Hard Spheres: the Gardner transition a the fullRSB solution

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- The simplest RFOT scheme
- Microscopic description of marginal stability: statistics of amorphous sphere packings.
- Infinite dimensional Hard Spheres. First steps.
- The 1RSB solution, phase diagram and inconsistencies.
- Stability of the 1RSB solution. The Gardner transition in the replicated theory.
- The 2RSB solution: a new approximate landscape.
- The fullRSB solution: analytical prediction for the critical exponents at jamming.
- The fullRSB marginal stability and the microscopic marginal stability.

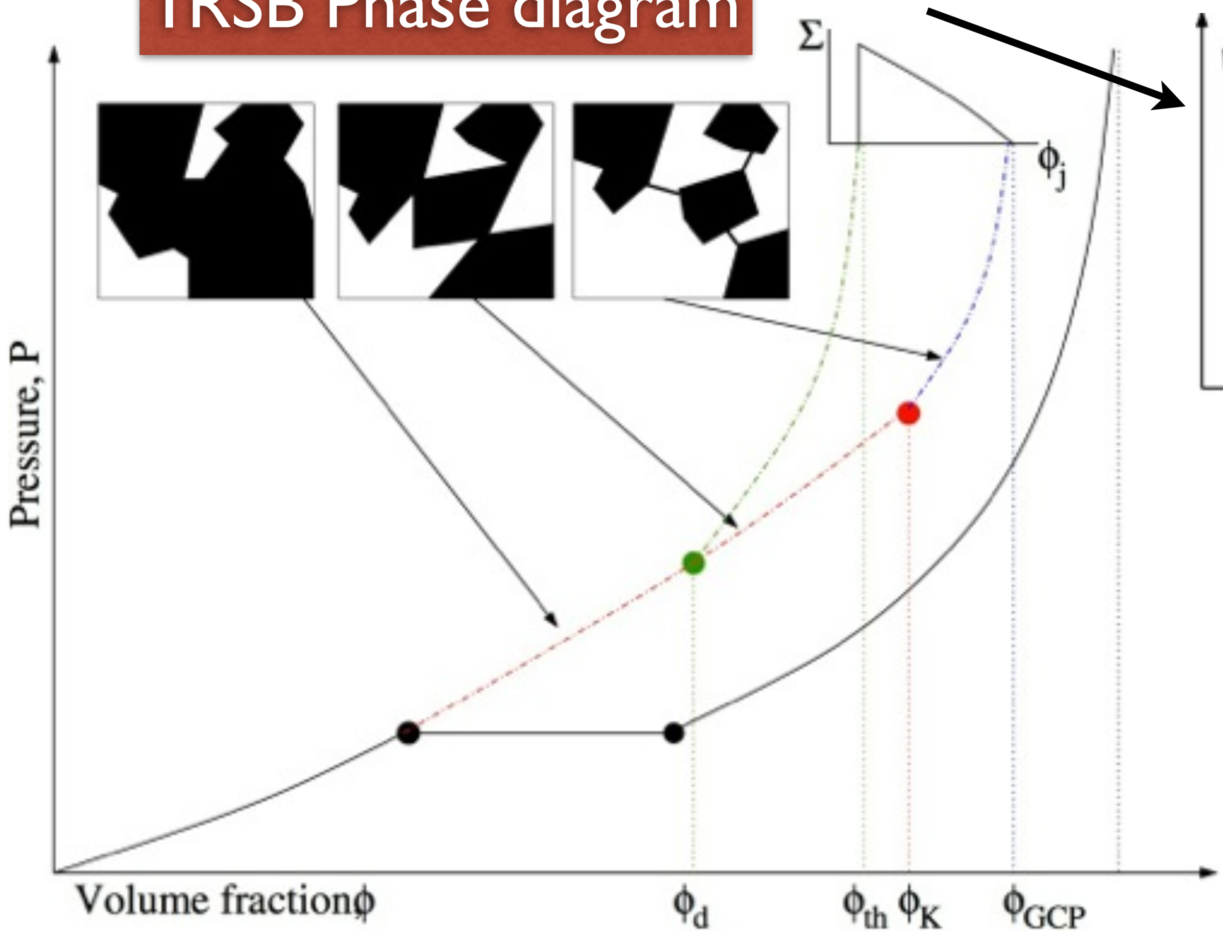
Idea

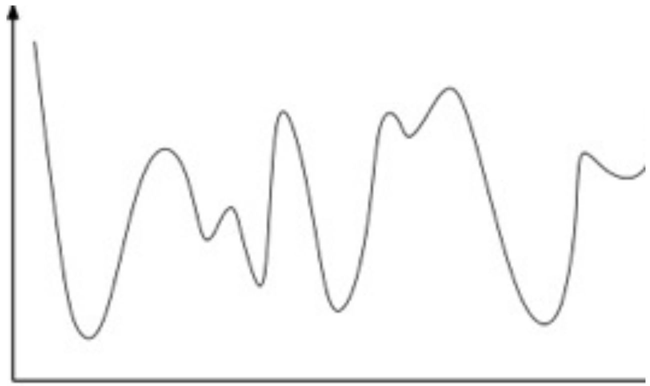
Think about jammed packings of hard spheres as the infinite pressure limit of metastable glassy states.

