Maximum independent sets in random d-regular graphs

Jian Ding, Allan Sly, and Nike Sun

Cargèse, Corsica 3 September 2014

Constraint satisfaction problem (**CSP**): given a collection of *variables* subject to *constraints*, find a **satisfying assignment**

CSPs are basic problems of both theoretical and practical interest

computational complexity theory, information theory

Constraint satisfaction problem (**CSP**): given a collection of *variables* subject to *constraints*, find a **satisfying assignment**

CSPs are basic problems of both theoretical and practical interest

computational complexity theory, information theory

A large subclass of CSPs is NP-complete or NP-hard —
best known algorithms have exponential runtime in worst case

k-SAT ($k \geqslant 3$), independent set, coloring, MAX-CUT

What about 'average' or 'typical' case?
— leads naturally to the consideration of **random CSPs** Levin '86

Boolean satisfiability: variables x_i taking values T or F
Each constraint is a clause (OR of literals): $x_1 \vee x_2 \vee \neg x_3$

A collection of clauses defines a **CNF** formula (AND of ORs)
— called k-**CNF** if each clause involves k literals

3-CNF: $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee \neg x_4 \vee x_5)$

A SAT solution is a variable assignment $\underline{x} \in \{\mathrm{T}, \mathrm{F}\}^n$ evaluating to T
— k-SAT is NP-complete for any $k \geqslant 3$ Cook '71, Levin '73

Boolean satisfiability: variables x_i taking values T or F
Each constraint is a clause (OR of literals): $x_1 \vee x_2 \vee \neg x_3$

A collection of clauses defines a **CNF** formula (AND of ORs)
— called ***k*-CNF** if each clause involves k literals

3-CNF: $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee \neg x_4 \vee x_5)$

A SAT solution is a variable assignment $\underline{x} \in \{\mathrm{T}, \mathrm{F}\}^n$ evaluating to T
— k-SAT is NP-complete for any $k \geqslant 3$ Cook '71, Levin '73

Natural choice for a **random** k-CNF: sample uniformly
from space of n-variable, m-clause formulas $(2n)^{mk}$ formulas

Boolean satisfiability: variables x_i taking values T or F
Each constraint is a clause (OR of literals): $x_1 \vee x_2 \vee \neg x_3$

A collection of clauses defines a **CNF** formula (AND of ORs)
— called ***k*-CNF** if each clause involves k literals

3-CNF: $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee \neg x_4 \vee x_5)$

A SAT solution is a variable assignment $\underline{x} \in \{\mathrm{T}, \mathrm{F}\}^n$ evaluating to T
— k-SAT is NP-complete for any $k \geqslant 3$ Cook '71, Levin '73

Natural choice for a **random** k-CNF: sample uniformly
from space of n-variable, m-clause formulas $(2n)^{mk}$ formulas

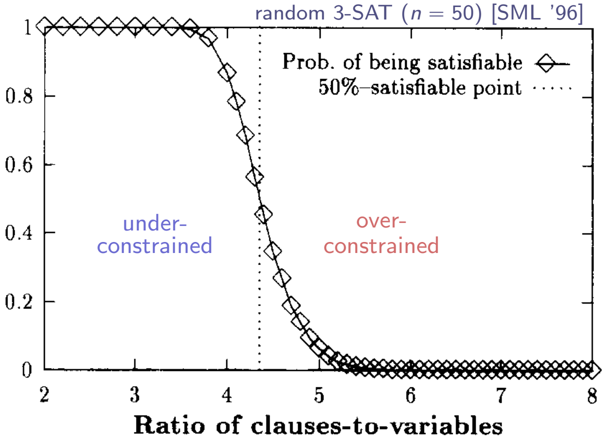
"Constraint parameter" is **clause density** $\alpha = m/n$

Early studies noted an empirical **SAT–UNSAT transition**

Cheeseman–Kanefsky–Taylor '91, Mitchell–Selman–Levesque '92, '96

Early studies noted an empirical **SAT–UNSAT transition**

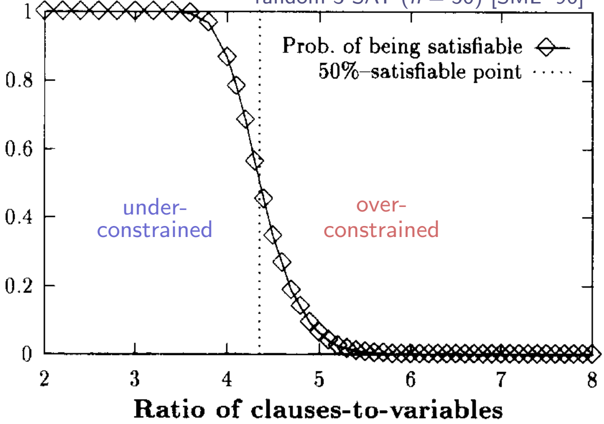
Cheeseman–Kanefsky–Taylor '91, Mitchell–Selman–Levesque '92, '96



Early studies noted an empirical **SAT–UNSAT transition**

Cheeseman–Kanefsky–Taylor '91, Mitchell–Selman–Levesque '92, '96

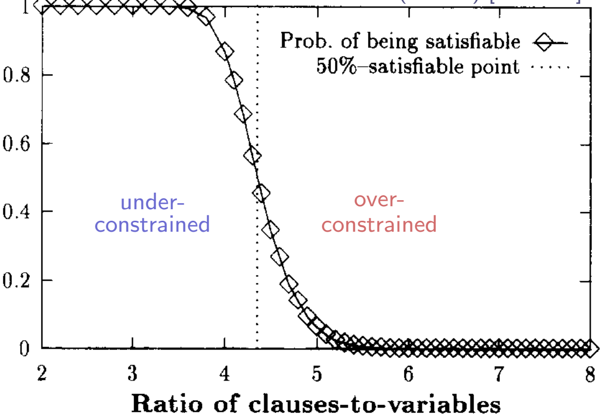
random 3-SAT ($n = 50$) [SML '96]



Early studies noted an empirical **SAT–UNSAT transition**

Cheeseman–Kanefsky–Taylor '91, Mitchell–Selman–Levesque '92, '96

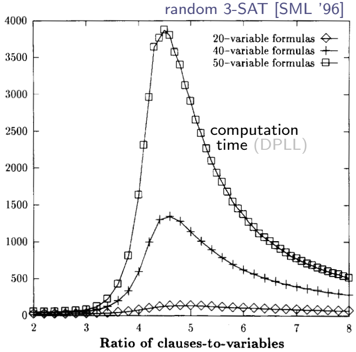
random 3-SAT ($n = 50$) [SML '96]



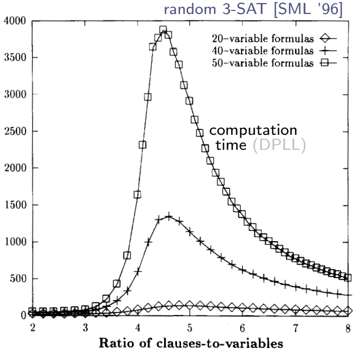
Remains major open problem to rigorously establish existence and location of sharp SAT–UNSAT transition for random k-SAT

"Hardest" problems seem to occur near SAT–UNSAT transition:

“Hardest” problems seem to occur near SAT–UNSAT transition:



"Hardest" problems seem to occur near SAT–UNSAT transition:



Understanding the SAT–UNSAT transition seems possibly a precursor to addressing the complexity behavior of random k-SAT

A major advance in the investigation of (random) CSPs was the realization that they may be regarded in the spin glass framework

Mézard–Parisi '85 (weighted matching), '86 (traveling salesman), Fu–Anderson '86 (graph partitioning)

— since these pioneering works, the study of CSPs as models of disordered systems has developed into a rich theory, yielding deep insights as well as novel algorithmic ideas

e.g. survey propagation [Mézard–Parisi–Zecchina '02]

A major advance in the investigation of (random) CSPs was the realization that they may be regarded in the spin glass framework

Mézard–Parisi '85 (weighted matching), '86 (traveling salesman), Fu–Anderson '86 (graph partitioning)

— since these pioneering works, the study of CSPs as models of disordered systems has developed into a rich theory, yielding deep insights as well as novel algorithmic ideas

e.g. survey propagation [Mézard–Parisi–Zecchina '02]

A notable consequence of the spin glass connection is an abundance of *exact mathematical predictions* for random CSPs

(concerning threshold phenomena, solution space geometry, . . .)

A major advance in the investigation of (random) CSPs was the realization that they may be regarded in the spin glass framework

Mézard–Parisi '85 (weighted matching), '86 (traveling salesman), Fu–Anderson '86 (graph partitioning)

— since these pioneering works, the study of CSPs as models of disordered systems has developed into a rich theory, yielding deep insights as well as novel algorithmic ideas

e.g. survey propagation [Mézard–Parisi–Zecchina '02]

A notable consequence of the spin glass connection is an abundance of *exact mathematical predictions* for random CSPs

(concerning threshold phenomena, solution space geometry, . . .)

Some predictions for *dense* graphs have been sucessfully proved;

Parisi formula for SK spin-glasses [Parisi '80 / Guerra '03, Talagrand '06]

$\zeta(2)$ limit of random assignments [Mézard–Parisi '87 / Aldous '00]

A major advance in the investigation of (random) CSPs was the realization that they may be regarded in the spin glass framework

Mézard–Parisi '85 (weighted matching), '86 (traveling salesman), Fu–Anderson '86 (graph partitioning)

— since these pioneering works, the study of CSPs as models of disordered systems has developed into a rich theory, yielding deep insights as well as novel algorithmic ideas

e.g. survey propagation [Mézard–Parisi–Zecchina '02]

A notable consequence of the spin glass connection is an abundance of *exact mathematical predictions* for random CSPs

(concerning threshold phenomena, solution space geometry, ...)

