Light, Polymers and Automobiles - Statistical Physics of Routing

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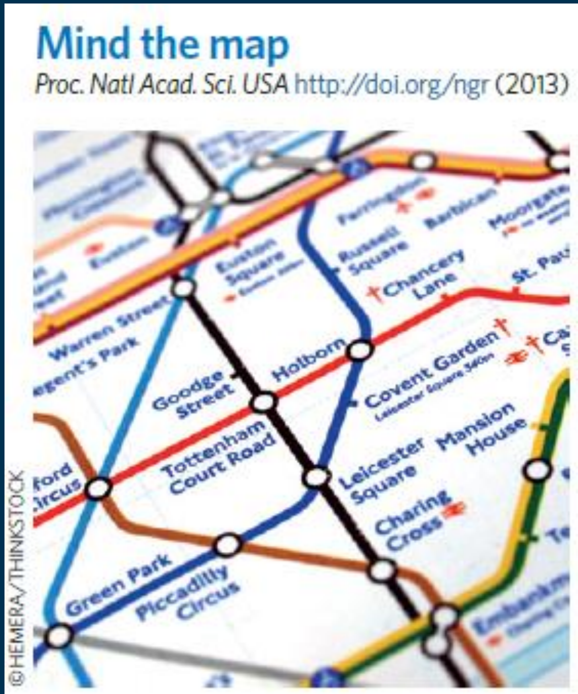
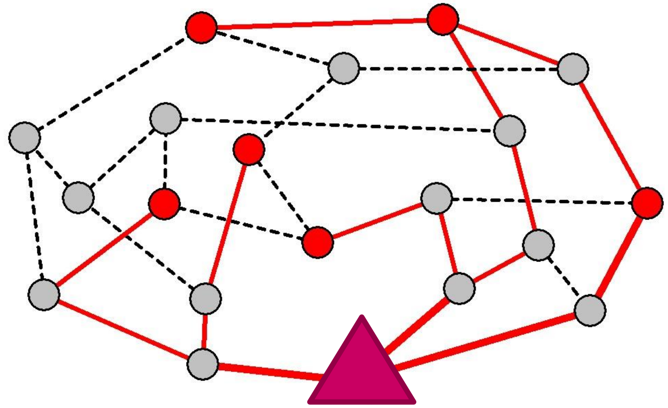


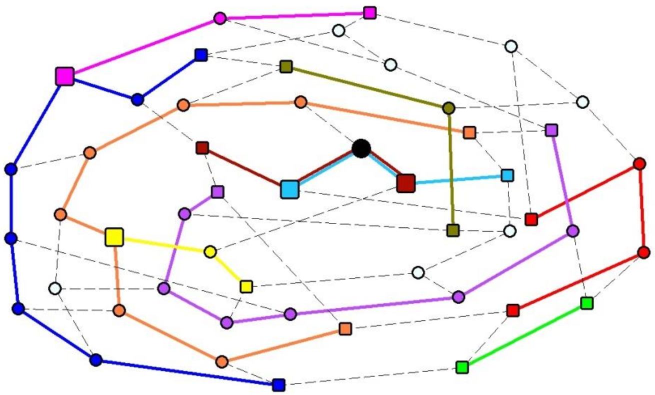
Outline

- Motivation – why routing?
- The models – two scenarios

① One universal source



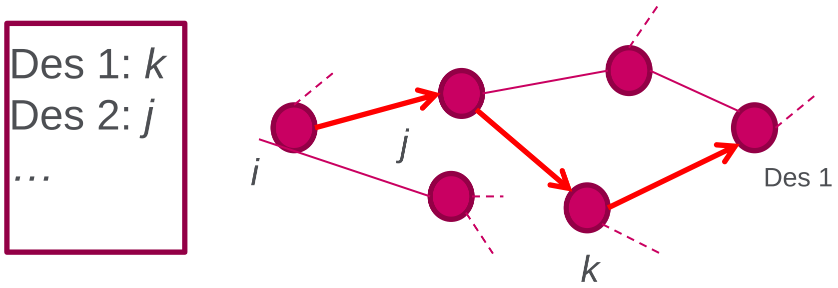
② Ordinary routing

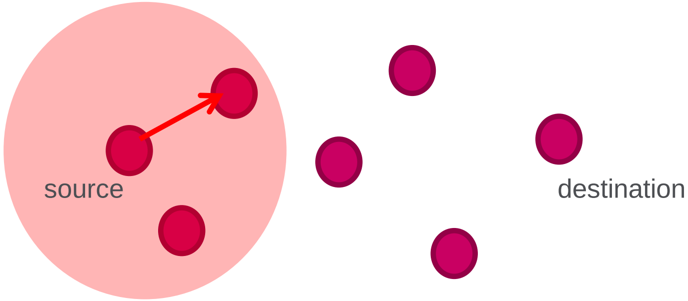


- Two approaches: cavity, replica and polymer methods
- Results: microscopic solutions, macroscopic phenomena
- Applications: e.g. subway, air traffic networks
- Conclusions

Why routing?

- **Are existing algorithms any good?**
- Routing tables computed by shortest-path, or minimal weight on path (e.g. Internet)
- Geographic routing (e.g. wireless networks)

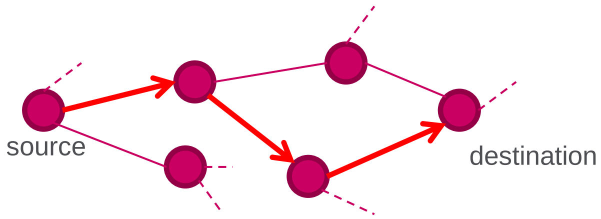




- **Insensitive to other path choices** → congestion, or low occupancy routers/stations for sparse traffic
- Heuristics- monitoring queue length → sub-optimal

Global optimization

1. A difficult problem with non-local variables

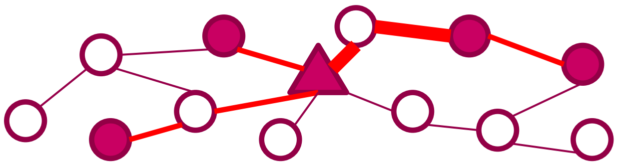


Unlike most combinatorial problems such as Graph coloring, Vertex cover, K-sat, etc.

2. Non-linear interaction between communications:

   avoid congestion → repulsion
   consolidate traffic → attraction

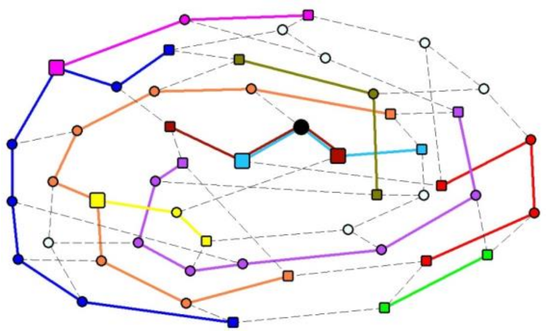
   paths interact with each other



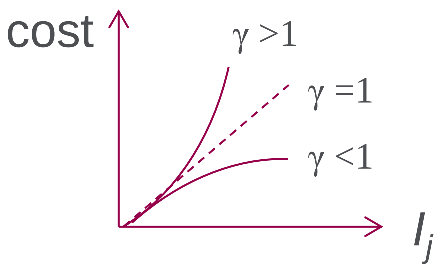
Interaction is absent in similar problems: spanning trees and Stenier trees

M.Bayati et al , PRL 101, 037208 (2008)

Communication Model



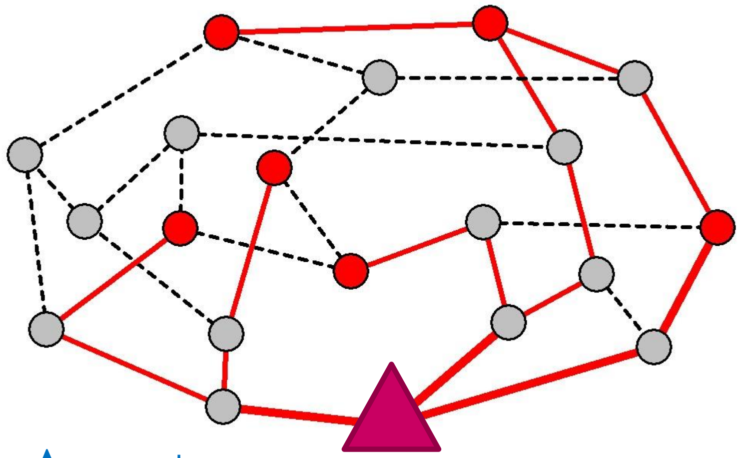
- N nodes ($i, j, k\ldots$)
- M communications (v,..)
  each with a fixed source and destination
- Denote, $\sigma_j^v = 1$ (communication v passes through node j)
  $\sigma_j^v = 0$ (otherwise)
- Traffic on j → $I_j = \Sigma_v \, \sigma_j^v$



- Find path configuration which **globally** minimizes
  $H=\Sigma_j \, (I_j)^\gamma$ or $H=\Sigma_{(ij)} \, (I_{ij})^\gamma$