Apply equilibrium statistical mechanics tools developed to study glassy systems

Mézard-Paris
Parisi-Zampo

1RSB Phase diagram

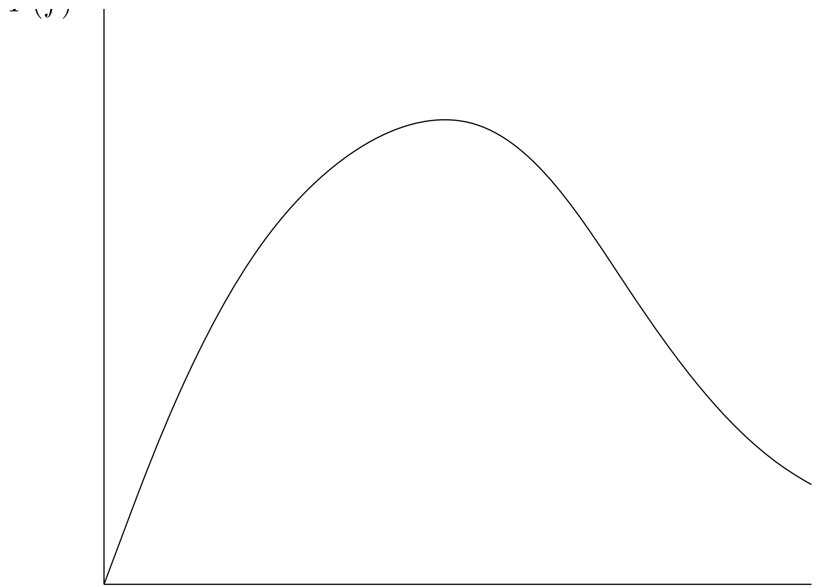




Mean field phase diagr

Up to now the theory
developed on the basis
hypothesis for the stru
metastables stat

The distribution of forces between spheres in contact displays a power law behavior at small forces



$$P(f) \sim f^{\theta}$$

Analogous to the pseudo-gap in the distribution of frozen fields in the Sherrington-Kirkpatrick model.

Wyart, Nagel, Muller

The $g(r)$ shows a power law divergence at the contact value of

$$g(r) \sim (r-1)^{-\gamma}$$

Given this and the fact that *the pack* *marginally stable*, a wonderful scaling of mean field critical exponents at j can be derived (Wyart)

- We can hope to do practical calculations in order to give first theoretical estimates for measured physical quantities (compute exponents!)
- Starting point of a *systematic* 1/d expansion
- The properties of hard spheres at jamming seem to depend slightly on the
  (It is not the same for the glass transition where the debate has not been solved ye

Charbonneau, Corwin, Paris

In the infinite dimensional limit the virial expansion takes a simple

$$S[\rho(x)] = \int \mathrm{d}x \rho(x) \left(1 - \log \rho(x)\right) + \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \rho(x) \rho(y) f(x,y) \qquad f(x) = -$$

Mayer

$\overline{x} = \{x_1, \ldots, x_m\}$

Monasson, I

The entropy is a function of the molecule density. $x_a = X + u_a$

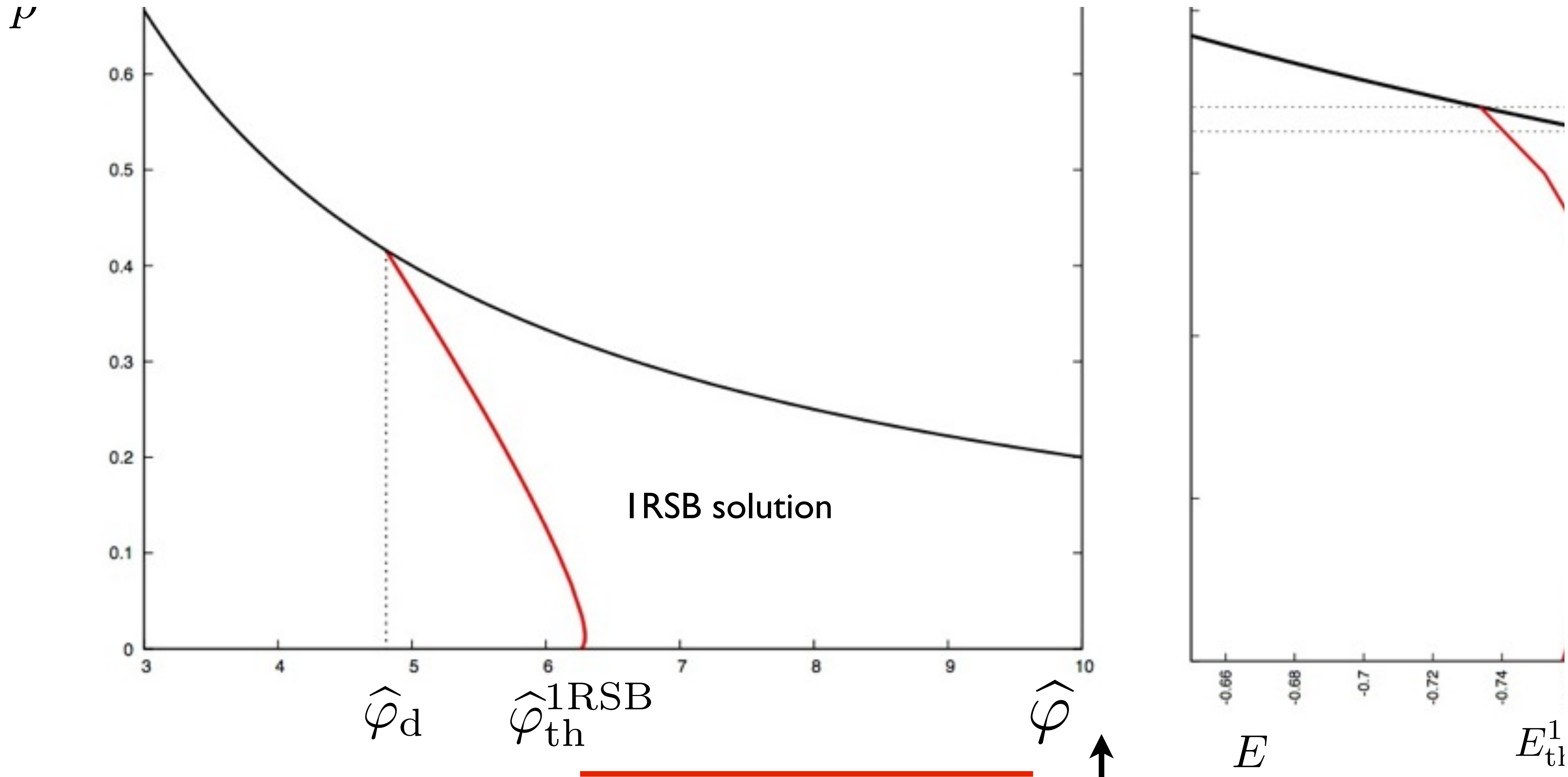
Translational and rotational invariance imply $\rho(\overline{x}) = \rho(q_{ab})$