Some predictions for *dense* graphs have been sucessfully proved;

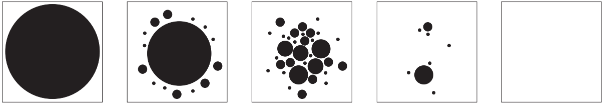
Parisi formula for SK spin-glasses [Parisi '80 / Guerra '03, Talagrand '06]

$\zeta(2)$ limit of random assignments [Mézard–Parisi '87 / Aldous '00]

rigorous understanding of *sparse* setting is comparatively lacking

This talk concerns the class of **sparse random CSPs** exhibiting (static) **replica symmetry breaking** (**RSB**)

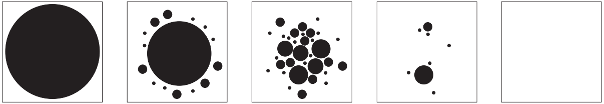
Solution space geometry has been investigated in several works, leading to this conjectural phase diagram:



Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07, Montanari–Ricci-Tersenghi–Semerjian '08

This talk concerns the class of **sparse random CSPs** exhibiting (static) **replica symmetry breaking** (**RSB**)

Solution space geometry has been investigated in several works, leading to this conjectural phase diagram:

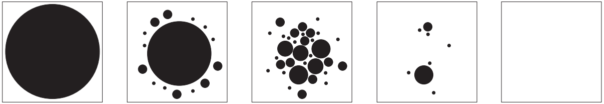


Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07, Montanari–Ricci-Tersenghi–Semerjian '08

— latest in significant body of literature including Monasson–Zecchina '96, Biroli–Monasson–Weigt '00, Mézard–Parisi–Zecchina '02, Mézard–Mora–Zecchina '05, Mézard–Palassini–Rivoire '05, Achlioptas–Ricci-Tersenghi '06

This talk concerns the class of **sparse random CSPs** exhibiting (static) **replica symmetry breaking** (**RSB**)

Solution space geometry has been investigated in several works, leading to this conjectural phase diagram:



Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07, Montanari–Ricci-Tersenghi–Semerjian '08

— latest in significant body of literature including Monasson–Zecchina '96, Biroli–Monasson–Weigt '00, Mézard–Parisi–Zecchina '02, Mézard–Mora–Zecchina '05, Mézard–Palassini–Rivoire '05, Achlioptas–Ricci-Tersenghi '06

We are interested in the rigorous computation of sharp satisfiability thresholds for this class of models

Prior rigorous work for sparse CSPs *without RSB*: the exact satisfiability threshold has been proved for several problems:

Prior rigorous work for sparse CSPs *without RSB*: the exact satisfiability threshold has been proved for several problems:

- 2-SAT transition   Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92
  scaling window: Bollobás–Borgs–Chayes–Kim–Wilson '01
- 1-in-k-SAT transition   Achlioptas–Chtcherba–Istrate–Moore '01
- k-XOR-SAT transition   Dubois–Mandler '02, Dietzfelbinger–Goerdt–
  –Mitzenmacher–Montanari–Pagh–Rink '10, Pittel–Sorkin '12

For sparse CSPs *with RSB*, threshold behavior long in question
Rigorous bounds on the SAT–UNSAT transition include:

For sparse CSPs *with RSB*, threshold behavior long in question
Rigorous bounds on the SAT–UNSAT transition include:

- random regular graph independent set — Bollobás '81, McKay '87, Frieze–Łuczak '92, Frieze–Suen '94, Wormald '95
- random graph coloring — Bollobás '88, Achlioptas–Naor '04, Coja-Oghlan–Vilenchik '13
- random k-NAE-SAT — Achlioptas–Moore '02, Coja-Oghlan–Zdeborová '12, Coja-Oghlan–Panagiotou '12
- random k-SAT — Kirousis et al. '97, Franz–Leone '03, Achlioptas–Peres '03, Coja-Oghlan–Panagiotou '13, Coja-Oghlan '14

For sparse CSPs *with RSB*, threshold behavior long in question
Rigorous bounds on the SAT–UNSAT transition include:

- random regular graph independent set — Bollobás '81, McKay '87, Frieze–Łuczak '92, Frieze–Suen '94, Wormald '95
- random graph coloring — Bollobás '88, Achlioptas–Naor '04, Coja-Oghlan–Vilenchik '13
- random k-NAE-SAT — Achlioptas–Moore '02, Coja-Oghlan–Zdeborová '12, Coja-Oghlan–Panagiotou '12
- random k-SAT — Kirousis et al. '97, Franz–Leone '03, Achlioptas–Peres '03, Coja-Oghlan–Panagiotou '13, Coja-Oghlan '14

(gap remains in all models: threshold existence not implied)

For sparse CSPs *with RSB*, threshold behavior long in question
Rigorous bounds on the SAT–UNSAT transition include:

- random regular graph independent set — Bollobás '81, McKay '87, Frieze–Łuczak '92, Frieze–Suen '94, Wormald '95
- random graph coloring — Bollobás '88, Achlioptas–Naor '04, Coja-Oghlan–Vilenchik '13
- random k-NAE-SAT — Achlioptas–Moore '02, Coja-Oghlan–Zdeborová '12, Coja-Oghlan–Panagiotou '12
- random k-SAT — Kirousis et al. '97, Franz–Leone '03, Achlioptas–Peres '03, Coja-Oghlan–Panagiotou '13, Coja-Oghlan '14

(gap remains in all models: threshold existence not implied)

Existence of threshold sequence (possibly non-convergent) — Friedgut '99

For sparse CSPs *with RSB*, threshold behavior long in question
Rigorous bounds on the SAT–UNSAT transition include:

- random regular graph independent set — Bollobás '81, McKay '87, Frieze–Łuczak '92, Frieze–Suen '94, Wormald '95
- random graph coloring — Bollobás '88, Achlioptas–Naor '04, Coja-Oghlan–Vilenchik '13
- random k-NAE-SAT — Achlioptas–Moore '02, Coja-Oghlan–Zdeborová '12, Coja-Oghlan–Panagiotou '12
- random k-SAT — Kirousis et al. '97, Franz–Leone '03, Achlioptas–Peres '03, Coja-Oghlan–Panagiotou '13, Coja-Oghlan '14

(gap remains in all models: threshold existence not implied)

Existence of threshold sequence (possibly non-convergent) — Friedgut '99

Existence of sharp threshold — Bayati–Gamarnik–Tetali '10
(cannot determine threshold location; does not cover random SAT)

Many problems within this class (including all on previous slide)
are believed to be described by the **1-RSB formalism**
(one-step replica symmetry breaking) Mézard–Parisi '01

Many problems within this class (including all on previous slide)
are believed to be described by the **1-RSB formalism**
(one-step replica symmetry breaking) Mézard–Parisi '01

For such problems, the *1-RSB cavity method* predicts the exact location of the SAT–UNSAT transition
Mézard–Parisi-Zecchina '02, Mertens–Mézard–Zecchina '06
(based on assumptions that are difficult to verify mathematically)

Many problems within this class (including all on previous slide)
are believed to be described by the **1-RSB formalism**
(one-step replica symmetry breaking) Mézard–Parisi '01

For such problems, the *1-RSB cavity method* predicts the exact location of the SAT–UNSAT transition
Mézard–Parisi-Zecchina '02, Mertens–Mézard–Zecchina '06
(based on assumptions that are difficult to verify mathematically)

Many problems within this class (including all on previous slide)
are believed to be described by the **1-RSB formalism**
(one-step replica symmetry breaking) Mézard–Parisi '01

For such problems, the *1-RSB cavity method* predicts the exact location of the SAT–UNSAT transition
Mézard–Parisi-Zecchina '02, Mertens–Mézard–Zecchina '06
(based on assumptions that are difficult to verify mathematically)

In our work we give rigorous verifications of the 1-RSB prediction for the SAT–UNSAT transition, for the following models:

Many problems within this class (including all on previous slide)
are believed to be described by the **1-RSB formalism**
(one-step replica symmetry breaking) Mézard–Parisi '01

For such problems, the *1-RSB cavity method* predicts the exact location of the SAT–UNSAT transition
Mézard–Parisi-Zecchina '02, Mertens–Mézard–Zecchina '06
(based on assumptions that are difficult to verify mathematically)

In our work we give rigorous verifications of the 1-RSB prediction for the SAT–UNSAT transition, for the following models:

- random regular k-NAE-SAT (next few slides)

Many problems within this class (including all on previous slide)
are believed to be described by the **1-RSB formalism**
(one-step replica symmetry breaking) Mézard–Parisi '01

For such problems, the *1-RSB cavity method* predicts the exact location of the SAT–UNSAT transition
Mézard–Parisi-Zecchina '02, Mertens–Mézard–Zecchina '06
(based on assumptions that are difficult to verify mathematically)

In our work we give rigorous verifications of the 1-RSB prediction for the SAT–UNSAT transition, for the following models:

- random regular k-NAE-SAT (next few slides)
- random regular graph independent set (rest of the talk)

boolean satisfiability

Random (Erdős–Rényi) k-CNF is uniform measure over all n-variable, m-clause k-CNF's

($(2n)^{mk}$ formulas; constraint structure is Erdős–Rényi hyper-graph)

Random regular k-CNF is uniform measure over all n-variable, m-clause k-CNF's with **fixed** variable degree $d = mk/n$

($2^{mk}(mk)!/(d!)^n$ formulas; constraint structure is regular hyper-graph)

"Constraint parameter" is clause density $\alpha = m/n$

Benchmark problem: SAT–UNSAT transition in random k-SAT

(UBD) Franco–Paull '83, Kirousis–Kranakis–Krizanc–Stamatiou '97;
(LBD) Chao–Franco '90, Achlioptas–Moore '02, Achlioptas–Peres '03,
Coja-Oghlan–Panagiotou '13, Coja-Oghlan '14 (gap remains in bounds)

Random k-SAT threshold is close to $2^k \log 2$, but the best known algorithmic lower bound is only $\asymp 2^k \log k / k$ Coja-Oghlan '10

First $\asymp 2^k$ LBD for random k-SAT achieved by non-algorithmic analysis of random ***k*-NAE-SAT**: Achlioptas–Moore '02
harder to satisfy, but easier to study, than SAT

A NAE-SAT solution is a SAT solution $\underline{x}$ such that $\neg\underline{x}$ is also SAT — eliminates TRUE/FALSE asymmetry of SAT; but believed to exhibit many of the same qualitative phenomena

Bounds on SAT–UNSAT in random (Erdős–Rényi) k-NAE-SAT:
AM '02, Coja-Oghlan–Zdeborová '12, Coja-Oghlan–Panagiotou '12
lower bounds (approx. halves) the SAT transition (gap remains in bounds)

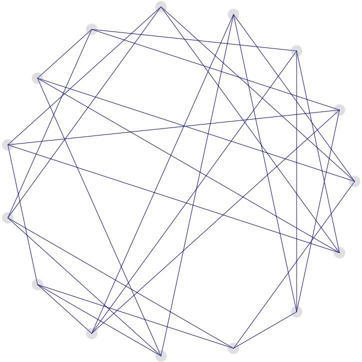
(main result for NAE-SAT)

THEOREM. Ding, Sly, S. [arXiv:1310.4784, STOC '14]
The random regular k-NAE-SAT problem has SAT–UNSAT transition at explicit threshold $\alpha_\star(k)$ for all $k \geqslant k_0$.