- γ >1 repulsion (between com.) → avoid congestion
- γ <1 attraction → aggregate traffic (to ↑ idle nodes)
- γ =1 no interaction, $H=\Sigma_{v\,j} \, \sigma_j^v$ → shortest path routing

Analytical approach ①



- Map the routing problem onto a model of resource allocation:

  Each node i has initial resource Λ_i

- Receiver (base station, router) $\Lambda_i = +\infty$
- Senders (e.g. com. nodes) $\Lambda_i = -1$
- others $\Lambda_i = 0$

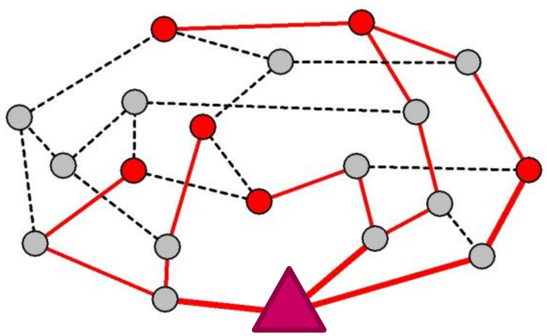
- Minimize $H=\Sigma_{(ij)} (I_{ij})^{\gamma}$
- Constraints: (i) final resource $R_i = \Lambda_i + \Sigma_{j \in \partial i} I_{ji} = 0$, all i

  (ii) currents are integers

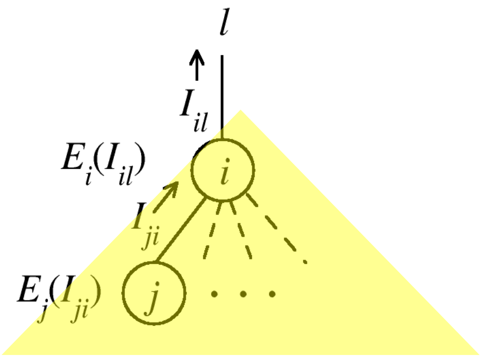
- Central router $\xrightarrow{\text{resource}}$ com. nodes (integer current)

→ each sender establishes a single path to the receiver

The cavity method



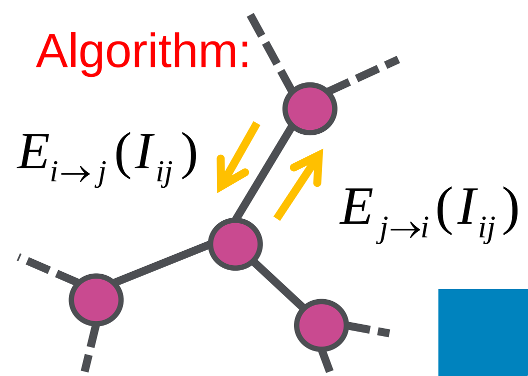
- $E_i(I_{il})$ = optimized energy of the tree terminated at node i without l
- At zero-temperature, we use the following recursion to obtain a stable $P[E_i(I_{il})]$



$$E_i(I_{il}) = \min_{\{\{I_{ji}\}|R_i=0\}} \left[|I_{il}|^\gamma + \sum_{j\in L_i\backslash\{l\}} E_j(I_{ji}) \right]$$

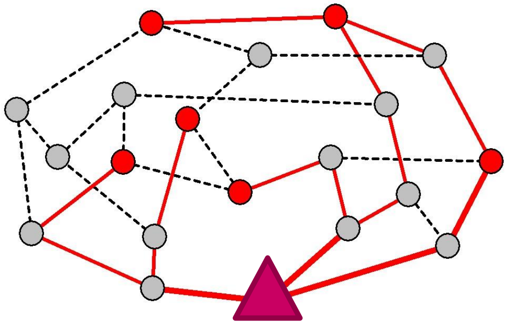
- However, constrained minimization over integer domain → difficult
- $\gamma>1$, we can show that $E_i(I_{il})$ is convex

→ computation greatly simplified



Yeung and Saad, PRL **108**, 208701 (2012); Yeung, IEEE Proc NETSTAT (2013)

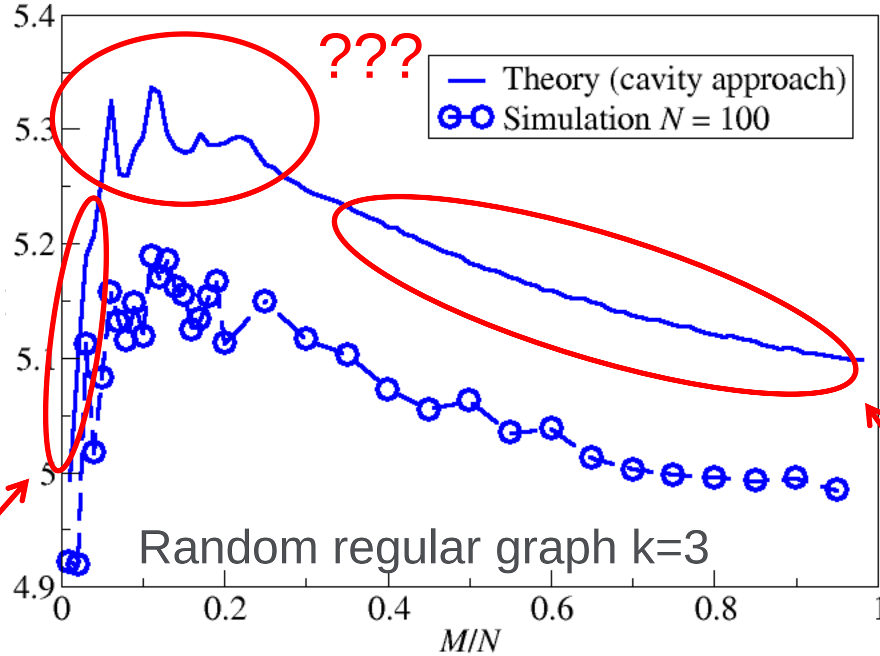
Results - Non-monotonic $\langle L \rangle$



$H=\Sigma_{(ij)} (I_{ij})^2$ i.e. γ =2 →avoid congestion

M – number of senders

Average path length per communication

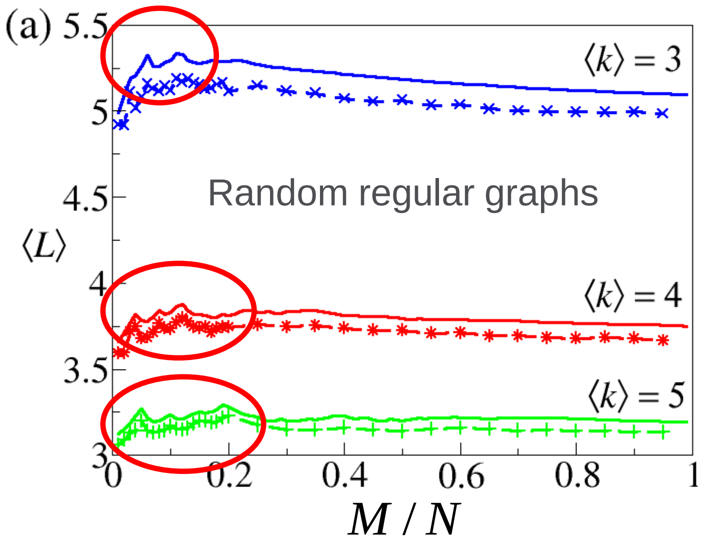


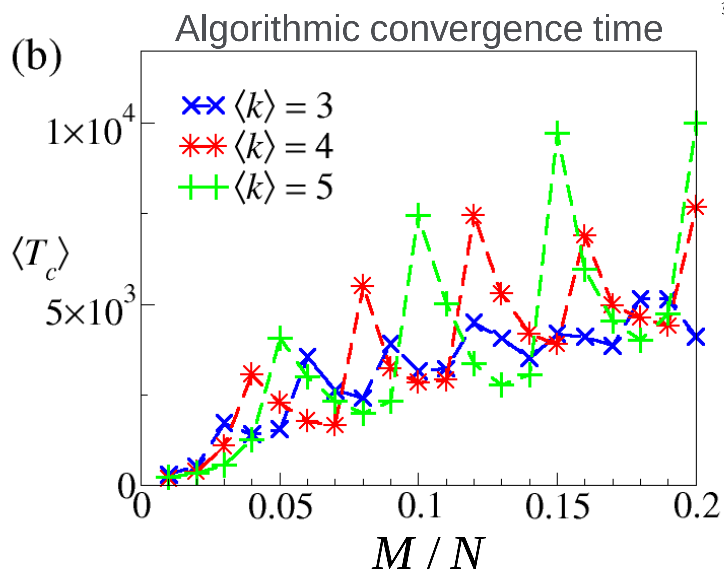
Small deviations between simulation - finite size effect, N ↑, deviation ↓

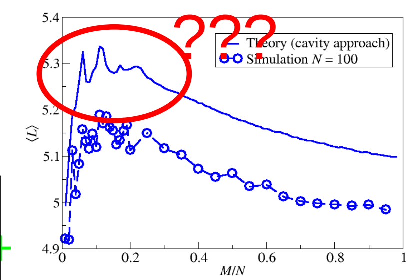
Initial ↑ in $\langle L \rangle$ - as short routes are being occupied longer routes are chosen