Write the saddle point equations to obtain the molecule density a
them to obtain the entropy.

This has been done by Kurchan, Parisi and Zamponi (2012)

A wonderful result is that if we take a Gaussian ansatz for th
molecule density we can obtain the correct entropy.

Gaussian ansatz $\Rightarrow$ Big simplification



packing fraction

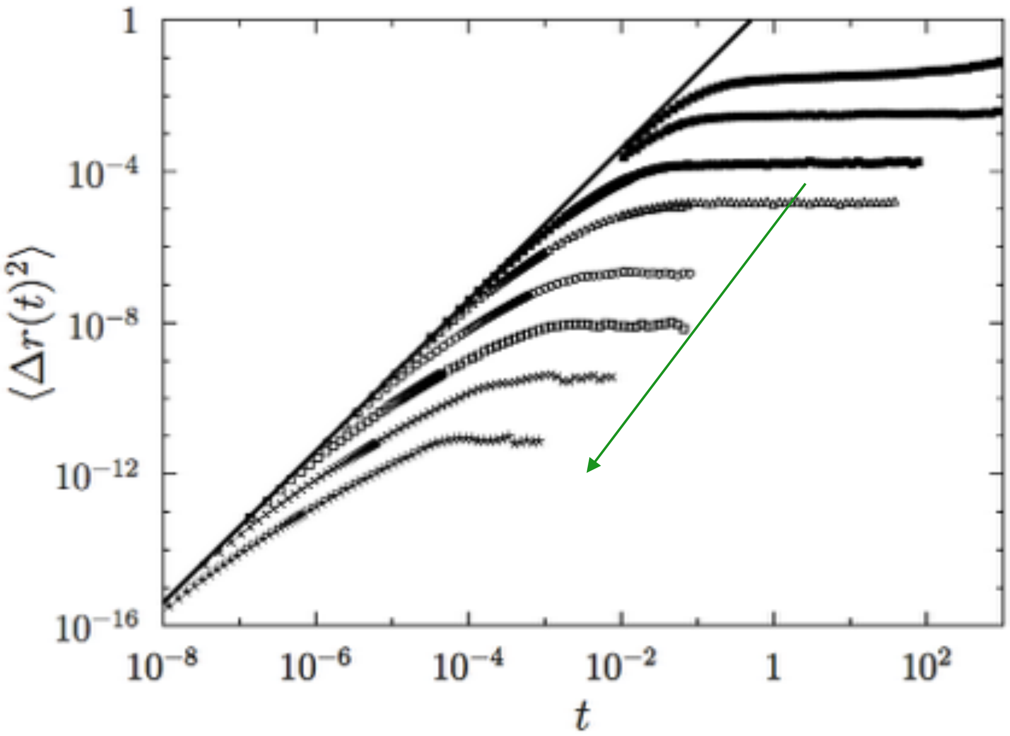
Parisi Zamponi (05-10)