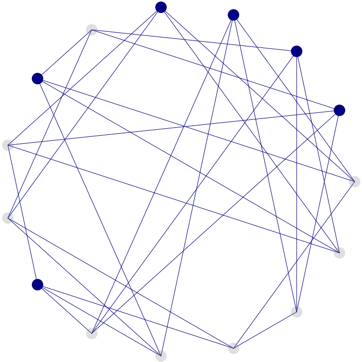
In simultaneous work, A. Coja-Oghlan [arXiv:1310.2728v1] considered a different symmetrization of random regular k-SAT, establishing a 1-RSB-type formula for a "quasi-satisfiability" threshold

independent sets

In an undirected graph, an **independent set**



In an undirected graph, an **independent set**



is a subset of vertices containing no neighbors

(equivalently, the complement is a vertex cover)

"Constraint parameter" of random SAT is clause density m/n

"Constraint parameter" of random SAT is clause density m/n

"Constraint parameter" of independent set is the set density —

"Constraint parameter" of random SAT is clause density m/n

"Constraint parameter" of independent set is the set density —
SAT–UNSAT corresponds to max-density (**independence ratio**)

“Constraint parameter” of random SAT is clause density m/n

“Constraint parameter” of independent set is the set density —
SAT–UNSAT corresponds to max-density (**independence ratio**)

The independence ratio is NP-hard to compute exactly; Karp ’72
in fact it is hard to approximate even on bounded-degree graphs
Papadimitriou–Yannakakis ’91 and PCP theorem

“Constraint parameter” of random SAT is clause density m/n

“Constraint parameter” of independent set is the set density — SAT–UNSAT corresponds to max-density (**independence ratio**)

The independence ratio is NP-hard to compute exactly; Karp ’72
in fact it is hard to approximate even on bounded-degree graphs
Papadimitriou–Yannakakis ’91 and PCP theorem

Randomize the problem by taking a **random graph** —

"Constraint parameter" of random SAT is clause density m/n

"Constraint parameter" of independent set is the set density — SAT–UNSAT corresponds to max-density (**independence ratio**)

The independence ratio is NP-hard to compute exactly; Karp '72

in fact it is hard to approximate even on bounded-degree graphs

Papadimitriou–Yannakakis '91 and PCP theorem

Randomize the problem by taking a **random graph** — let $\mathbf{A}_n \equiv$ MAX-IND-SET size in random graph G_n on n vertices:

"Constraint parameter" of random SAT is clause density m/n

"Constraint parameter" of independent set is the set density — SAT–UNSAT corresponds to max-density (**independence ratio**)

The independence ratio is NP-hard to compute exactly; Karp '72
in fact it is hard to approximate even on bounded-degree graphs
Papadimitriou–Yannakakis '91 and PCP theorem

Randomize the problem by taking a **random graph** — let
$\mathbf{A}_n \equiv$ MAX-IND-SET size in random graph G_n on n vertices:
for natural ensembles G_n, **what are the asymptotics of $\mathbf{A}_n$?**
dense ER graph $G_{n,p}$, sparse ER graph $G_{n,d/n}$,
(uniform) random regular graph $\mathscr{G}_{n,d}$

"Constraint parameter" of random SAT is clause density m/n

"Constraint parameter" of independent set is the set density — SAT–UNSAT corresponds to max-density (**independence ratio**)

The independence ratio is NP-hard to compute exactly; Karp '72
in fact it is hard to approximate even on bounded-degree graphs
Papadimitriou–Yannakakis '91 and PCP theorem

Randomize the problem by taking a **random graph** — let
$\mathbf{A}_n \equiv$ MAX-IND-SET size in random graph G_n on n vertices:
for natural ensembles G_n, **what are the asymptotics of $\mathbf{A}_n$?**
dense ER graph $G_{n,p}$, sparse ER graph $G_{n,d/n}$,
(uniform) random regular graph $\mathscr{G}_{n,d}$

Sharpness of the SAT–UNSAT transition
corresponds to *concentration* of the random variable $\mathbf{A}_n$

Previous work on random graph independent sets:

Previous work on random graph independent sets:

Dense Erdős–Rényi ensemble $G_{n,p}$ Grimmett–McDiarmid '75

Previous work on random graph independent sets:

Dense Erdős–Rényi ensemble $G_{n,p}$ Grimmett–McDiarmid '75

Previous work on random graph independent sets:

Dense Erdős–Rényi ensemble $G_{n,p}$ Grimmett–McDiarmid '75

Sparse Erdős–Rényi $G_{n,d/n}$; random d-regular $\mathscr{G}_{n,d}$

Previous work on random graph independent sets:

Dense Erdős–Rényi ensemble $G_{n,p}$ Grimmett–McDiarmid '75

Sparse Erdős–Rényi $G_{n,d/n}$; random d-regular $\mathscr{G}_{n,d}$

(UBD) Bollobás '81, McKay '87;
(LBD) Frieze–Łuczak '92, Frieze–Suen '94, Wormald '95
(threshold around $2(\log d)/d$, but gap remains)

Previous work on random graph independent sets:

Dense Erdős–Rényi ensemble $G_{n,p}$ Grimmett–McDiarmid '75

Sparse Erdős–Rényi $G_{n,d/n}$; random d-regular $\mathscr{G}_{n,d}$

(UBD) Bollobás '81, McKay '87;
(LBD) Frieze–Łuczak '92, Frieze–Suen '94, Wormald '95
(threshold around $2(\log d)/d$, but gap remains)

Classical argument with martingale bound ('80s) implies the transition sharpens: $\mathbf{A}_n$ has $O(n^{1/2})$ fluctuations about $\mathbb{E}A_n$

Previous work on random graph independent sets:

Dense Erdős–Rényi ensemble $G_{n,p}$ Grimmett–McDiarmid '75

Sparse Erdős–Rényi $G_{n,d/n}$; random d-regular $\mathscr{G}_{n,d}$

(UBD) Bollobás '81, McKay '87;
(LBD) Frieze–Łuczak '92, Frieze–Suen '94, Wormald '95
(threshold around $2(\log d)/d$, but gap remains)

Classical argument with martingale bound ('80s) implies the transition sharpens: $\mathbf{A}_n$ has $O(n^{1/2})$ fluctuations about $\mathbb{E}A_n$

Existence of limiting threshold location $\mathbf{A}_n/n \to \alpha_\star$ proved, but with no information on the actual value Bayati–Gamarnik–Tetali '10

(main result for MAX-IND-SET)

(main result for MAX-IND-SET)

THEOREM. Ding, Sly, S. [arXiv:1310.4787]

The maximum independent set size $\mathbf{A}_n$ *in the (uniformly) random d-regular graph* $\mathscr{G}_{n,d}$

(main result for MAX-IND-SET)

THEOREM. Ding, Sly, S. [arXiv:1310.4787]

The maximum independent set size $\mathbf{A}_n$ *in the (uniformly) random d-regular graph* $\mathscr{G}_{n,d}$ *has* $O(1)$ *fluctuations around*

$$n\alpha_\star - c_\star \log n$$

for explicit $\alpha_\star(d)$ *and* $c_\star(d)$*, provided* $d \geqslant d_0$*.*

Explicit formula for independent set threshold:

Explicit formula for independent set threshold: first define

$$\phi(q) \equiv -\log[1 - q(1 - 1/\lambda)] - (d/2 - 1)\log[1 - q^2(1 - 1/\lambda)] - \alpha \log \lambda$$

Explicit formula for independent set threshold: first define

$$\phi(q) \equiv -\log[1-q(1-1/\lambda)] \\ -(d/2-1)\log[1-q^2(1-1/\lambda)] - \alpha\log\lambda$$

$$\text{with } \lambda(q) \equiv q\frac{1-(1-q)^{d-1}}{(1-q)^d}$$

$$\text{and } \alpha(q) \equiv q\frac{1-q+dq/[2\lambda(q)]}{1-q^2(1-1/\lambda(q))}$$

Explicit formula for independent set threshold: first define

$$\phi(q) \equiv -\log[1 - q(1 - 1/\lambda)] - (d/2 - 1)\log[1 - q^2(1 - 1/\lambda)] - \alpha \log \lambda$$

$$\text{with } \lambda(q) \equiv q\frac{1 - (1 - q)^{d-1}}{(1 - q)^d}$$

$$\text{and } \alpha(q) \equiv q\frac{1 - q + dq/[2\lambda(q)]}{1 - q^2(1 - 1/\lambda(q))}$$

Solve for the largest zero $q_\star \leqslant 2(\log d)/d$ of $\phi(q)$:

Explicit formula for independent set threshold: first define

$$\phi(q) \equiv -\log[1 - q(1 - 1/\lambda)] \\ - (d/2 - 1)\log[1 - q^2(1 - 1/\lambda)] - \alpha \log \lambda$$

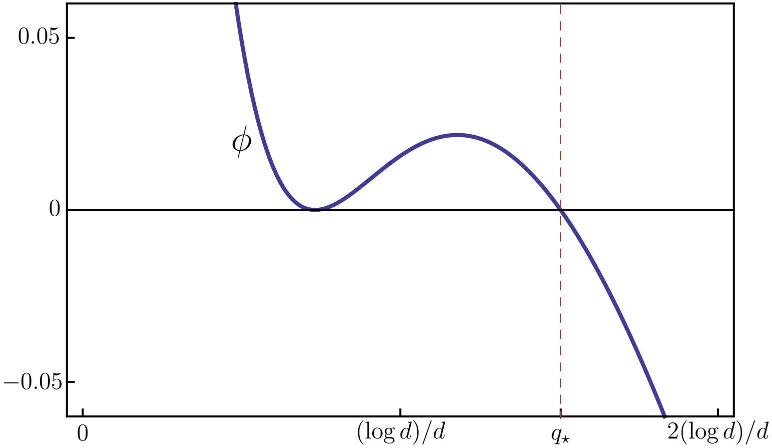
$$\text{with } \lambda(q) \equiv q\frac{1 - (1 - q)^{d-1}}{(1 - q)^d}$$

$$\text{and } \alpha(q) \equiv q\frac{1 - q + dq/[2\lambda(q)]}{1 - q^2(1 - 1/\lambda(q))}$$