Final ↓ in $\langle L \rangle$ - when traffic is dense, everywhere is congested

Results - balanced receiver

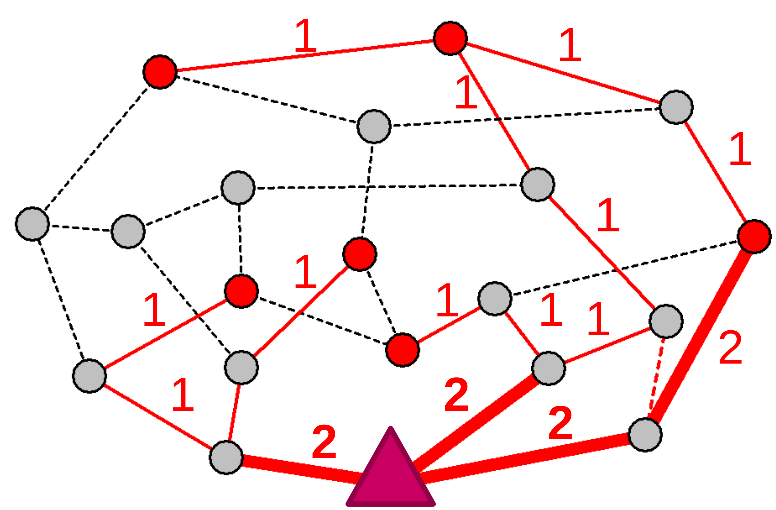




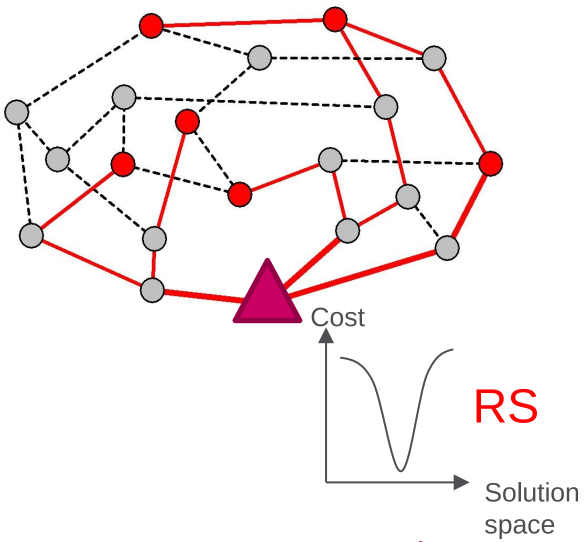


Example:
M=6, $\langle k \rangle$=3
$H=\Sigma_{(ij)} (I_{ij})^2$

- Small peaks in $\langle L \rangle$ are multiples of $\langle k \rangle$, balance traffic around receiver
- Consequence →peaks occur in convergence time $\langle T_c \rangle$
- Studied - random network, scale-free networks, qualitatively similar behavior

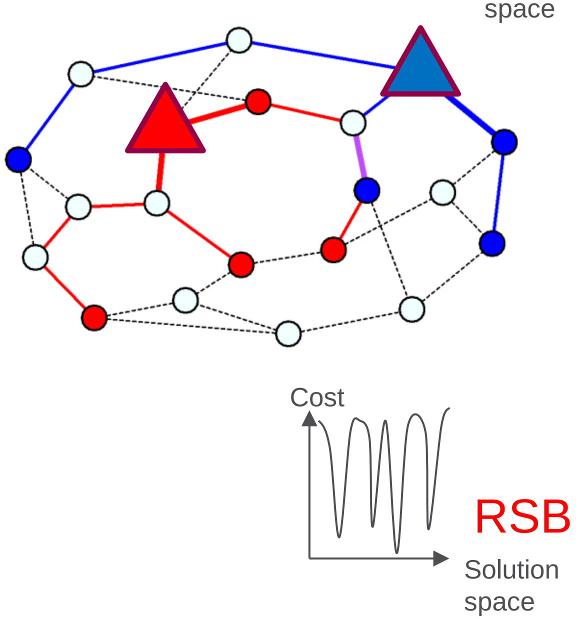


RS/RSB multiple router types



One receiver "type"

- $H=\Sigma_{(ij)} (I_{ij})^{\gamma}$,$\gamma>1$
- $E_j(I_{ji})$ is convex
- RS for any M/N



Two receiver "types": A & B

Senders with $\Lambda^{A} = -1$ or $\Lambda^{B} = -1$

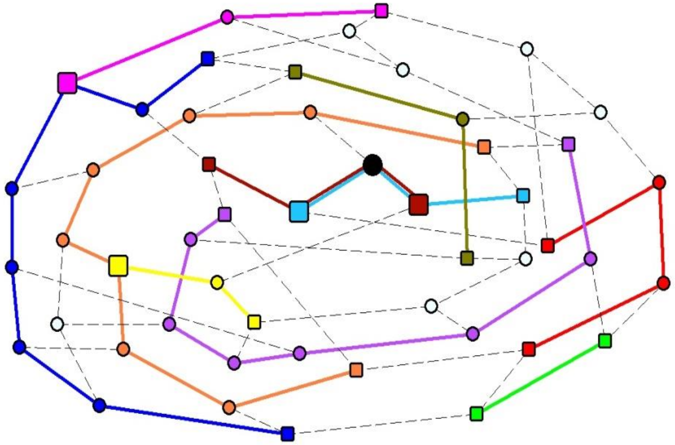
$H=\Sigma_{(ij)} (|I_{ij}^{A}|+|I_{ij}^{B}|)^{\gamma}$,$\gamma>1$

$E_j(I_{ij}^{A}, I_{ij}^{B})$ not always convex

Experiments exhibit RSB-like behavior

Node disjoint routing ②

- Random communicating pairs $\mu=1 \ldots M$
- Routes do not cross



Node i has initial resource Λ^{μ}_{i} :

- Receiver $\Lambda^{\mu}_{i} = -1$
- Senders $\Lambda^{\mu}_{i} = +1$
- others $\Lambda^{\mu}_{i} = 0$

Currents:

- Route μ passes through (i,j) $i \rightarrow j$ $I^{\mu}_{ij} = +1$
- Route μ passes through (i,j) $j \rightarrow i$ $I^{\mu}_{ij} = -1$
- otherwise $I^{\mu}_{ij} = 0$

- Minimize $H=\Sigma_{(ij)} f(\Sigma_{\mu} |I^{\mu}_{ij}|)$ - but no crossing
- Constraints:(i) final resource $\Lambda^{\mu}_{i} + \Sigma_{j\in\partial i} I^{\mu}_{ji} = 0$, all i, μ

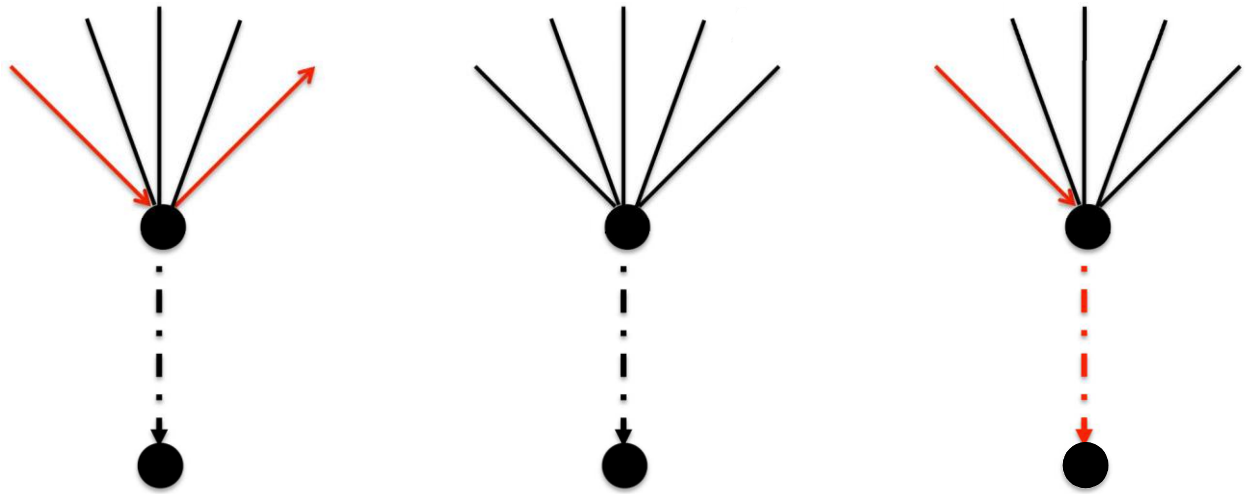
  (ii) currents are integers and $I^{\mu}_{ij} = -I^{\mu}_{ji}$

