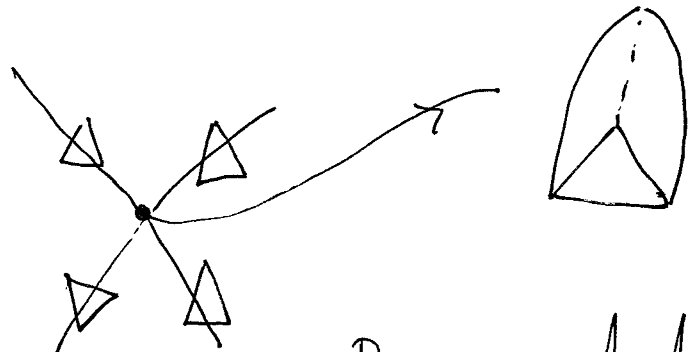
$\boxed{\text{entries}}$ $\circ \longrightarrow \circ$

graph: 1D simpl. complex

4D $\det(I - A)$

A: incidence matrix of orientation of 3D simplicial complex

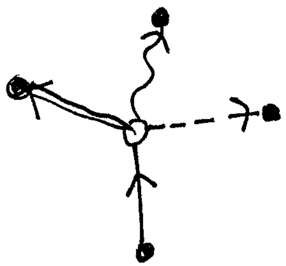
$\boxed{\text{entries}}$ directed tetrahedrons



<u>orientation</u>: each tetrahedron has 4 colored $\triangle$

<u>Representation</u>: $\circ$ vertex inside tetrahedron

$\bullet$ vertex inside $\triangle$

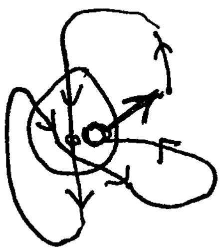


WALKS

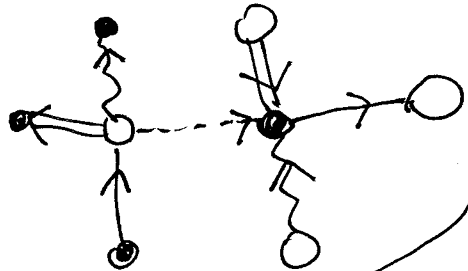
2D

closed connected
aperiodic
no beginning

set of directed
edges and
"vertex connectors"
with obvious
properties



Branching and annihilation process on △
through : closed connected aperiodic no beginning



set of directed tetrahedrons and
△ connectors

FEASIBILITY PROPERTY : Process described from
triangle t : at each time there is at most one
active t- connector
- whose leaving arc in the process
- whose entering arc in the process

Active : at least 1 arc in process,
— " — not in process.