Solve for the largest zero $q_\star \leqslant 2(\log d)/d$ of $\phi(q)$:
then $\mathbf{A}_n - n\alpha_\star - c_\star \log n$ is a tight random variable with

$$\alpha_\star = \alpha(q_\star) \text{ and } c_\star = (2\log \lambda(q_\star))^{-1}$$

the function $\phi(q)$ for $d = 100$



(some remarks)

(some remarks)

Our thresholds match the 1-RSB predictions made by physicists

(NAE-SAT) Castellani–Napolano–Ricci-Tersenghi–Zecchina '03,
Dall'Asta–Ramezanpour–Zecchina '08;
(independent set) Rivoire '05, Hartmann–Weigt '05,
Barbier–Krząkała–Zdeborová–Zhang '13

(some remarks)

Our thresholds match the 1-RSB predictions made by physicists

(NAE-SAT) Castellani–Napolano–Ricci-Tersenghi–Zecchina '03,
Dall'Asta–Ramezanpour–Zecchina '08;
(independent set) Rivoire '05, Hartmann–Weigt '05,
Barbier–Krząkała–Zdeborová–Zhang '13

These predictions were derived with the **survey propagation** (**SP**) method introduced by Mézard–Parisi–Zecchina '02, '05

see also Braunstein–Mézard–Zecchina '05,
Maneva–Mossel–Wainwright '07

(some remarks)

Our thresholds match the 1-RSB predictions made by physicists

(NAE-SAT) Castellani–Napolano–Ricci-Tersenghi–Zecchina '03,
Dall'Asta–Ramezanpour–Zecchina '08;
(independent set) Rivoire '05, Hartmann–Weigt '05,
Barbier–Krząkała–Zdeborová–Zhang '13

These predictions were derived with the **survey propagation** (**SP**) method introduced by Mézard–Parisi–Zecchina '02, '05

see also Braunstein–Mézard–Zecchina '05,
Maneva–Mossel–Wainwright '07

Our method of proof gives some rigorous validation to the 1-RSB & SP heuristics for these models

RSB and moment method

(probabilistic methods for rigorously bounding the SAT–UNSAT transition)

The SAT–UNSAT transition is the threshold for positivity of the random variable $Z_\alpha \equiv \#$ solutions at constraint level α

(# independent sets of density α in $\mathscr{G}_{n,d}$)

(probabilistic methods for rigorously bounding the SAT–UNSAT transition)

The SAT–UNSAT transition is the threshold for positivity of the random variable $Z_\alpha \equiv \#$ solutions at constraint level α

(# independent sets of density α in $\mathscr{G}_{n,d}$)

Upper bound is given by the *1st moment threshold* α_1 where $\mathbb{E}Z_\alpha$ crosses from exponentially large to exponentially small

(probabilistic methods for rigorously bounding the SAT–UNSAT transition)

The SAT–UNSAT transition is the threshold for positivity of the random variable $Z_\alpha \equiv \#$ solutions at constraint level α

(# independent sets of density α in $\mathscr{G}_{n,d}$)

Upper bound is given by the *1st moment threshold* α_1 where $\mathbb{E}Z_\alpha$ crosses from exponentially large to exponentially small

Lower bound: algorithmic analysis meets with barriers; and the (non-constructive) *2nd moment* approach often does much better:

e.g. Achlioptas–Moore '02

(probabilistic methods for rigorously bounding the SAT–UNSAT transition)

The SAT–UNSAT transition is the threshold for positivity of the random variable $Z_\alpha \equiv \#$ solutions at constraint level α

($\#$ independent sets of density α in $\mathscr{G}_{n,d}$)

Upper bound is given by the *1st moment threshold* α_1 where $\mathbb{E}Z_\alpha$ crosses from exponentially large to exponentially small

Lower bound: algorithmic analysis meets with barriers; and the (non-constructive) *2nd moment* approach often does much better:

e.g. Achlioptas–Moore '02

2nd moment LBD: $\mathbb{P}(Z > 0) \geqslant \dfrac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]}$ (apply with $Z = Z_\alpha$)

$$\mathbb{P}(Z > 0) \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]}$$

$$\mathbb{P}(Z > 0) \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]} = \frac{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid}) \times \mathbb{P}(\underline{\tau} \text{ valid})}{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid AND } \underline{\tau} \text{ valid})}$$

$$\mathbb{P}(Z > 0) \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]} = \frac{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid}) \times \mathbb{P}(\underline{\tau} \text{ valid})}{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid AND } \underline{\tau} \text{ valid})}$$

$\mathbb{E}[Z^2]$ has contribution $\mathbb{E}Z$ from *exactly-identical* pairs $\underline{\sigma} = \underline{\tau}$;
so contribution from *near-identical* pairs is clearly at least $\mathbb{E}Z$

$$\mathbb{P}(Z > 0) \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]} = \frac{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid}) \times \mathbb{P}(\underline{\tau} \text{ valid})}{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid AND } \underline{\tau} \text{ valid})}$$

$\mathbb{E}[Z^2]$ has contribution $\mathbb{E}Z$ from *exactly-identical* pairs $\underline{\sigma} = \underline{\tau}$;
so contribution from *near-identical* pairs is clearly at least $\mathbb{E}Z$

In a sparse CSPs, a typical solution has $\asymp n$ *unforced* variables, indicating exponential-size *clusters* of near-identical solutions: near-identical contribution to $\mathbb{E}[Z^2]$ is $\approx (\mathbb{E}Z) \times$ (avg. cluster size)

$$\mathbb{P}(Z > 0) \geqslant \frac{(\mathbb{E}Z)^2}{\mathbb{E}[Z^2]} = \frac{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid}) \times \mathbb{P}(\underline{\tau} \text{ valid})}{\sum_{\underline{\sigma}} \sum_{\underline{\tau}} \mathbb{P}(\underline{\sigma} \text{ valid AND } \underline{\tau} \text{ valid})}$$

$\mathbb{E}[Z^2]$ has contribution $\mathbb{E}Z$ from *exactly-identical* pairs $\underline{\sigma} = \underline{\tau}$;
so contribution from *near-identical* pairs is clearly at least $\mathbb{E}Z$

In a sparse CSPs, a typical solution has $\asymp n$ *unforced* variables, indicating exponential-size *clusters* of near-identical solutions: near-identical contribution to $\mathbb{E}[Z^2]$ is $\approx (\mathbb{E}Z) \times$ (avg. cluster size)

If (avg. cluster size) $\gg \mathbb{E}Z$ then 2nd moment method fails — occurs if avg. cluster size does not decrease fast enough as α increases towards the 1st moment threshold

An independent set at density $\alpha \in (0, 1)$ must have a positive fraction π of *unoccupied vertices with a single occupied neighbor*

An independent set at density $\alpha \in (0,1)$ must have a positive fraction π of *unoccupied vertices with a single occupied neighbor*

Such vertices are *unforced*, indicating a *cluster* of size $\geqslant 2^{n\pi}$

An independent set at density $\alpha \in (0,1)$ must have a positive fraction π of *unoccupied vertices with a single occupied neighbor*

Such vertices are *unforced*, indicating a *cluster* of size $\geqslant 2^{n\pi}$

Issue is that π *stays positive even above 1st moment threshold* — 2nd moment begins to fail strictly below the 1st moment threshold

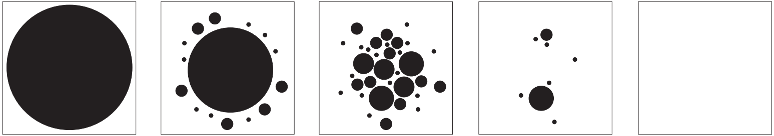
An independent set at density $\alpha \in (0, 1)$ must have a positive fraction π of *unoccupied vertices with a single occupied neighbor*

Such vertices are *unforced*, indicating a *cluster* of size $\geqslant 2^{n\pi}$

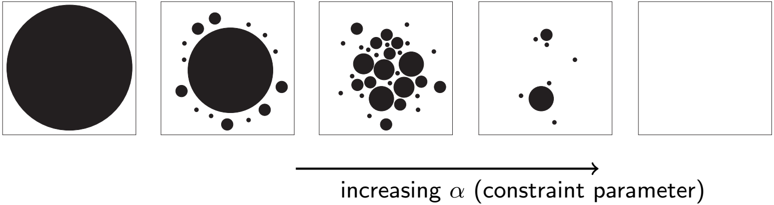
Issue is that *π stays positive even above 1st moment threshold* — 2nd moment begins to fail strictly below the 1st moment threshold

In regime (α_2, α_1), $\mathbb{E}Z \gg 1$ but $\mathbb{E}[Z^2] \gg (\mathbb{E}Z)^2$ — that is to say, Z is highly non-concentrated, and the 1st/2nd moment method yields no information about its typical behavior

conjectural phase diagram of a random CSP: Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07, Montanari–Ricci-Tersenghi–Semerjian '08

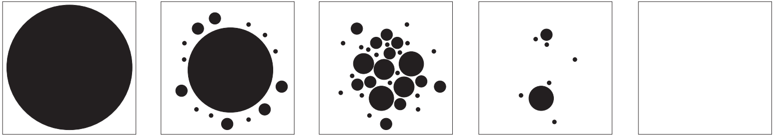


conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08



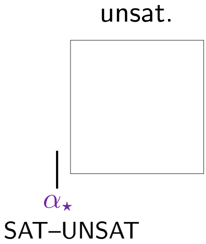
conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08

black disk = solution cluster



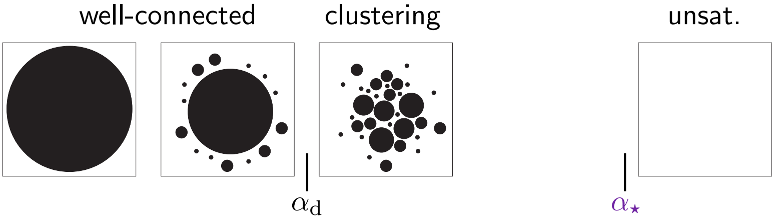
conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08

conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08

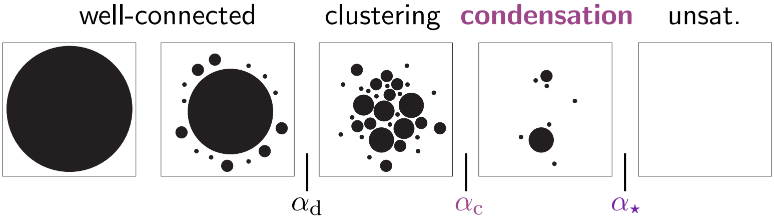


conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08

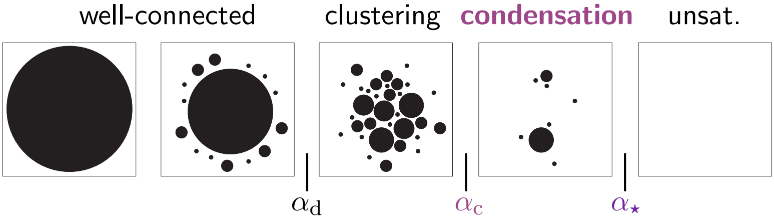
conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08



conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08

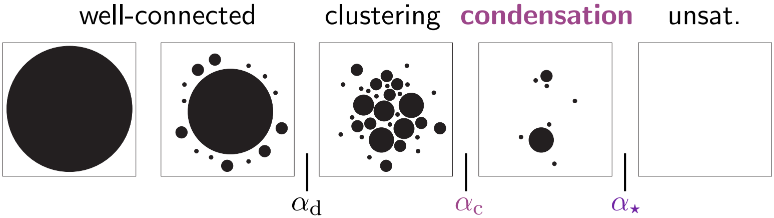


conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08



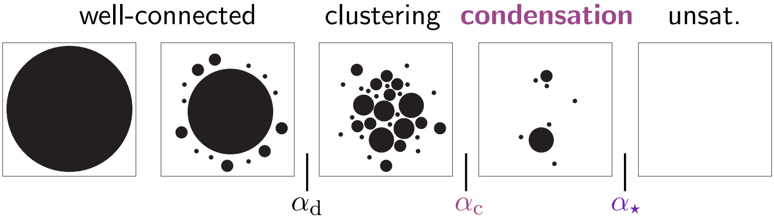
$$\mathbb{E}Z_\alpha = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\#\text{ clusters of that size at level }\alpha]}_{\exp\{n\Sigma_\alpha(s)\};\text{ compute by 1-RSB methods}};$$

conjectural phase diagram of a random CSP:
Krzạkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08



$$\mathbb{E}Z_\alpha = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\#\text{ clusters of that size at level }\alpha]}_{\exp\{n\Sigma_\alpha(s)\};\text{ compute by 1-RSB methods}};$$

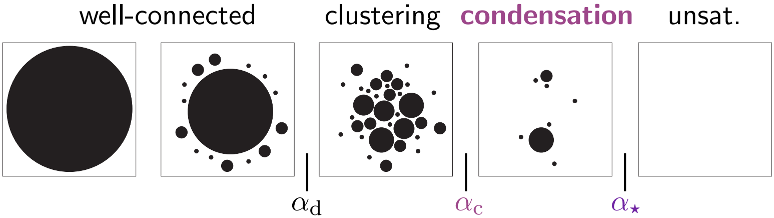
conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08



$$\mathbb{E}Z_\alpha = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\#\text{ clusters of that size at level } \alpha]}_{\exp\{n\Sigma_\alpha(s)\};\ \text{compute by 1-RSB methods}};$$

1st moment dominated by $s_\star(\alpha) = \operatorname{argmax}_s [s + \Sigma_\alpha(s)]$

conjectural phase diagram of a random CSP:
Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Montanari–Ricci-Tersenghi–Semerjian '08



$$\mathbb{E}Z_\alpha = \sum \underbrace{(\text{cluster size})}_{\exp\{ns\}} \times \underbrace{\mathbb{E}[\#\text{ clusters of that size at level }\alpha]}_{\exp\{n\Sigma_\alpha(s)\};\text{ compute by 1-RSB methods}};$$

1st moment dominated by $s_\star(\alpha) = \mathrm{argmax}_s[s + \Sigma_\alpha(s)]$

Condensation: $\Sigma_\alpha(s_\star(\alpha))$ is negative, meaning the 1st moment is dominated by extremely atypical clusters, but $\max \Sigma_\alpha$ is positive, meaning did not yet reach satisfiability threshold

1-RSB and proof approach

Independent set expected to be 1-RSB on graphs of high degree, vs. full-RSB on graphs of low degree

Barbier–Krząkała–Zdeborová–Zhang '13

Independent set expected to be 1-RSB on graphs of high degree, vs. full-RSB on graphs of low degree

Barbier–Krząkała–Zdeborová–Zhang '13

1-RSB says **clusters are RS** though individual solutions are not — moment method should succeed on **number of clusters**

Independent set expected to be 1-RSB on graphs of high degree, vs. full-RSB on graphs of low degree

Barbier–Krząkała–Zdeborová–Zhang '13

1-RSB says **clusters are RS** though individual solutions are not — moment method should succeed on **number of clusters**

Previous attempts to implement this suggestion failed to locate exact threshold due to reliance on inexact proxies for clusters

Coja-Oghlan–Panagiotou '12 (random NAE-SAT)

Independent set expected to be 1-RSB on graphs of high degree, vs. full-RSB on graphs of low degree

Barbier–Krząkała–Zdeborová–Zhang '13

1-RSB says **clusters are RS** though individual solutions are not — moment method should succeed on **number of clusters**

Previous attempts to implement this suggestion failed to locate exact threshold due to reliance on inexact proxies for clusters

Coja-Oghlan–Panagiotou '12 (random NAE-SAT)

Main novelty in our approach is a simple combinatorial model for clusters of large independent sets (clusters of NAE-SAT solutions)

Independent set expected to be 1-RSB on graphs of high degree, vs. full-RSB on graphs of low degree

Barbier–Krząkała–Zdeborová–Zhang '13

1-RSB says **clusters are RS** though individual solutions are not — moment method should succeed on **number of clusters**

Previous attempts to implement this suggestion failed to locate exact threshold due to reliance on inexact proxies for clusters

Coja-Oghlan–Panagiotou '12 (random NAE-SAT)

Main novelty in our approach is a simple combinatorial model for clusters of large independent sets (clusters of NAE-SAT solutions)

We show the moment method locates the sharp transition for this model, proving the result and validating the 1-RSB hypothesis

Modeling clusters of large independent sets:

Modeling clusters of large independent sets:
In independent set, let $0 \equiv$ unoccupied; $1 \equiv$ occupied

Modeling clusters of large independent sets:
In independent set, let 0 $\equiv$ unoccupied; 1 $\equiv$ occupied

Typical independent set has linear number of 0's with a single neighboring 1:

Modeling clusters of large independent sets:
In independent set, let 0 $\equiv$ unoccupied; 1 $\equiv$ occupied

Typical independent set has linear number of 0's with a single neighboring 1: results in exponential-sized clusters of independent sets joined by neighboring (0 — 1) swaps

Modeling clusters of large independent sets:
In independent set, let 0 $\equiv$ unoccupied; 1 $\equiv$ occupied

Typical independent set has linear number of 0's with a single neighboring 1: results in exponential-sized clusters of independent sets joined by neighboring (0 — 1) swaps

Chains of swaps can and will occur;

Modeling clusters of large independent sets:
In independent set, let 0 $\equiv$ unoccupied; 1 $\equiv$ occupied

Typical independent set has linear number of 0's with a single neighboring 1: results in exponential-sized clusters of independent sets joined by neighboring (0 — 1) swaps

Chains of swaps can and will occur; but near threshold ($\approx 2(\log d)/d$) they propagate like a subcritical branching process

Modeling clusters of large independent sets:
In independent set, let 0 $\equiv$ unoccupied; 1 $\equiv$ occupied

Typical independent set has linear number of 0's with a single neighboring 1: results in exponential-sized clusters of independent sets joined by neighboring (0 — 1) swaps

Chains of swaps can and will occur; but near threshold ($\approx 2(\log d)/d$) they propagate like a subcritical branching process

Cluster model defined by **coarsening** (projection) from original:

Modeling clusters of large independent sets:
In independent set, let 0 $\equiv$ unoccupied; 1 $\equiv$ occupied

Typical independent set has linear number of 0's with a single neighboring 1: results in exponential-sized clusters of independent sets joined by neighboring (0 — 1) swaps

Chains of swaps can and will occur; but near threshold ($\approx 2(\log d)/d$) they propagate like a subcritical branching process

Cluster model defined by **coarsening** (projection) from original:

- Relabel all neighboring (0 — 1) swaps with (f =○= f)

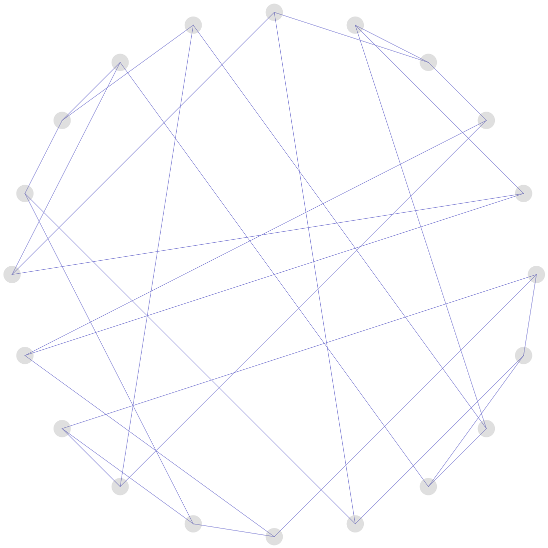
Modeling clusters of large independent sets:
In independent set, let $\mathtt{0} \equiv$ unoccupied; $\mathtt{1} \equiv$ occupied

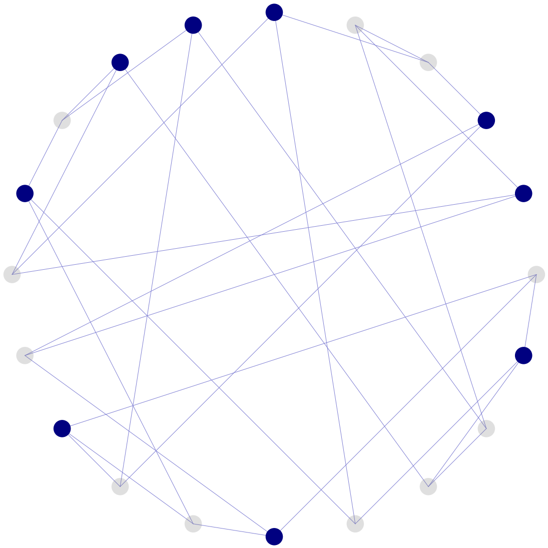
Typical independent set has linear number of 0's with a single neighboring 1: results in exponential-sized clusters of independent sets joined by neighboring ($\mathtt{0}$ — $\mathtt{1}$) swaps