The cavity method

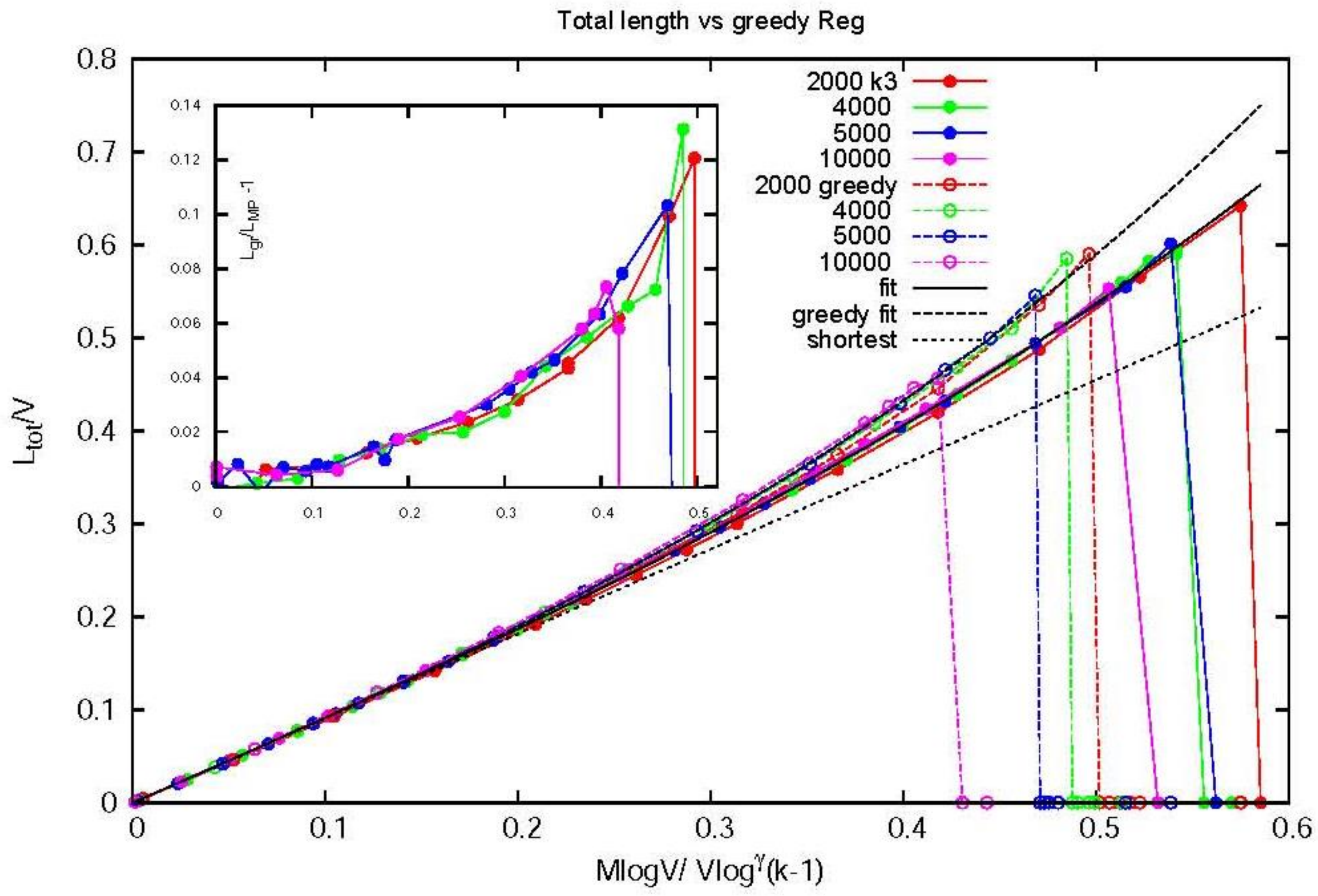
- At zero-temperature, we use recursion relation to obtain a stable $P[E_{ij}(\{I_{ij}\})]$ where $\{I_{ij}\}= I^1_{ij}, I^2_{ij}, \ldots, I^M_{ij}$
- $f(I) = I$ for $I=0,1$ and ∞ otherwise

$$E_{ij}(\{I_{ij}\}) = \min_{\{\{I_{ki}\}|R_i=0\}} \left[\sum_{k \in L_i \backslash \{j\}} E_{ki}(\{I_{ki}\}) \right] + f(\| \{I_{il}\} \|)$$

- Messages reduced from 3^M to $2M+1$



Results



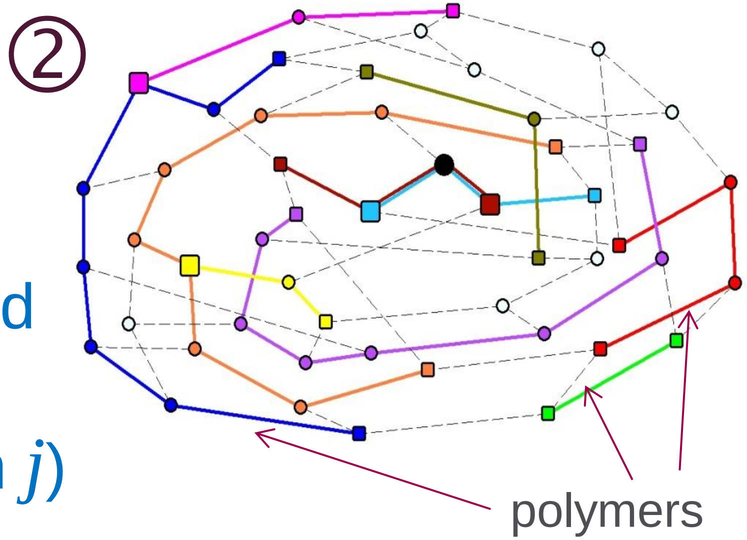
De Bacco, Franz, Saad, Yeung JSTAT P07009 (2014)

Do you see the light?

- Node disjoint routing is important for optical networks
- Task: accommodate more communications per wavelength
- Same wavelength communications cannot share an edge/vertex
- Approaches used: greedy algorithms, integer-linear programming…
- Greedy algorithms (such as breadth first-search) usually calculate shortest path and remove nodes from the network

General routing - analytical approach

- More complicated, cannot map to resource allocation



- Use model of interacting polymers
  - communication → polymer with fixed ends
  - $\sigma_j^\nu = 1$ (if polymer ν passes through j)
    $\sigma_j^\nu = 0$ (otherwise)
  - $I_j = \Sigma_\nu \, \sigma_j^\nu$ (no. of polymers passing through j)
  - minimize $H = \Sigma_j \, (I_j)^\gamma$, of any γ
- We use polymer method+ replica

Analytical approach

- Replica approach – averaging topology, start/end

$$\log Z = \lim_{n \to 0} \frac{Z^n - 1}{n}$$

- Polymer method– p-component spin such that $\langle S_a{}^2 \rangle = 1$ and $\sum_a S_a{}^2 = p$, when $p \to 0$,
- The expansion of

$$\left(\Pi_i \int d\mu(\boldsymbol{S}_i) \right) \Pi_{(kl)} (1 + A_{kl}\ \boldsymbol{S}_k \cdot \boldsymbol{S}_l)$$

results in $S_{ka} S_{la} S_{la} S_{ja} S_{ja} S_{ra} S_{ra} \ldots$ describing a self-avoiding loop/path between 2 ends

M. Daoud et al (and P. G. de Gennes) Macromolecules 8, 804 (1976)

Related works

- Polymer method+ replica approach was used to study travelling salesman problem (Difference: one path, no polymer interaction)
- Cavity approach was used to study interacting polymers (Diff: only neighboring interactions considered, here we consider overlapping interaction)
- Here: polymer + replica approach to solve a system of polymers with overlapping interaction

→recursion + message passing algorithms (for any γ)

M. Mezard, G. Parisi, J. Physique 47, 1284 (1986)
A. Montanari, M. Muller, M. Mezard, PRL 92, 185509 (2004)
E. Marinari, R. Monasson, JSTAT P09004 (2004); EM, RM, G. Semerjian. EPL 73, 8 (2006)

The algorithm

$$a^{\nu}_{j\to i} = \begin{cases} \min\limits_{l\in\mathcal{L}_j\setminus\{i\}} \left[a^{\nu}_{l\to j}\right] - \min\left[-\phi'(\lambda^{\nu *}_j) - \epsilon^{\nu}_j, \min\limits_{\substack{l,r\in\mathcal{L}_j\setminus\{i\}\\ l\neq r}} \left[a^{\nu}_{l\to j} + b^{\nu}_{r\to j}\right]\right], & \Lambda^{\nu}_j = 0 \\ -\min\limits_{l\in\mathcal{L}_j\setminus\{i\}} \left[b^{\nu}_{l\to j}\right], & \Lambda^{\nu}_j = 1 \\ \infty, & \Lambda^{\nu}_j = -1 \end{cases}$$

$$b^{\nu}_{j\to i} = \begin{cases} \min\limits_{l\in\mathcal{L}_j\setminus\{i\}} \left[b^{\nu}_{l\to j}\right] - \min\left[-\phi'(\lambda^{\nu *}_j) - \epsilon^{\nu}_j, \min\limits_{\substack{l,r\in\mathcal{L}_j\setminus\{i\}\\ l\neq r}} \left[a^{\nu}_{l\to j} + b^{\nu}_{r\to j}\right]\right], & \Lambda^{\nu}_j = 0 \\ \infty, & \Lambda^{\nu}_j = 1 \\ 0, & \Lambda^{\nu}_j = -1 \end{cases}$$