J-line: jammed states

1. The jammed packings are expected to be isostatic.

The 1RSB ansatz gives hyperstatic packings

2. Scaling of the Debye-Waller factor with the pressure



From 1RSB: $\Delta_{EA} \sim$

Instead it was observed that

$$\Delta_{EA} \sim p^{-\kappa} \qquad \kappa \sim$$

3. The distribution of forces show a power law behavior

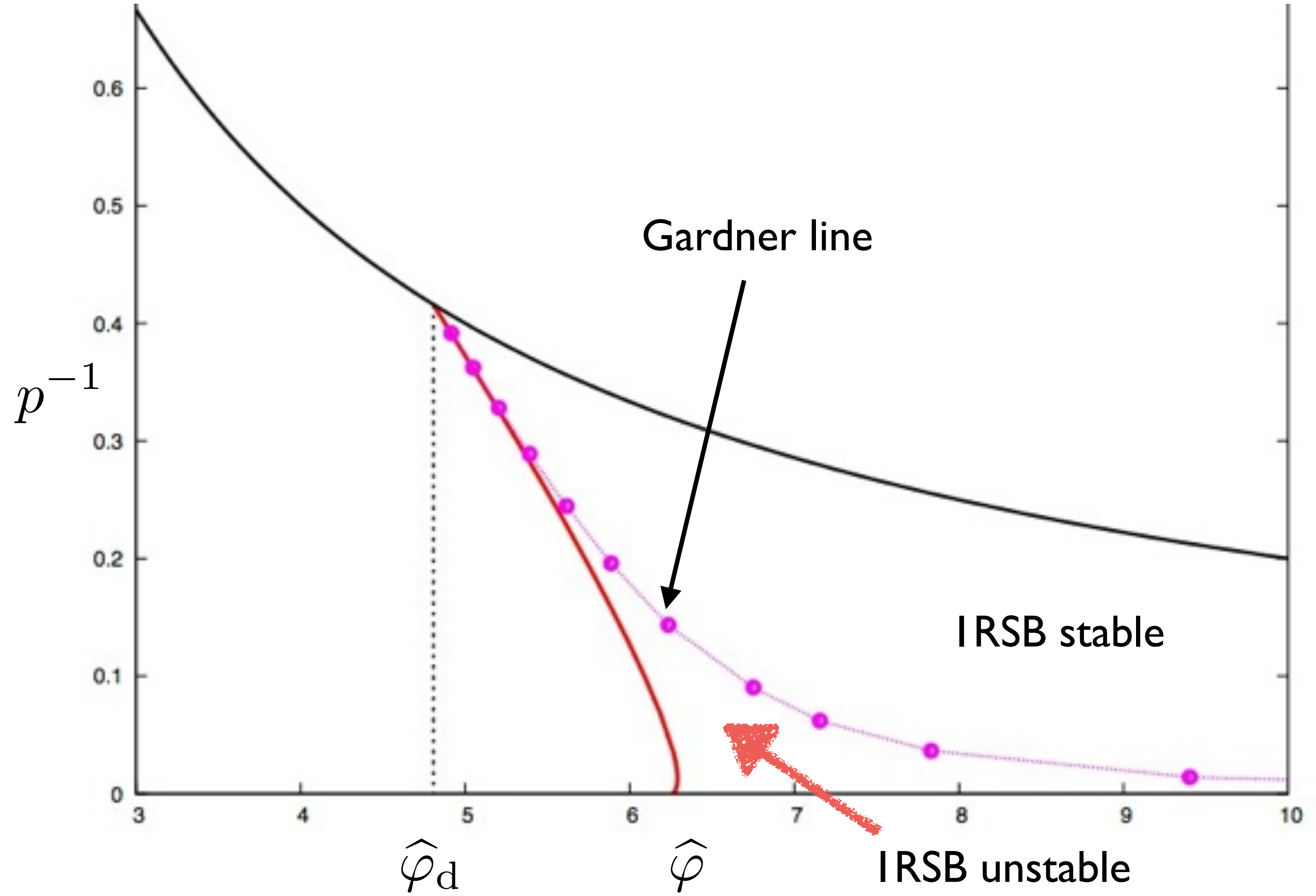
Entropic term

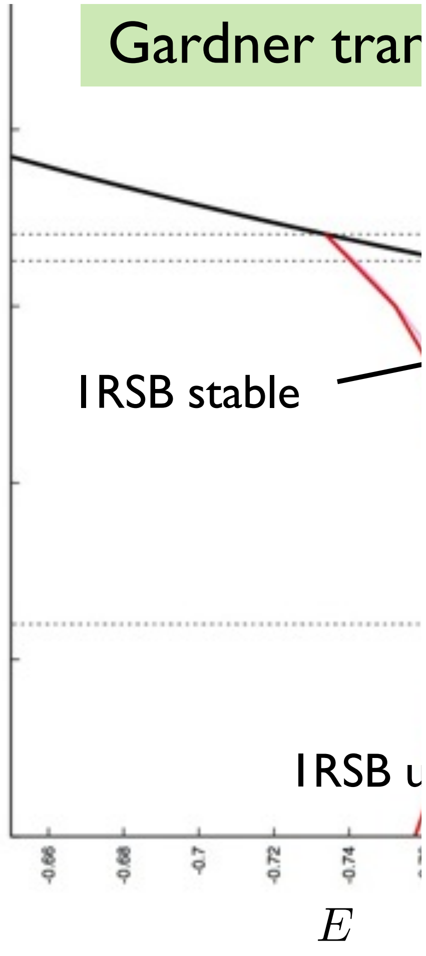
Interaction term

where $\mathcal{F}[\hat{v}] = \lim_{n \to 0} \sum_{n_1,\ldots,n_m : \sum_a n_a = n} \frac{n!}{n_1! \ldots n_m!} \exp \left[-\frac{1}{2} \sum_{a=1}^{m} v_{aa} \frac{n_a}{n} + \frac{1}{2} \sum_{a,b=1}^{m} v_{\ldots}\right.$