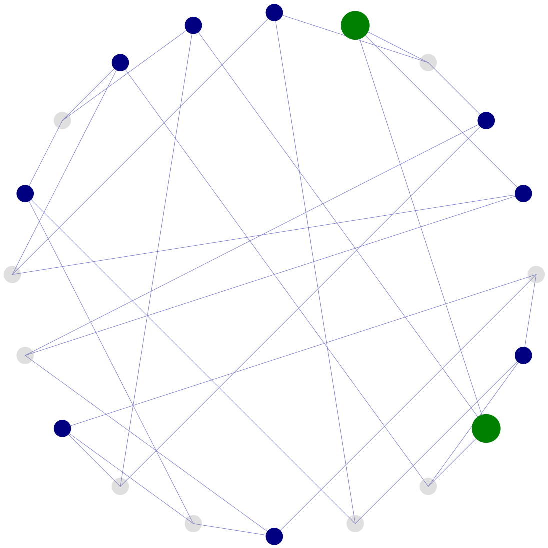
Chains of swaps can and will occur; but near threshold ($\approx 2(\log d)/d$) they propagate like a subcritical branching process

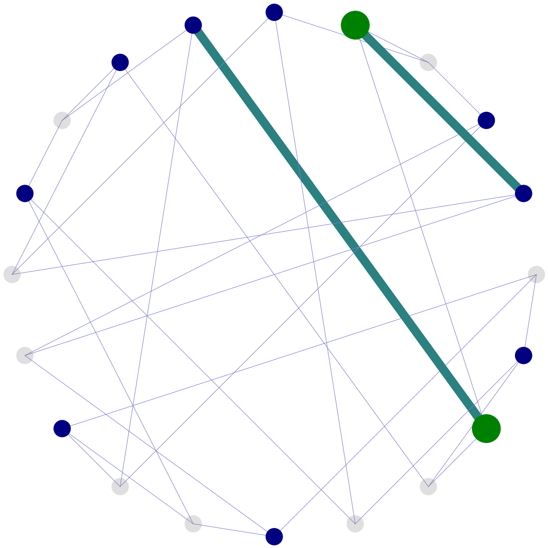
Cluster model defined by **coarsening** (projection) from original:

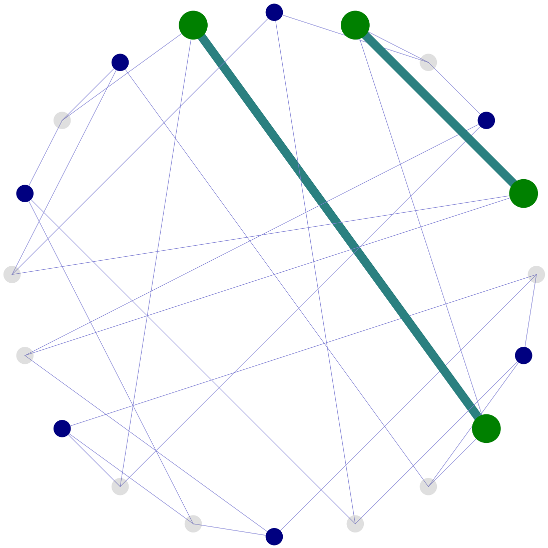
- Relabel all neighboring ($\mathtt{0}$ — $\mathtt{1}$) swaps with ($\mathtt{f} \mathrel{=\!\circ\!=} \mathtt{f}$)
- Operation may result in formation of new ($\mathtt{0}$ — $\mathtt{1}$) swaps; iterate until none remain