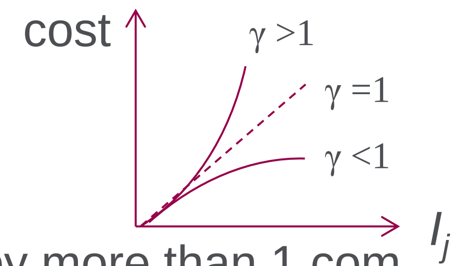
and $\lambda^{\nu *}_j$ is given by $\lambda^{\nu *}_j = g^{-1}_{\nu}(0)$, such that $g_{\nu}(x)$ is given by

$$g_{\nu}(x) = x - \frac{1}{M} - \frac{1}{M}\sum_{\mu\neq\nu}\left\{|\Lambda_{\mu}| + (1-|\Lambda_{\mu}|)\Theta\left(-\phi'(x) - \epsilon^{\mu}_j - \min\limits_{\substack{l,r\in\mathcal{L}_j\\ l\neq r}} \left[a^{\mu}_{l\to j} + b^{\mu}_{r\to j}\right]\right)\right\}$$

Extensions

- Edge cost
- Weighted edge costs
- Combination of edge/node costs
- Directed edges

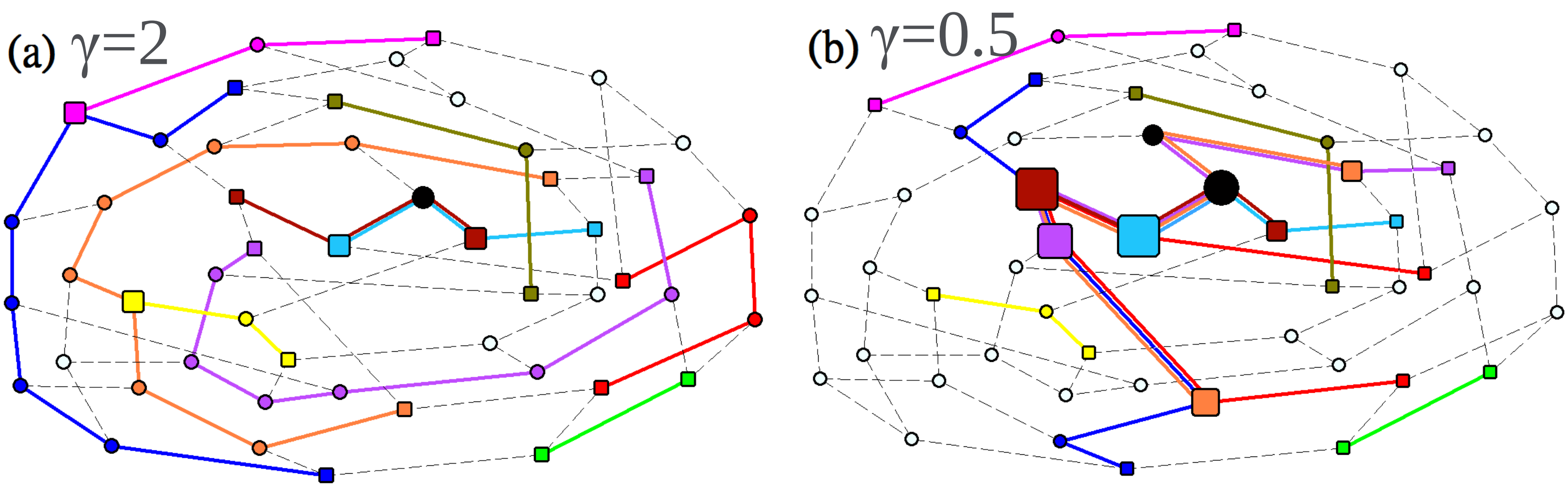
Results – Microscopic solution convex vs. concave cost



- source/destination of a communication
- shared by more than 1 com.

Size of node ∝ traffic

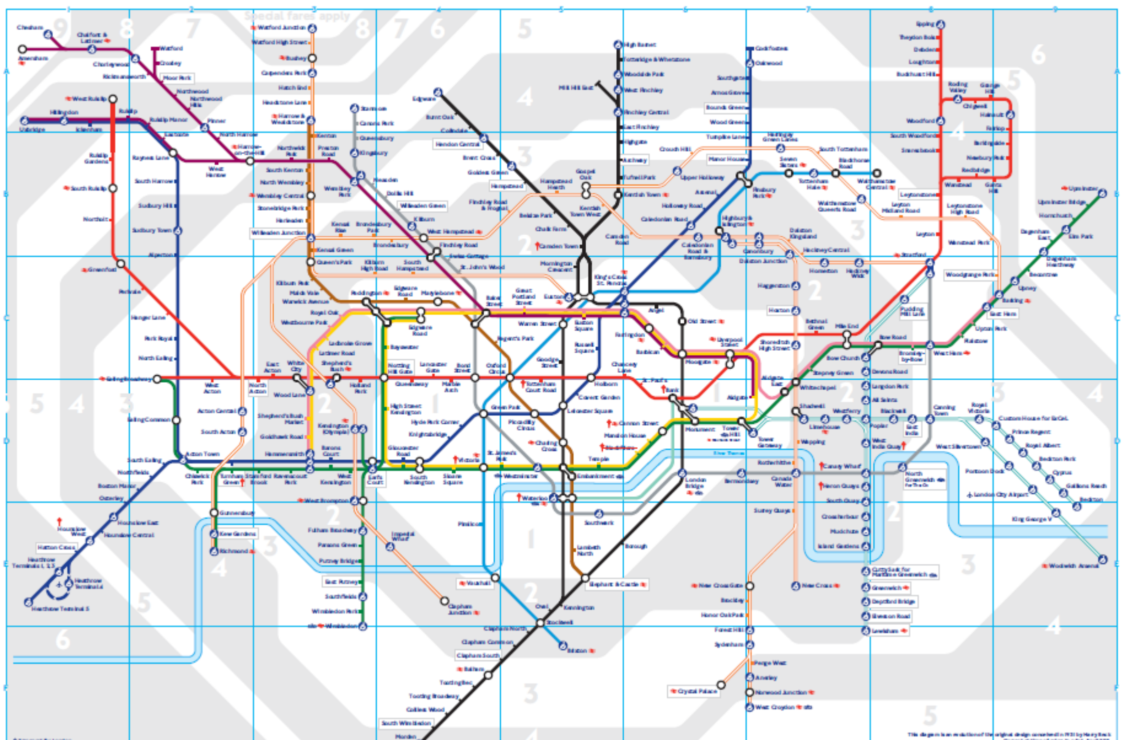
N=50, M=10



- γ >1 repulsion (between com.) → avoid congestion
- γ <1 attraction → aggregate traffic (to ↑ idle nodes) → to save energy

London subway network

- 275 stations
- Each polymer/communication – Oyster card recorded real passengers source/destination pair