- Take the 1RSB ansatz $\alpha_{ab}^{\mathrm{1RSB}} = \widehat{\alpha}_1 (m\delta_{ab} - 1)$
- Plug it into the expression for the entropy. Find 1RSB saddle point equa[...]
  Obtain the solution. Study its stability.
- Obtain a stability matrix $M_{ab;cd}$ (Simple replica structure)
- Study its eigenvalues. In particular: *replicon eigenvalue*
- Expand at the third order to obtain the cubic terms and the value of the
  exponent parameter (MCT dynamical exponents)

$$\lambda = \frac{w_2}{w_1}$$ (Original motivation for the work)



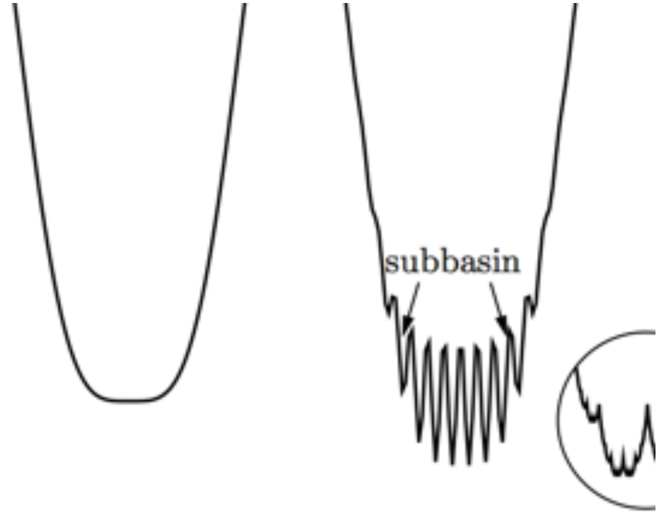


The instability affects the Jamming part of the phase

We hope to cure IRSB inconsistencies going beyond the IRSB

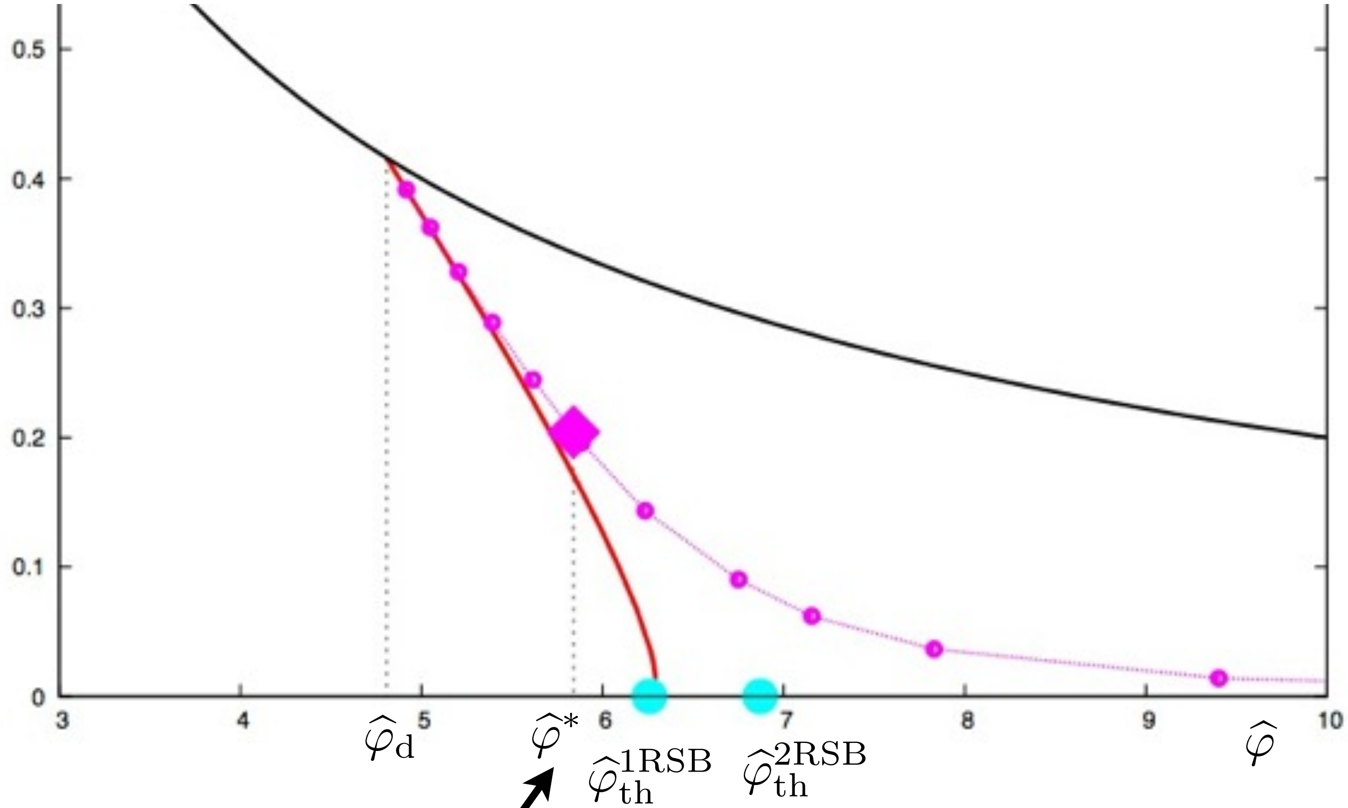
Kurchan Parisi U

- Gardner (85) showed that the equilibrium states undergo a fullRSB transition at very low temperature (below the Kauzmann point)
- Montanari and Ricci-Tersenghi (03) however showed that the Gardner phase affects strongly off equilibrium states. It may be relevant for off-equilibrium dynamics!
- Renewal interest from Krzakala and Zdeborova (13) who were able to follow adiabatically glassy states showing their instability.
- Rizzo(13): fullRSB solution.



