Spectral Properties of the Quantum Random Energy Model

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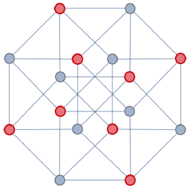


Hamming cube: $\mathcal{Q}_N := \{-1,1\}^N$

- configuration space of N spins

Laplacian on $\mathcal{Q}_N$: $(-\Delta\psi)(\sigma) := N\psi(\sigma) - \sum_{j=1}^N \psi(F_j\sigma)$

- Spin flip: $F_j\sigma = (\sigma_1, \dots, -\sigma_j, \dots, \sigma_N)$

  Hence the Laplacian acts as a **transversal magnetic field**: $-\Delta = N - \sum_{j=1}^N \sigma_j^x$

- Eigenvalues: $2|A|, \quad A \subset \{1,\dots,N\}$ Degeneracies: $\binom{N}{|A|}$

  Normalized Eigenvectors: $f_A(\sigma) = \frac{1}{\sqrt{2}^N} \prod_{j\in A} \sigma_j$

Perturbation by a multiplication operator U:

$$\boxed{H = -\Delta + \kappa\, U}$$

- $U = U(\sigma_1^z, \dots, \sigma_j^z)$; Coupling constant $\kappa \geq 0$; $\|U\|_\infty \approx \mathcal{O}(N)$
- **In this talk:** $\boxed{U(\sigma) = \sqrt{N}\, g(\sigma)}$ with $\{g(\sigma)\}_{\sigma\in Q_N}$ i.i.d. standard Gaussian r.v. **REM**

Some motivations and related questions

1. Adiabatic Quantum Optimization: Farhi/Goldstone/Gutmann/Snipser '01, ...

Question: Find minimum in a complex energy landscape $U(\sigma)$

e.g. REM, Exact Cover 3, ...

Idea: Evolve the ground state through adiabatic quantum evolution, i.e. $i\,\partial_t \psi_t = H(t/\tau)\,\psi_t$ generated by

$$\boxed{H(s) := (1-s)(-\Delta) + s\,U\,, \qquad s \in [0,1]}$$

Required time : $\tau \approx c\,\Delta_{min}^{-2}$

2. Mean field model for localization transition in disordered N particle systems

Altshuler '06

3. Evolutionary Genetics: Rugged fitness landscape for quasispecies ...

Schuster/Eigner '77, Baake/Wagner '01, ...

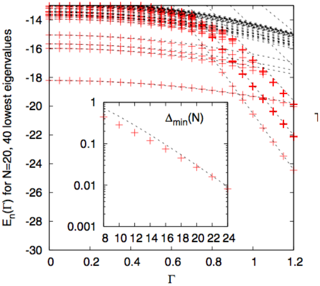
Predicted properties for QREM

Predicted low-energy spectrum:

$\widehat{H} = \Gamma\,(-\Delta - N) + U/\sqrt{2}, \quad$ i.e. $\kappa = (\sqrt{2}\,\Gamma)^{-1}$

Jörg/Krzakala/Kurchan/Maggs '08

Presilla/Ostilli '10, ...



First order phase transition of the ground state at $\kappa_c = \frac{1}{\sqrt{2\ln 2}}$:

$\kappa < \kappa_c$: Extended ground state with non-random ground-state energy

$$E_0 = -\kappa^2 + o(1)$$

$\kappa > \kappa_c$: Low lying eigenstates are concentrated on lowest values of U.

In particular: $E_0 = N + \kappa \min U + \mathcal{O}(1)$

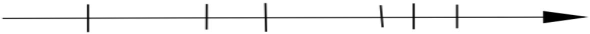
$\kappa = \kappa_c$: Energy gap $\Delta_{min} = E_1 - E_0$ vanishes exponentially in N

Known properties of the **REM**: $U(\sigma) = \sqrt{N}\, g(\sigma)$

- Location of the minimum:

$$U_0 := \min U = -\kappa_c^{-1}\, N + \mathcal{O}(\ln N)$$

- The extreme values $U_0 \leq U_1 \leq \ldots$ form a **Poisson process** about $-\kappa_c^{-1}\, N + \mathcal{O}(\ln N)$ of exponentially increasing intensity.



Perturbation theory:

- Fate of localized states: $\langle \delta_\sigma , H \delta_\sigma \rangle = N + \kappa\, U(\sigma)$.
- Fate of delocalized states: $\langle f_A , U\, f_A \rangle = \frac{1}{2^N} \sum_\sigma U(\sigma) = \mathcal{O}(\sqrt{N}\, 2^{-N/2})$.

2. Low-energy regime of the QREM in case $\kappa < \kappa_c$

Theorem (Case $\kappa < \kappa_c$)

For any $\varepsilon > 0$ and except for events of exponentially small probability, the eigenvalues of H below $\left(1 - \frac{\kappa}{\kappa_c} - \varepsilon\right) N$ are within balls centred at

$$\boxed{2n - \frac{\kappa^2}{1 - \frac{2n}{N}}, \qquad n \in \{0, 1, \dots\}},$$

of radius $\mathcal{O}\left(N^{-\frac{1}{2}+\delta}\right)$ with $\delta > 0$ arbitrary.