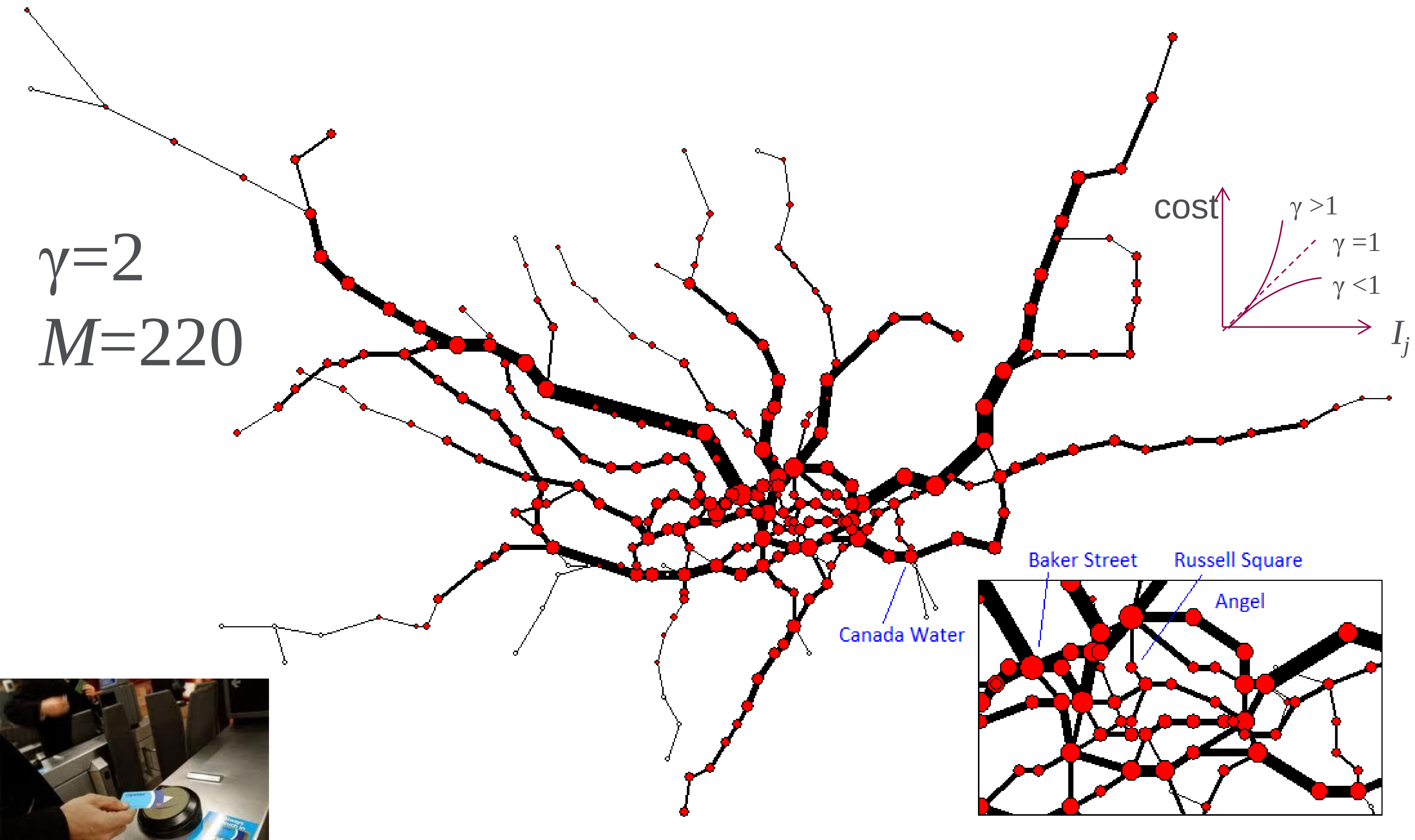


London tube map

Oyster card

Results – London subway with real source destination pairs recorded by Oyster card

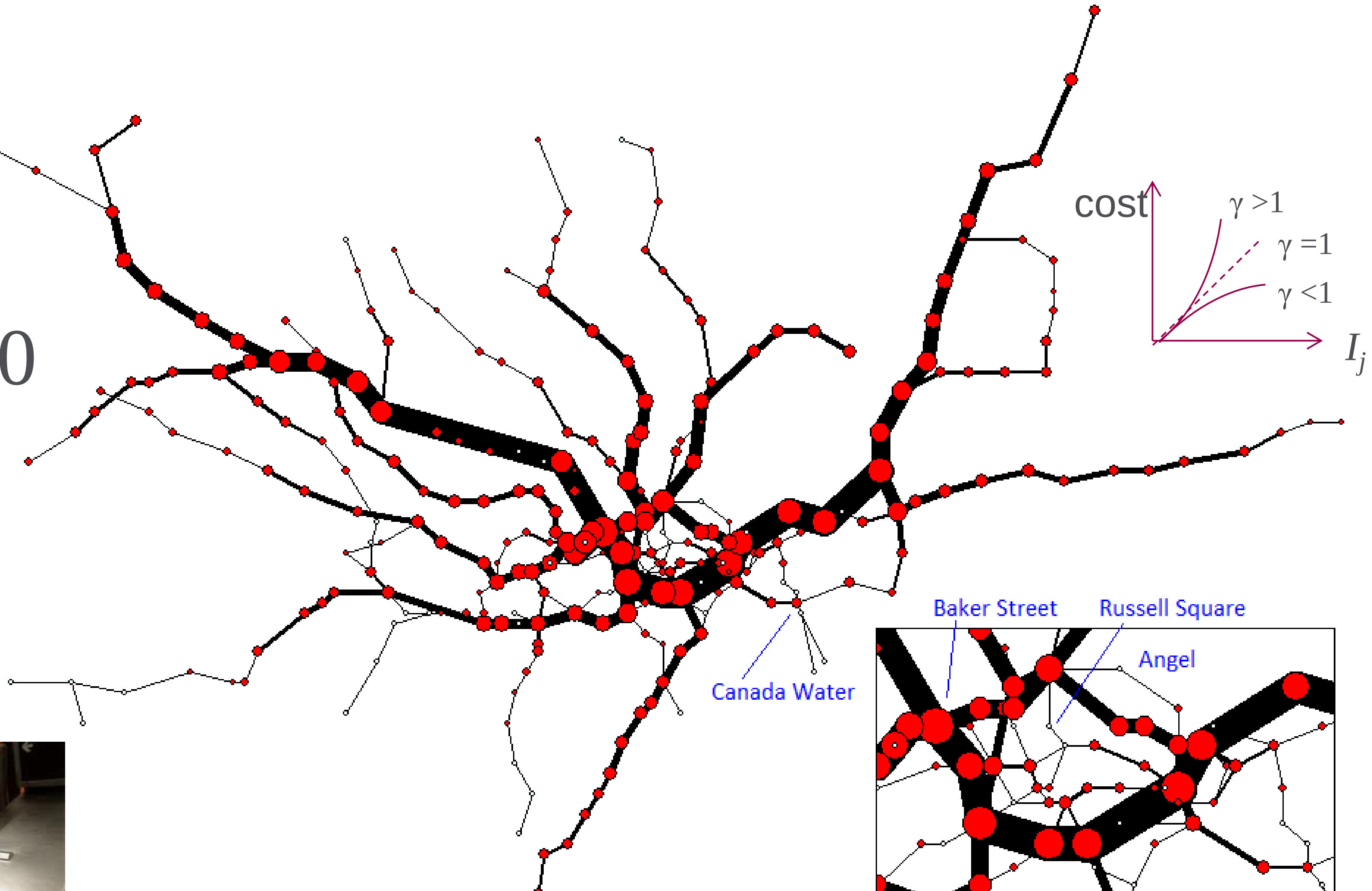
γ=2
M=220



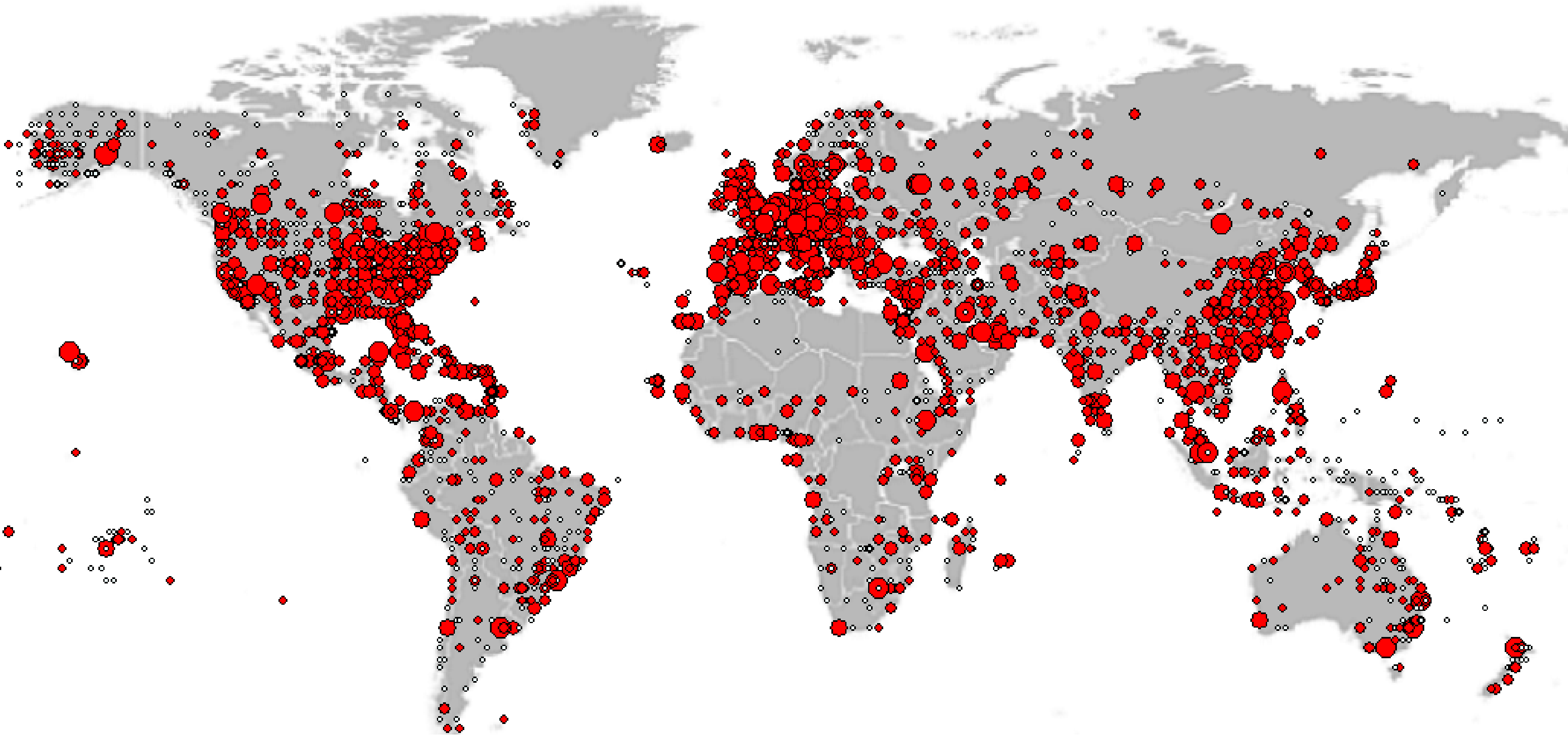
Results – London subway with real source destination pairs recorded by Oyster card