Implicatior

- If in the Gardner pha
fullRSB solution we h
modes
- The fullRSB solution
stable. Could the full
stability reproduce th
microscopic marginal
jammed packings?

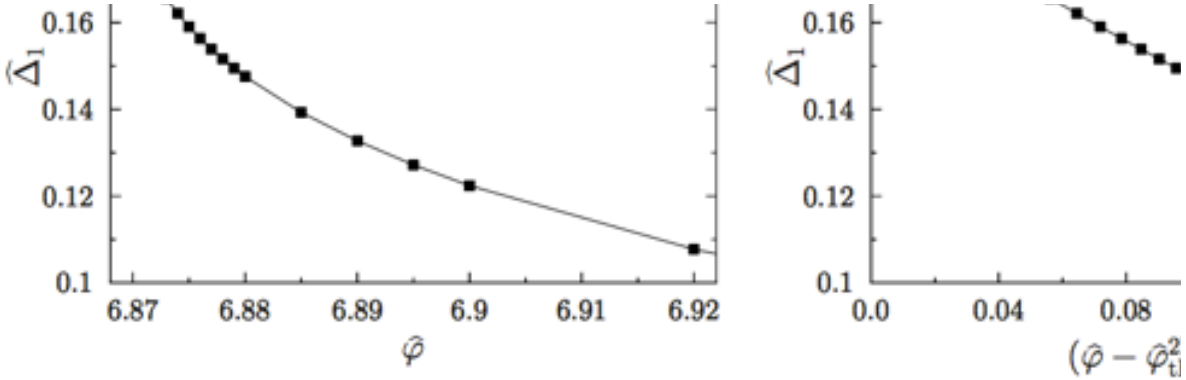


$$\boxed{\lambda = \frac{w_2}{w_1} > m}$$

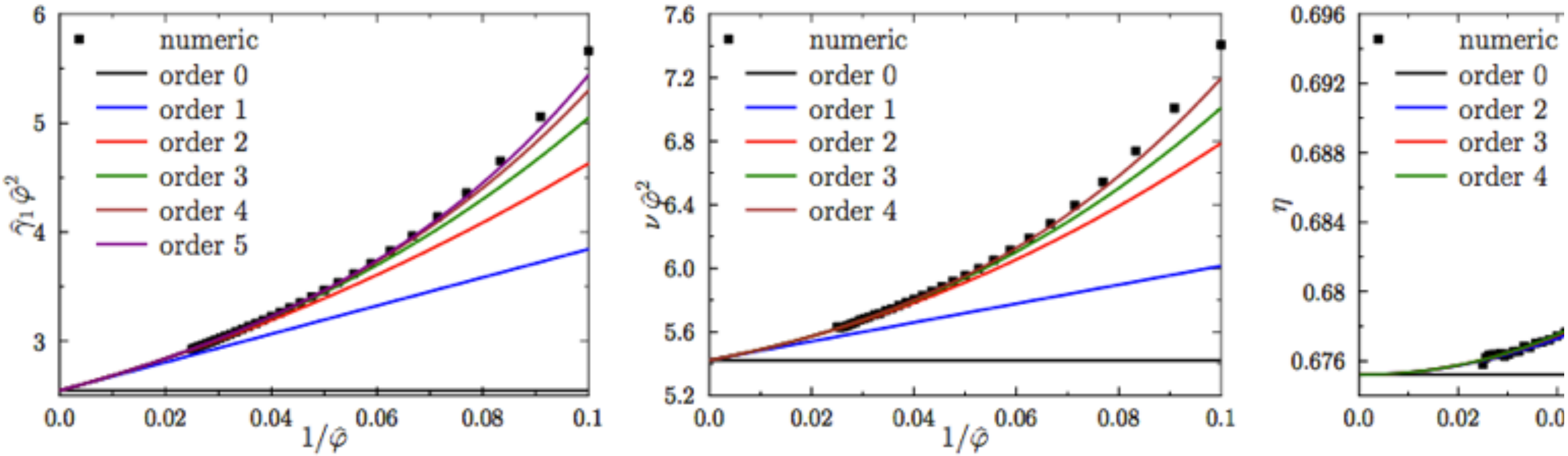
Rizzo (13) in the
Ising p-spin m

Doing a perturbative 2RSB calculation just below the instability line shows that a 2RSB solution (and a fullRSB one) can be found only above a certain packing fraction