There are exactly $\binom{N}{n}$ eigenvalues in each ball and their eigenfunctions are delocalized:

$$\boxed{\|\psi_E\|_\infty^2 \leq 2^{-N} e^{\Gamma\left(\frac{x_E}{2}\right)N}}$$

where $\Gamma(x) := -x \ln x - (1-x)\ln(1-x)$ and $x_E := \frac{E}{N} + \frac{\kappa}{\kappa_c} + \varepsilon$.

Sketch of the proof – delocalization regime

Step 1: *Hypercontractivity of the Laplacian*

$$|\psi_E(\sigma)|^2 \le \langle \delta_\sigma \, , P_{(-\infty,E]}(H)\, \delta_\sigma \rangle \;=\; \inf_{t>0} \, e^{tE} \langle \delta_\sigma \, , e^{-tH} \, \delta_\sigma \rangle$$
$$= \inf_{t>0} \, e^{t(E-\kappa U_0)} \langle \delta_\sigma \, , e^{t\Delta} \, \delta_\sigma \rangle \;=\; 2^{-N} \, e^{\Gamma\left(\frac{x_E}{2}\right) N} \, .$$

Step 2: *Reduction of fluctuations*

Projection on centre of band and its complement:

$$Q_\varepsilon := 1 - P_\varepsilon \;:=\; 1_{[N(1-\varepsilon), N(1+\varepsilon)]}(-\Delta) \, .$$

Note: $\dim P_\varepsilon \le 2^N \, e^{-\varepsilon^2 N/2}$ – take $\varepsilon = \mathcal{O}\left(N^{-\frac{1}{2}+\delta}\right)$.

Lemma

There exist constants $C, c < \infty$ such that for any $\varepsilon > 0$ and any $\lambda > 0$:

$$\mathbb{P}\left(\left| \|P_\varepsilon U P_\varepsilon\| - \mathbb{E}\left[\|P_\varepsilon U P_\varepsilon\|\right] \right| > \lambda \sqrt{\frac{\dim P_\varepsilon}{2^N}} \right) \leq C\, e^{-c\lambda^2}$$

$$\mathbb{E}\left[\|P_\varepsilon U P_\varepsilon\|\right] \leq C\, N \sqrt{\frac{\dim P_\varepsilon}{2^N}} = C\, N\, e^{-\varepsilon^2 N/4}.$$

1. Concentration of measure using Talagrand inequality:

   Lipschitz continuity of $F : \mathbb{R}^{\mathcal{Q}_N} \to \mathbb{R}$, $\quad F(U) := \|P_\varepsilon U P_\varepsilon\|$:

$$F(U) - F(U') \leq \langle \psi, U\psi \rangle - \langle \psi, U'\psi \rangle$$
$$\leq \|U - U'\|_2 \|\psi\|_\infty \leq \|U - U'\|_2 \sqrt{\frac{\dim P_\varepsilon}{2^N}}.$$

2. Moment method to estimate $\mathbb{E}\left[\|P_\varepsilon U P_\varepsilon\|\right] \leq \left(\mathbb{E}\left[\mathrm{Tr}(P_\varepsilon U P_\varepsilon)^{2N}\right]\right)^{1/2N} \ldots$

Step 3: *Schur complement formula*

$$P_\varepsilon (H - z)^{-1} P_\varepsilon \;=\; \left(P_\varepsilon H P_\varepsilon - z - \kappa^2 P_\varepsilon U Q_\varepsilon (Q_\varepsilon H Q_\varepsilon - z)^{-1} Q_\varepsilon U P_\varepsilon\right)^{-1}$$

. . . and using Step 2:

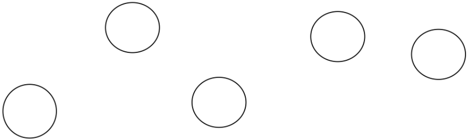
$$P_\varepsilon U Q_\varepsilon (Q_\varepsilon H Q_\varepsilon - z)^{-1} Q_\varepsilon U P_\varepsilon \approx \frac{N}{N - z} P_\varepsilon + \mathcal{O}\left(N^{-\frac{1}{2}+\delta}\right) .$$

Low-energy regime of the QREM

Main idea: *Geometric decomposition*

For energies below $E_\delta := \left(1 - \frac{\kappa}{\kappa_c} + \delta\right) N$ the localized eigenstates originate in **large negative deviation sites**:

$$\boxed{X_\delta := \left\{ \sigma \,\middle|\, \kappa\, U(\sigma) < -\frac{\kappa}{\kappa_c} N + \delta N \right\}}$$



For $\delta > 0$ small enough and except for events of exponentially small probability (e.e.p.):

- X_δ consists of isolated points which are separated by a distance greater than $2\gamma N$ with some $\gamma > 0$.
- On balls $B_{\gamma,\sigma} := \{\sigma' \mid \text{dist}(\sigma, \sigma') < \gamma N\}$ the potential is larger than $-\epsilon N$ aside from at σ.