$\gamma=0.5$

$M=220$

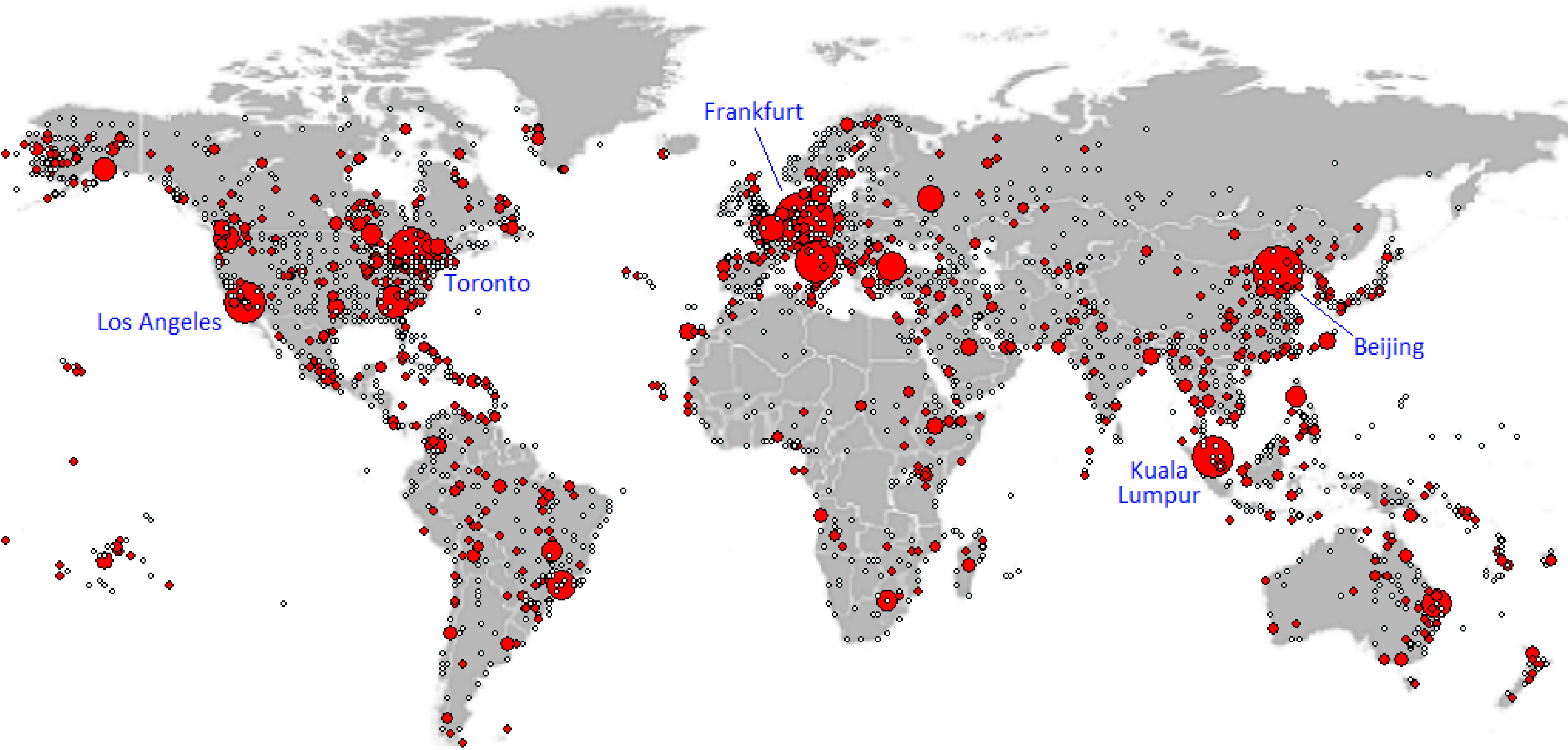


Results – Airport network



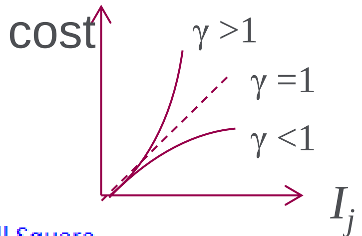
γ=2, M=300

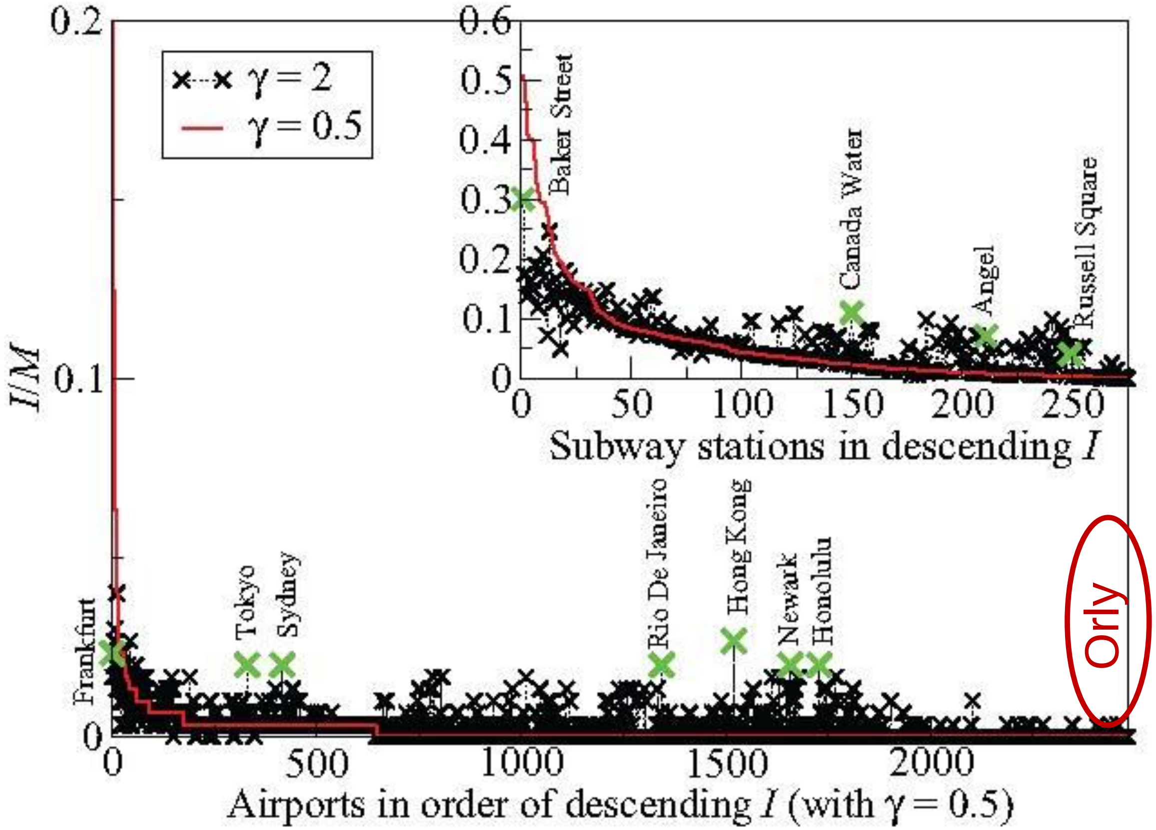
Results – Airport network

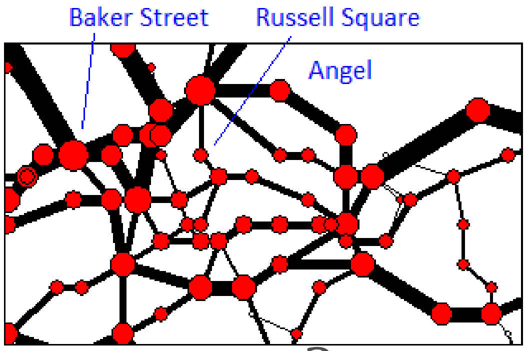


γ=0.5, M=300

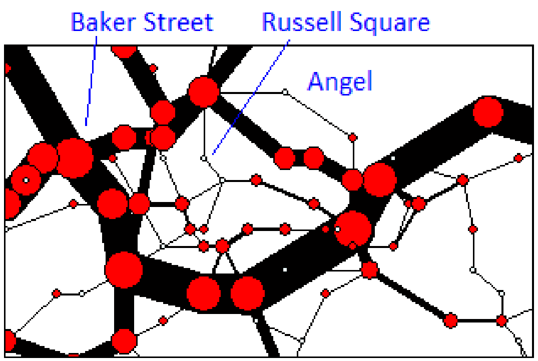
Results – comparison of traffic







γ=2

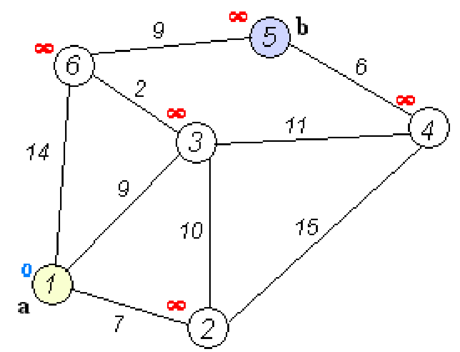


γ=0.5

- γ=2 vs γ=0.5
  - Overloaded station/airport has lower traffic
  - Underloaded station /airport has higher traffic