Behavior at the 2RSB threshold



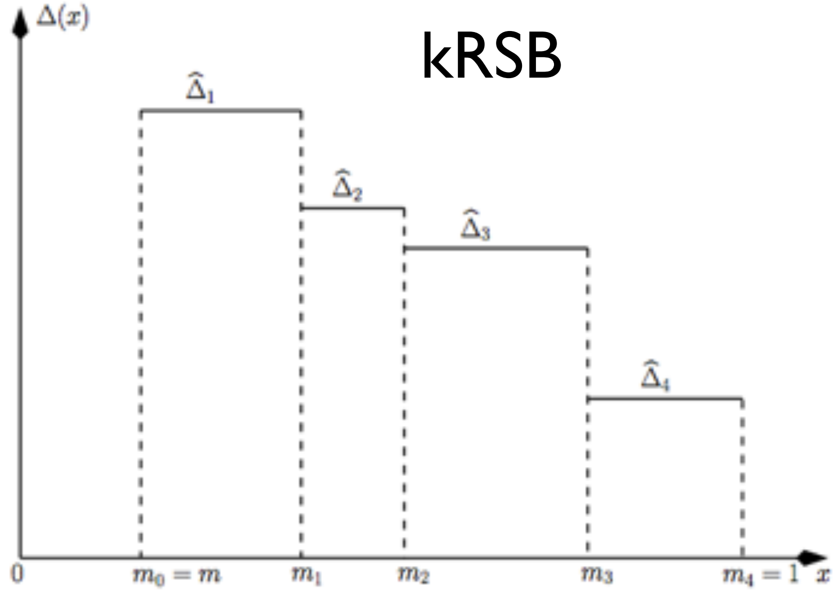
Behavior in the high density limit

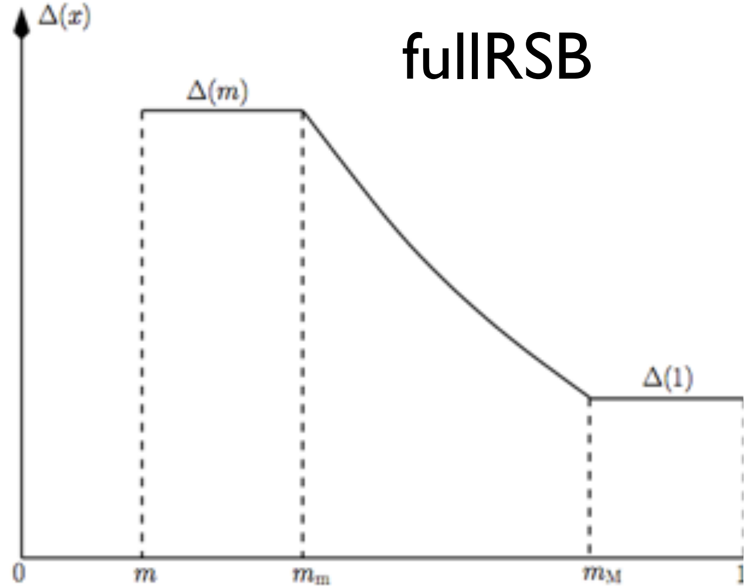


$$\widehat{\Delta} = \begin{pmatrix} 0 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 \\ \widehat{\Delta}_1 & 0 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & 0 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & 0 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & 0 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & 0 & \widehat{\Delta}_1 & \widehat{\Delta}_1 \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & 0 & \widehat{\Delta}_1 \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & 0 \end{pmatrix}$$

$$\boxed{\Delta_{ab} \to \Delta(x)}$$

$$\widehat{\Delta} = \begin{pmatrix} 0 & \widehat{\Delta}_2 & \widehat{\Delta}_2 & \widehat{\Delta}_2 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta} \\ \widehat{\Delta}_2 & 0 & \widehat{\Delta}_2 & \widehat{\Delta}_2 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta} \\ \widehat{\Delta}_2 & \widehat{\Delta}_2 & 0 & \widehat{\Delta}_2 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta} \\ \widehat{\Delta}_2 & \widehat{\Delta}_2 & \widehat{\Delta}_2 & 0 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta} \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & 0 & \widehat{\Delta}_2 & \widehat{\Delta} \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_2 & 0 & \widehat{\Delta} \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_2 & \widehat{\Delta}_2 & 0 \\ \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_1 & \widehat{\Delta}_2 & \widehat{\Delta}_2 & \widehat{\Delta} \end{pmatrix}$$





The mean square displacement profile depends on m in a complicated way

We are interested in the behavior of the Debye-Waller factor

$$\Delta_{EA} = \Delta(1) \sim m^{?}$$

Reminder: n

$$f(1,h) = \log \Theta \left[\frac{h}{\sqrt{2\Delta(1)}} \right]$$

$$\frac{\partial f(x,h)}{\partial x} = -\frac{1}{2}\dot{q}(x) \left[\frac{\partial^2 f(x,h)}{\partial h^2} \right.$$

$$f(1,h) = \log(2\cosh(\beta h))$$

Once the expression of the fullRSB entropy is produced, one has to optimize it over the mean square displacement profile.