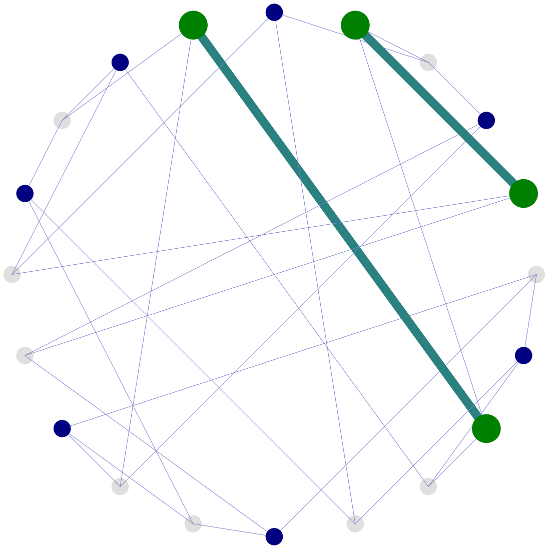


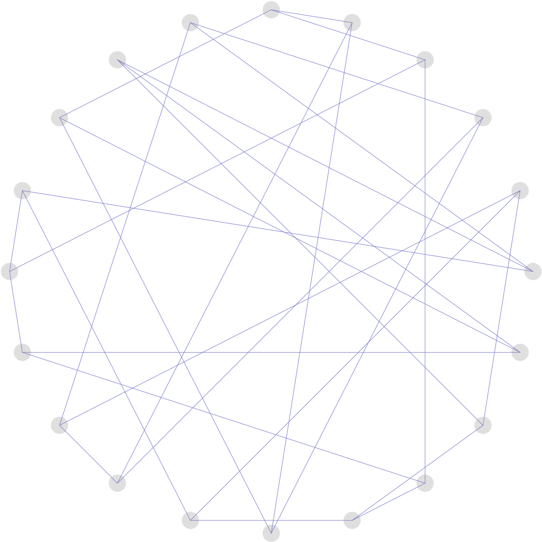


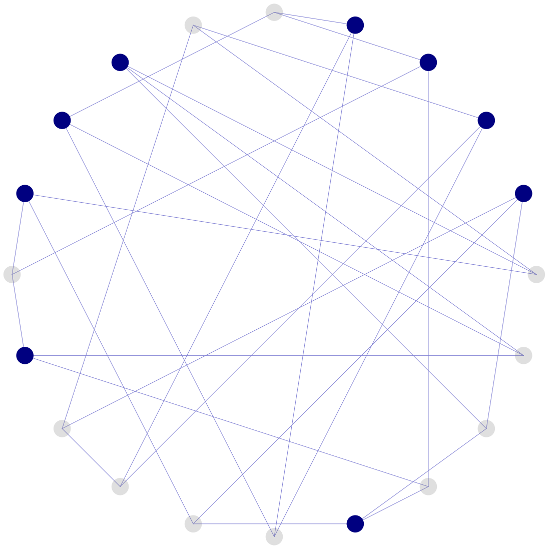


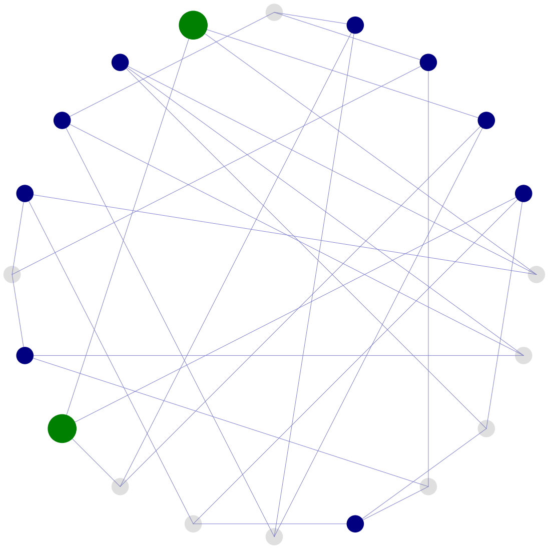


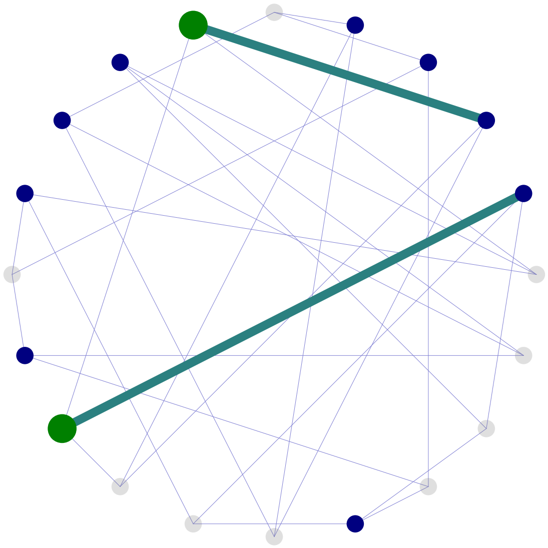


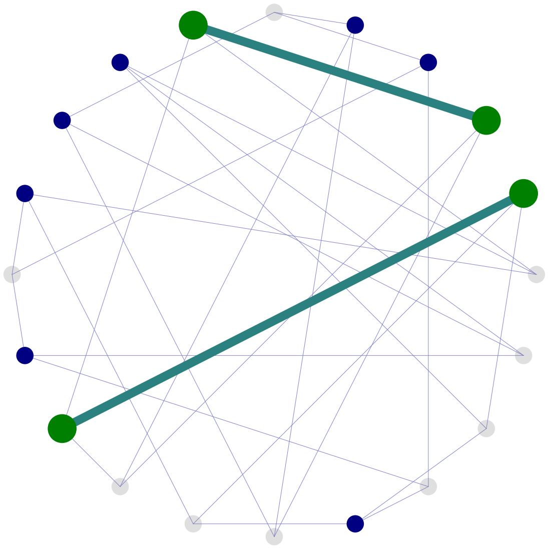


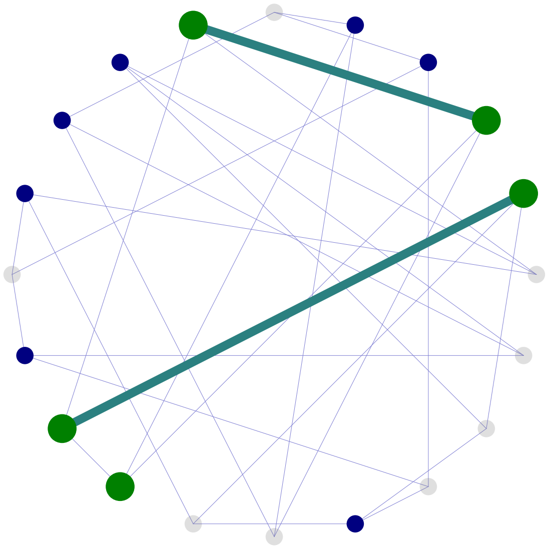


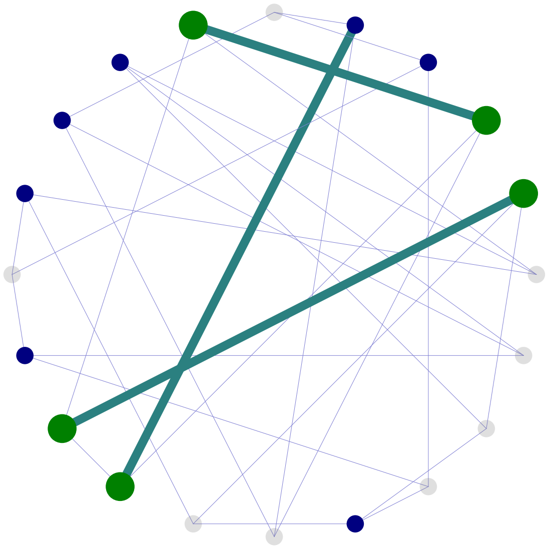


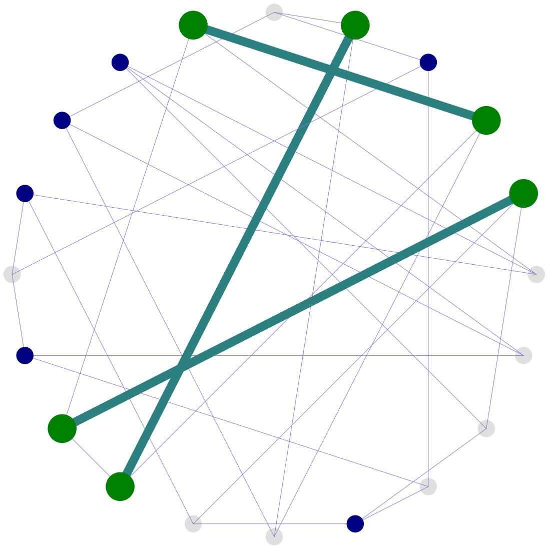


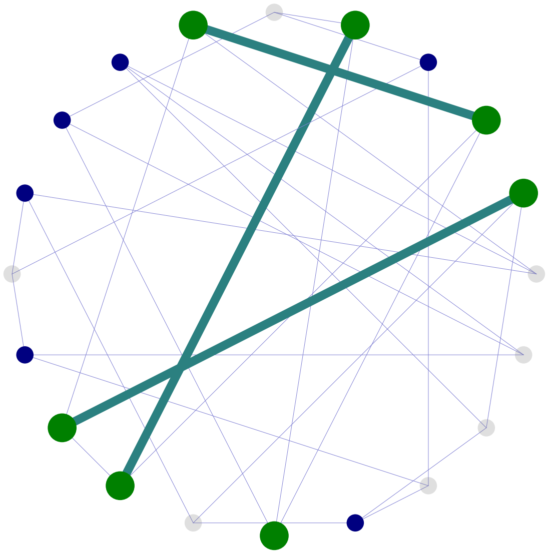


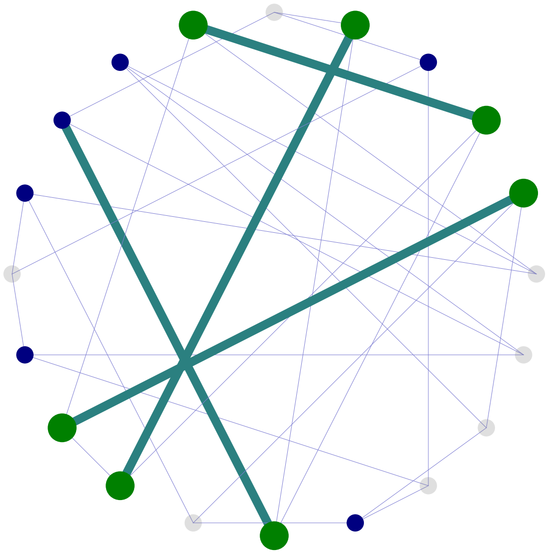


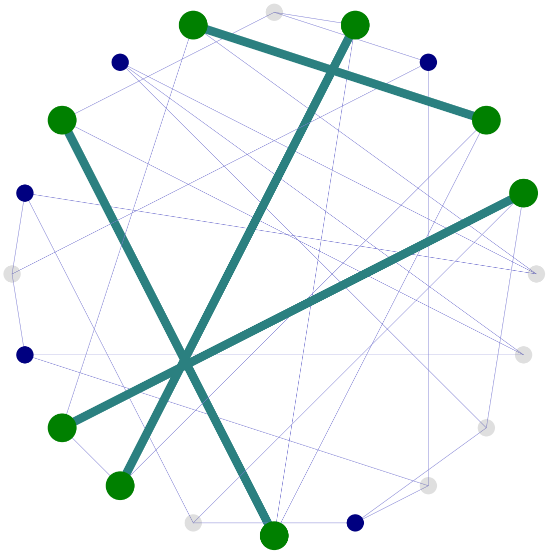


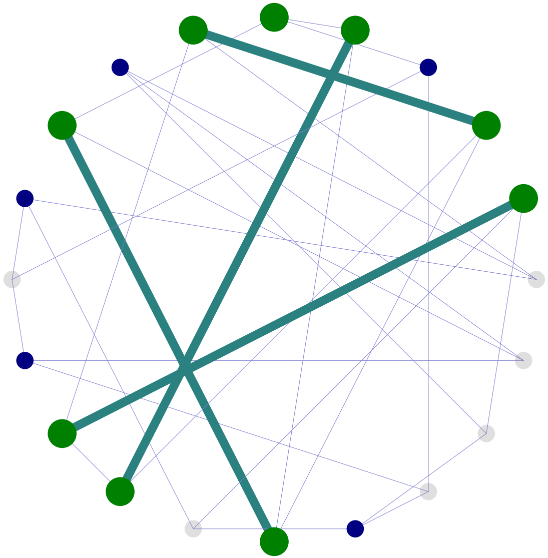


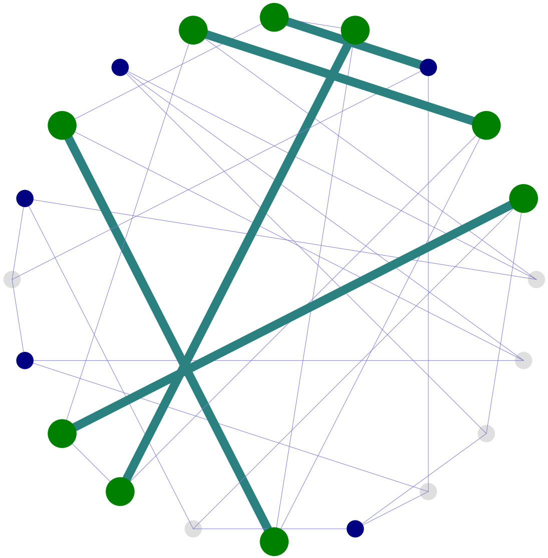


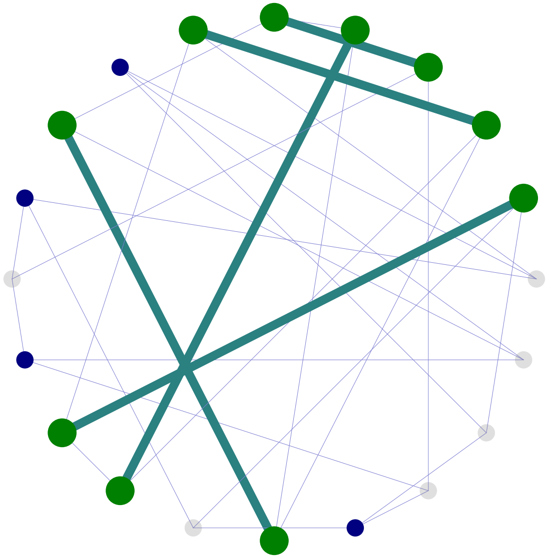


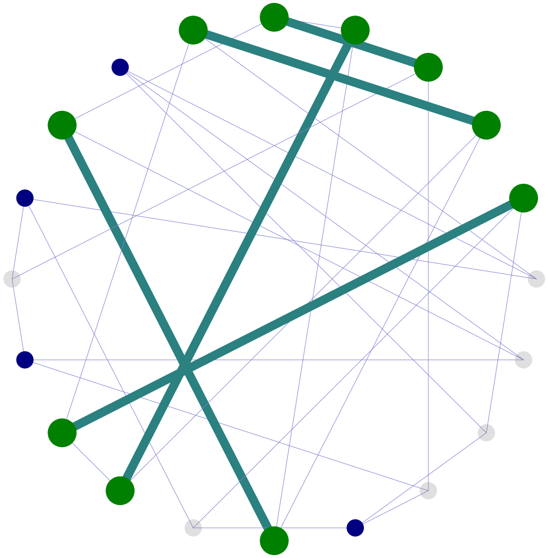












Analogues of coarsening procedure appeared in previous analyses of solution space geometry for other CSPs —
also termed **whitening** or **warning propagation**

Parisi '02, Achlioptas–Ricci-Tersenghi '06,
Maneva–Mossel–Wainwright '07, Maneva–Sinclair '08

Analogues of coarsening procedure appeared in previous analyses of solution space geometry for other CSPs —
also termed **whitening** or **warning propagation**

Parisi '02, Achlioptas–Ricci-Tersenghi '06,
Maneva–Mossel–Wainwright '07, Maneva–Sinclair '08

Key observation is that coarsened configurations themselves essentially form a graphical model, *forgetting the coarsening*

Analogues of coarsening procedure appeared in previous analyses of solution space geometry for other CSPs —
also termed **whitening** or **warning propagation**

Parisi '02, Achlioptas–Ricci-Tersenghi '06,
Maneva–Mossel–Wainwright '07, Maneva–Sinclair '08

Key observation is that coarsened configurations themselves essentially form a graphical model, *forgetting the coarsening*

Clusters of large independent sets are encoded by configurations of 0's, 1's, and $(\mathtt{f} \;\text{—o—}\; \mathtt{f})$ pairs satisfying some simple local rules:

Analogues of coarsening procedure appeared in previous analyses of solution space geometry for other CSPs —
also termed **whitening** or **warning propagation**