Low-energy regime of the QREM

Theorem

E.e.p. and for $\delta > 0$ sufficiently small, there is some $\gamma > 0$ such that all eigenvalues of H below $E_\delta = \left(1 - \frac{\kappa}{\kappa_c} + \delta\right) N$ coincide up to an exponentially small error with those of

$$\widehat{H}_\delta := H_R \oplus \bigoplus_{\sigma \in X_\delta} H_{B_{\gamma,\sigma}} .$$

where $R := Q_N \setminus \bigcup_{\sigma \in X_\delta} B_{\gamma,\sigma}$.

- Low energy spectrum of H_R looks like H in the delocalisation regime
- Low energy spectrum of $H_{B_{\gamma,\sigma}}$ is explicit consisting of exactly one eigenstate below E_δ ...

Known properties of Laplacian on $B_{\gamma,\sigma}$:

$$\boxed{E_0(-\Delta_{B_{\gamma,\sigma}}) = N(1 - 2\sqrt{\gamma(1-\gamma)}) + o(N)}$$

Adding a large negative potential κU at σ and some more moderate background elsewhere, rank-one analysis yields:

- $$\boxed{E_0(H_{B_{\gamma,\sigma}}) \;=\; N + \kappa U(\sigma) - s_\gamma(N + \kappa U(\sigma)) + \mathcal{O}(N^{-1/2})}$$

  where s_γ is the self-energy of the Laplacian on a ball of radius γN.
- for the corresponding normalised ground state:

  $$\sum_{\sigma' \in \partial B_{\gamma,\sigma}} \left|\psi_0(\sigma')\right|^2 \;\leq\; e^{-L_\gamma N} \qquad \text{for some } L_\gamma > 0.$$

  $$\left|\psi_0(\sigma)\right|^2 \geq 1 - \mathcal{O}(N^{-1})$$
- $H_{B_{\gamma,\sigma}}$ has a spectral gap of $\mathcal{O}(N)$ above the ground state.

3. Comment on adiabatic quantum optimization

Study $\quad i\,\partial_t \psi_t = H(t/\tau)\,\psi_t$ with $\psi_0(\sigma) = 1/\sqrt{2^N}$ and

$$\boxed{H(s) := (1-s)(-\Delta) + s\,\kappa\,U\,, \qquad s \in [0,1]\,.}$$

1. Farhi/Goldstone/Gutmann/Negaj '06

   Let $\sigma_0 \in \mathcal{Q}_N$ be minimizing configuration for $\{U(\sigma)\}$ and

   $$|\langle \psi_\tau\,, \delta_{\sigma_0}\rangle|^2 \;\geq\; b\,.$$

   Then $$\tau \geq \frac{2^N\,b - 2\sqrt{2^N}}{4\sqrt{\sum_\sigma (U(\sigma) - U(\sigma_0))^2}} \;\approx\; \mathcal{O}(2^{N/2})\,.$$

2. Adiabatic theorem of Jansen/Ruskai/Seiler '07 as used in Farhi/Goldstone/Gosset/Gutmann/Shor '10 yields:

   Typically, the minimum ground-state gap of $H(s)$ along the path $s \in [0,1]$ is exponentially small in N.

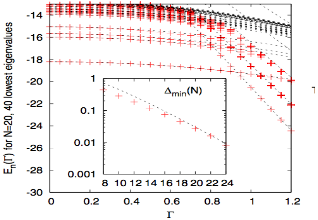
4. Conclusion:

1. Complete description of the **low-energy** spectrum of the QREM

   ...and generalisations to non-gaussian r.v.'s

   Ground-state phase transition
   at $\kappa = \kappa_c$ with
   an exponentially closing gap.

   Jörg/Krzakala/Kurchan/Maggs '08



2. **Open problem:** **Resonant delocalisation conjecture** in QREM with eigenfunctions possibly violating ergodicity are expected to occur closer to centre of band within renormalised gaps of Laplacian.

   Laumann/Pal/Scardicchio '14

Jansen/Ruskai/Seiler '07

Theorem

Let $H(s)$, $s \in [0,1]$, be a twice differentiable family of self-adjoint operators with non-degenerate ground-state eigenvectors ϕ_s and ground-state gaps $\gamma(s)$. Then the solution of

$$i\,\partial_t \psi_t = H(t/\tau)\,\psi_t\,, \qquad \psi_0 = \phi_0\,,$$

satisfies:

$$\sqrt{1 - |\langle \psi_\tau, \phi_1 \rangle|^2} \le \frac{1}{\tau}\left[\frac{1}{\gamma(0)^2}\|H'(0)\| + \frac{1}{\gamma(1)^2}\|H'(1)\| + \int_0^1 \frac{7}{\gamma(s)^3}\|H'(s)\| + \frac{1}{\gamma(s)^2}\|H''(s)\|ds\right].$$