Comparison with Dijkstra algorithm

Comparison of energy *E* and path length *L* obtained by polymers-inspired (P) and Dijkstra (D) algorithms



	γ=2		γ=0.5	
	$\frac{E_P - E_D}{E_D}$	$\frac{L_P - L_D}{L_D}$	$\frac{E_P - E_D}{E_D}$	$\frac{L_P - L_D}{L_D}$
London subway	−20.5 ± 0.5%	+5.8 ± 0.1%	−4.0± 0.1%	+5.8 ± 0.3%
Global airport	−56.0 ± 2.0%	+6.2 ± 0.2%	−9.5 ± 0.2%	+8.6 ± 1.2%

and with a Multi-Commodity flow algorithm

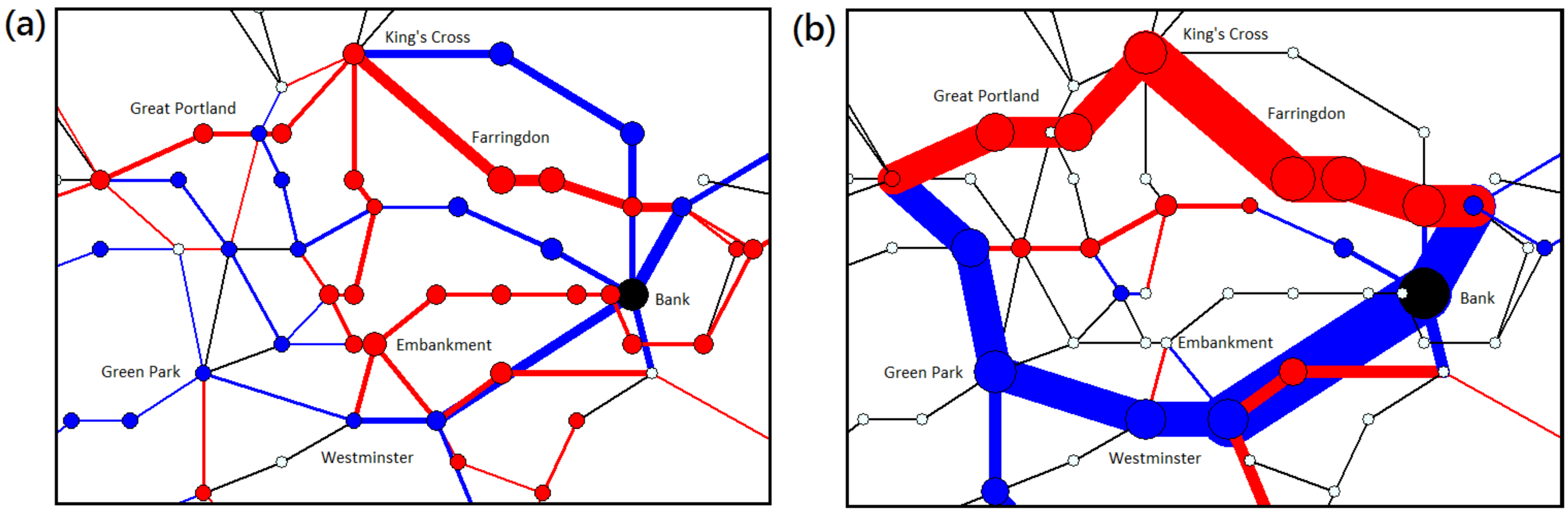
Comparison of energy *E* and path length *L* obtained by polymers-inspired (P) and Multi-Commodity flow (MC) algorithms (Awerbuch, Khandekar (2007) with optimal α)

Based on node-weighted shortest paths d_i using total current I_i; rerouting longest paths below edge capacity

$$d_i = \frac{e^{\alpha I_i}}{\sum_j e^{\alpha I_j}}$$

	γ=2		γ=0.5
	$\frac{E_P - E_{MC}(\alpha)}{E_{MC}(\alpha)}$	$\frac{L_P - L_{MC}(\alpha)}{L_{MC}(\alpha)}$	No algorithm identified for comparison
London subway	−0.7 ± 0.04%	+0.72 ± 0.10%	
Global airport	−3.9 ± 0.59%	+0.90 ± 0.64%	

Results - Change of Optimal Traffic & Adaptation to Topology Change



$\gamma=2$ $\gamma=0.5$

- After the removal of station "Bank" ( ) …

\- Size of node, thickness of edges $\propto$ traffic

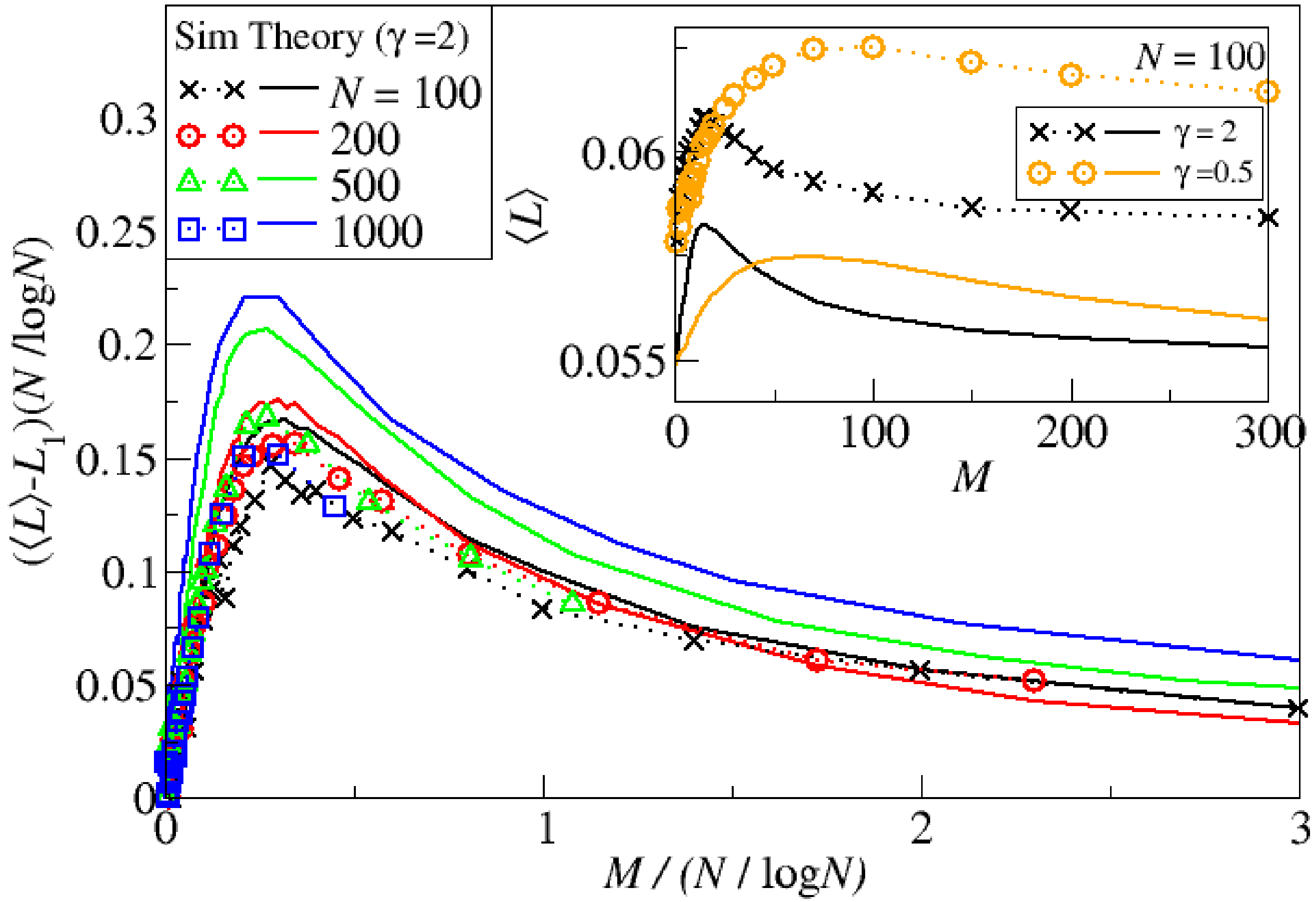
- traffic ↑ - traffic ↓ - no change

- $\gamma=2$ has smaller, yet more extensive, changes on individual nodes and edges

Macroscopic behavior

- Data collapse of $\langle L \rangle$ vs M for different N

- $\log N \propto$ typical distance
- $M \log N / N \propto$ average traffic per node



Conclusion

- We employed statistical physics of disordered system to study routing problems
- **Microscopically**, we derive a traffic-sensitive optimization algorithms
- **Macroscopically**, we observe interesting phenomena: non-monotonic path length, balanced receiver, different routing patterns, phase transitions
- **Extensions**: Best-response, Nash equilibrium, time
- **Applications**: routing in communication networks, transportation networks (traffic), optical networks

[1] C. H. Yeung and D. Saad, PRL 108, 208701 (2012)
[2] C. H. Yeung, D. Saad, K.Y.M. Wong, PNAS 110, 13717 (2013)
[3] C. De Bacco, S. Franz, D. Saad, C.H.Yeung JSTAT P07009 (2014)