Parisi '02, Achlioptas–Ricci-Tersenghi '06,
Maneva–Mossel–Wainwright '07, Maneva–Sinclair '08

Key observation is that coarsened configurations themselves essentially form a graphical model, *forgetting the coarsening*

Clusters of large independent sets are encoded by configurations of 0's, 1's, and (f ⊸⊸ f) pairs satisfying some simple local rules:
1 must neighbor all 0's,

Analogues of coarsening procedure appeared in previous analyses of solution space geometry for other CSPs —
also termed **whitening** or **warning propagation**

Parisi '02, Achlioptas–Ricci-Tersenghi '06,
Maneva–Mossel–Wainwright '07, Maneva–Sinclair '08

Key observation is that coarsened configurations themselves essentially form a graphical model, *forgetting the coarsening*

Clusters of large independent sets are encoded by configurations of 0's, 1's, and (f ═o═ f) pairs satisfying some simple local rules:
1 must neighbor all 0's, while each 0 must neighbor *at least two* 1's

Analogues of coarsening procedure appeared in previous analyses of solution space geometry for other CSPs —
also termed **whitening** or **warning propagation**

Parisi '02, Achlioptas–Ricci-Tersenghi '06,
Maneva–Mossel–Wainwright '07, Maneva–Sinclair '08

Key observation is that coarsened configurations themselves essentially form a graphical model, *forgetting the coarsening*

Clusters of large independent sets are encoded by configurations of 0's, 1's, and (f — f) pairs satisfying some simple local rules:
1 must neighbor all 0's, while each 0 must neighbor *at least two* 1's

Let $\mathbf{Z}_\alpha = \#$ valid 0/1/f configurations on $\mathscr{G}_{n,d}$ with
(number of 1's) $+ \frac{1}{2}$(number of f's) $= n\alpha$
— $\mathbf{Z}_\alpha$ counts clusters in the space of density-α independent sets

back to replica symmetry

We started from a model (independent sets) exhibiting RSB

We started from a model (independent sets) exhibiting RSB

The coarsening procedure led us to a graphical model of clusters

We started from a model (independent sets) exhibiting RSB

The coarsening procedure led us to a graphical model of clusters

If indeed the model is 1-RSB, the clusters should be RS,
meaning moment method should locate the sharp threshold

We started from a model (independent sets) exhibiting RSB

The coarsening procedure led us to a graphical model of clusters

If indeed the model is 1-RSB, the clusters should be RS,
meaning moment method should locate the sharp threshold

RS (belief propagation) prediction for cluster model corresponds exactly to 1-RSB (survey propagation) prediction for original model

observed in SAT context by various authors
incl. Maneva–Mossel–Wainwright '07

We started from a model (independent sets) exhibiting RSB

The coarsening procedure led us to a graphical model of clusters

If indeed the model is 1-RSB, the clusters should be RS, meaning moment method should locate the sharp threshold

RS (belief propagation) prediction for cluster model corresponds exactly to 1-RSB (survey propagation) prediction for original model

observed in SAT context by various authors
incl. Maneva–Mossel–Wainwright '07

Much of the technical work goes into actually proving that the moment method succeeds for the cluster model ...

$\mathbf{Z}_\alpha$ counts clusters *restricted to α-hyperplane*: handle by introducing fugacity λ to act as Lagrange multiplier:

$$\mathbf{Z}(\lambda) = \sum_\alpha \lambda^{n\alpha} \mathbf{Z}_\alpha; \quad \mathbb{E}\mathbf{Z}(\lambda) \asymp \exp\{n\varphi(\lambda)\}$$

$\mathbf{Z}_\alpha$ counts clusters *restricted to α-hyperplane*: handle by introducing fugacity λ to act as Lagrange multiplier:

$$\mathbf{Z}(\lambda) = \sum_\alpha \lambda^{n\alpha} \mathbf{Z}_\alpha; \quad \mathbb{E}\mathbf{Z}(\lambda) \asymp \exp\{n\varphi(\lambda)\}$$

Given α, set λ_α so that $\mathbb{E}\mathbf{Z}(\lambda)$ is dominated by α-hyperplane contribution — $\log \mathbb{E}\mathbf{Z}(\lambda)$ decays *quadratically* around α, so

$$\mathbb{E}\mathbf{Z}_\alpha \asymp \frac{\mathbb{E}\mathbf{Z}(\lambda)/\lambda^{n\alpha}}{n^{1/2}}$$

$\mathbf{Z}_\alpha$ counts clusters *restricted to α-hyperplane*: handle by introducing fugacity λ to act as Lagrange multiplier:

$$\mathbf{Z}(\lambda) = \sum_\alpha \lambda^{n\alpha} \mathbf{Z}_\alpha; \quad \mathbb{E}\mathbf{Z}(\lambda) \asymp \exp\{n\varphi(\lambda)\}$$

Given α, set λ_α so that $\mathbb{E}\mathbf{Z}(\lambda)$ is dominated by α-hyperplane contribution — $\log \mathbb{E}\mathbf{Z}(\lambda)$ decays *quadratically* around α, so

$$\mathbb{E}\mathbf{Z}_\alpha \asymp \frac{\mathbb{E}\mathbf{Z}(\lambda)/\lambda^{n\alpha}}{n^{1/2}} \asymp \frac{\exp\{n\psi(\alpha)\}}{n^{1/2}}$$

with $\psi(\alpha) = \varphi(\lambda_\alpha) - \alpha \log \lambda_\alpha$ and $\psi'(\alpha) = -\log \lambda_\alpha$

$\mathbf{Z}_\alpha$ counts clusters *restricted to α-hyperplane*: handle by introducing fugacity λ to act as Lagrange multiplier:

$$\mathbf{Z}(\lambda) = \sum_\alpha \lambda^{n\alpha} \mathbf{Z}_\alpha; \quad \mathbb{E}\mathbf{Z}(\lambda) \asymp \exp\{n\varphi(\lambda)\}$$

Given α, set λ_α so that $\mathbb{E}\mathbf{Z}(\lambda)$ is dominated by α-hyperplane contribution — $\log \mathbb{E}\mathbf{Z}(\lambda)$ decays *quadratically* around α, so

$$\mathbb{E}\mathbf{Z}_\alpha \asymp \frac{\mathbb{E}\mathbf{Z}(\lambda)/\lambda^{n\alpha}}{n^{1/2}} \asymp \frac{\exp\{n\psi(\alpha)\}}{n^{1/2}}$$

with $\psi(\alpha) = \varphi(\lambda_\alpha) - \alpha \log \lambda_\alpha$ and $\psi'(\alpha) = -\log \lambda_\alpha$

We prove $\mathbb{E}\mathbf{Z}_\alpha \asymp 1$ determines the true threshold: $n\alpha_\star - c_\star \log n$ where $\psi(\alpha_\star) = 0$, and $c_\star = 1/(2 \log \lambda_{\alpha_\star})$ corrects for $n^{1/2}$ factor

$\mathbf{Z}_\alpha$ counts clusters *restricted to α-hyperplane*: handle by introducing fugacity λ to act as Lagrange multiplier:

$$\mathbf{Z}(\lambda) = \sum_\alpha \lambda^{n\alpha} \mathbf{Z}_\alpha; \quad \mathbb{E}\mathbf{Z}(\lambda) \asymp \exp\{n\varphi(\lambda)\}$$

Given α, set λ_α so that $\mathbb{E}\mathbf{Z}(\lambda)$ is dominated by α-hyperplane contribution — $\log \mathbb{E}\mathbf{Z}(\lambda)$ decays *quadratically* around α, so

$$\mathbb{E}\mathbf{Z}_\alpha \asymp \frac{\mathbb{E}\mathbf{Z}(\lambda)/\lambda^{n\alpha}}{n^{1/2}} \asymp \frac{\exp\{n\psi(\alpha)\}}{n^{1/2}}$$

$$\text{with } \psi(\alpha) = \varphi(\lambda_\alpha) - \alpha \log \lambda_\alpha \text{ and } \psi'(\alpha) = -\log \lambda_\alpha$$

We prove $\mathbb{E}\mathbf{Z}_\alpha \asymp 1$ determines the true threshold: $n\alpha_\star - c_\star \log n$ where $\psi(\alpha_\star) = 0$, and $c_\star = 1/(2 \log \lambda_{\alpha_\star})$ corrects for $n^{1/2}$ factor

Establishing constant-order fluctuations about $n\alpha_\star - c_\star \log n$ requires further work (variance decomposition by Fourier analysis)

Possible further directions:

Possible further directions:

Extension to q-coloring?

Possible further directions:

Extension to q-coloring? To k-SAT, or to Erdős–Rényi graphs?

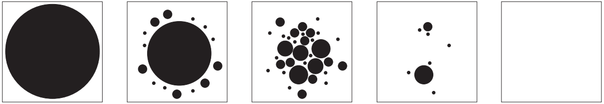
Possible further directions:

Extension to q-coloring? To k-SAT, or to Erdős–Rényi graphs?
Requires improved methods for replica symmetric models

Possible further directions:

Extension to q-coloring? To k-SAT, or to Erdős–Rényi graphs?
Requires improved methods for replica symmetric models

Other aspects of the RSB phase diagram?

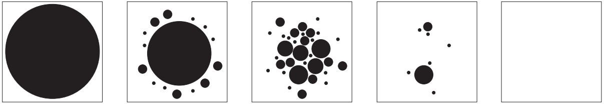


see Bapst–Coja-Oghlan–Hetterich–Rassmann–Vilenchik '14
for condensation phase transition in random graph coloring

Possible further directions:

Extension to q-coloring? To k-SAT, or to Erdős–Rényi graphs?
Requires improved methods for replica symmetric models

Other aspects of the RSB phase diagram?



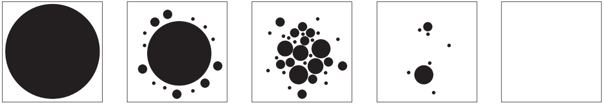
see Bapst–Coja-Oghlan–Hetterich–Rassmann–Vilenchik '14
for condensation phase transition in random graph coloring

Models with higher levels of RSB, e.g. MAX-CUT

Possible further directions:

Extension to q-coloring? To k-SAT, or to Erdős–Rényi graphs?
Requires improved methods for replica symmetric models

Other aspects of the RSB phase diagram?



see Bapst–Coja-Oghlan–Hetterich–Rassmann–Vilenchik '14
for condensation phase transition in random graph coloring

Models with higher levels of RSB, e.g. MAX-CUT

Thank you!