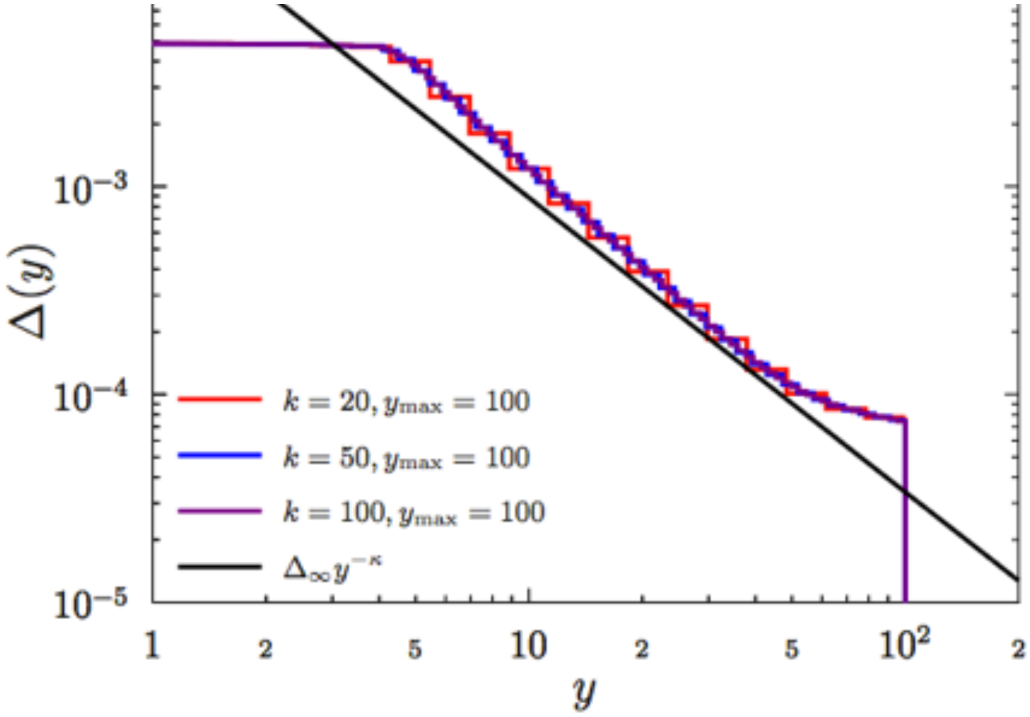
Find the variational equations. We must enforce the Parisi equations.

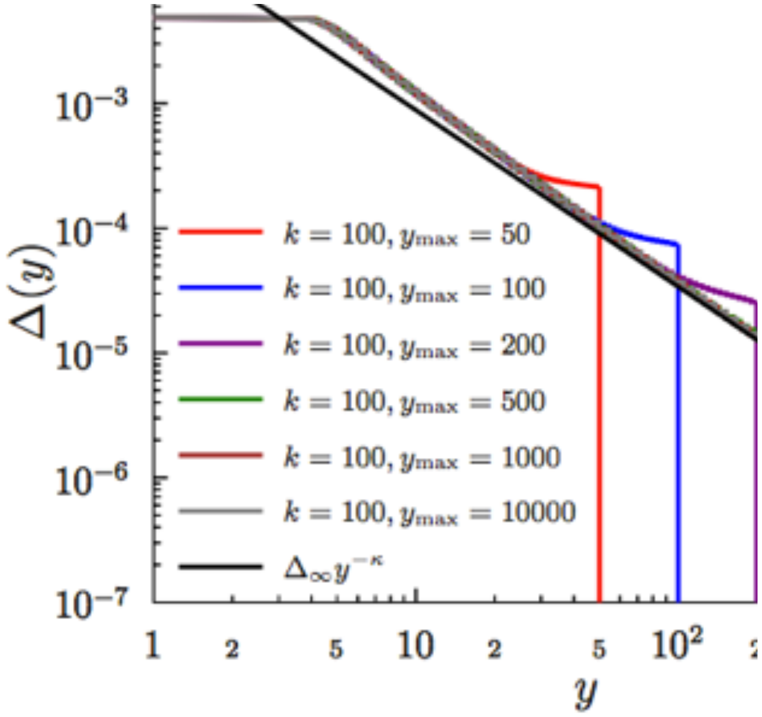
- Find the variational kRSB equations $\longrightarrow$ Find their continuum lin
- Add Lagrange multipliers to the action and then take the variatio with respect to everything.

At the end check that the two methods give the same eqns.

How to solve the equations

- Put them on a grid and discretize everything.
- <u>Solve numerically the kRSB equations for successive values of k.</u>





$$y = x/m \qquad [m, 1] \to [1, 1/m]$$

$$\boxed{\Delta(y) \sim y^{-\kappa} \qquad \kappa \simeq 1.416}$$

functions that appear in them.

$$\Delta(y) \sim y^{-\kappa}$$

Asymptotic power law behavior of the mean square displacement profile

$$\widehat{j}(y,h) = -\frac{c}{2y} J(hy^b/\sqrt{\gamma_\infty})$$

Roughly, the solution of the Parisi equation

$$\widehat{P}(y,h) \sim \begin{cases} y^c p_0(hy^c) & \text{for } h \sim -y^{-c} \\ y^a p_1(hy^b) & \text{for } |h| \sim y^{-b} \\ p_2(h) & \text{for } h \gg y^{-b} \end{cases}$$

Analogous to the distribution of fields in the SK model

Connected with the g(r) in the g

$$b = \frac{1+c}{2}$$

Solve for the exponents.

$a = 0.29213\ldots$	$b = 0.70787\ldots$	$c = 0.41574\ldots$
$\gamma = 0.41269\ldots$	$\theta = 0.42311\ldots$	$\kappa = 1.41574\ldots$

$$\gamma = \frac{a}{b} \qquad \theta = \frac{c-a}{b-c} \qquad \kappa = 1 + c$$

$\gamma =$

a

Within the fullRSB ansatz

$$1 - 2q(1) + \int_{-\infty}^{\infty} \mathrm{d}y P(1,h) m^4(1.h) =$$

$$= T^2 \int_{-\infty}^{\infty} \mathrm{d}h P(1,h) [m'(1,h)]^2 \leq T^2$$

$$l = 1 - \beta^2 (1 - 2q_{EA}$$

$$q_{EA} = \frac{1}{N} \sum_{i=1}^{N} m_i^2$$

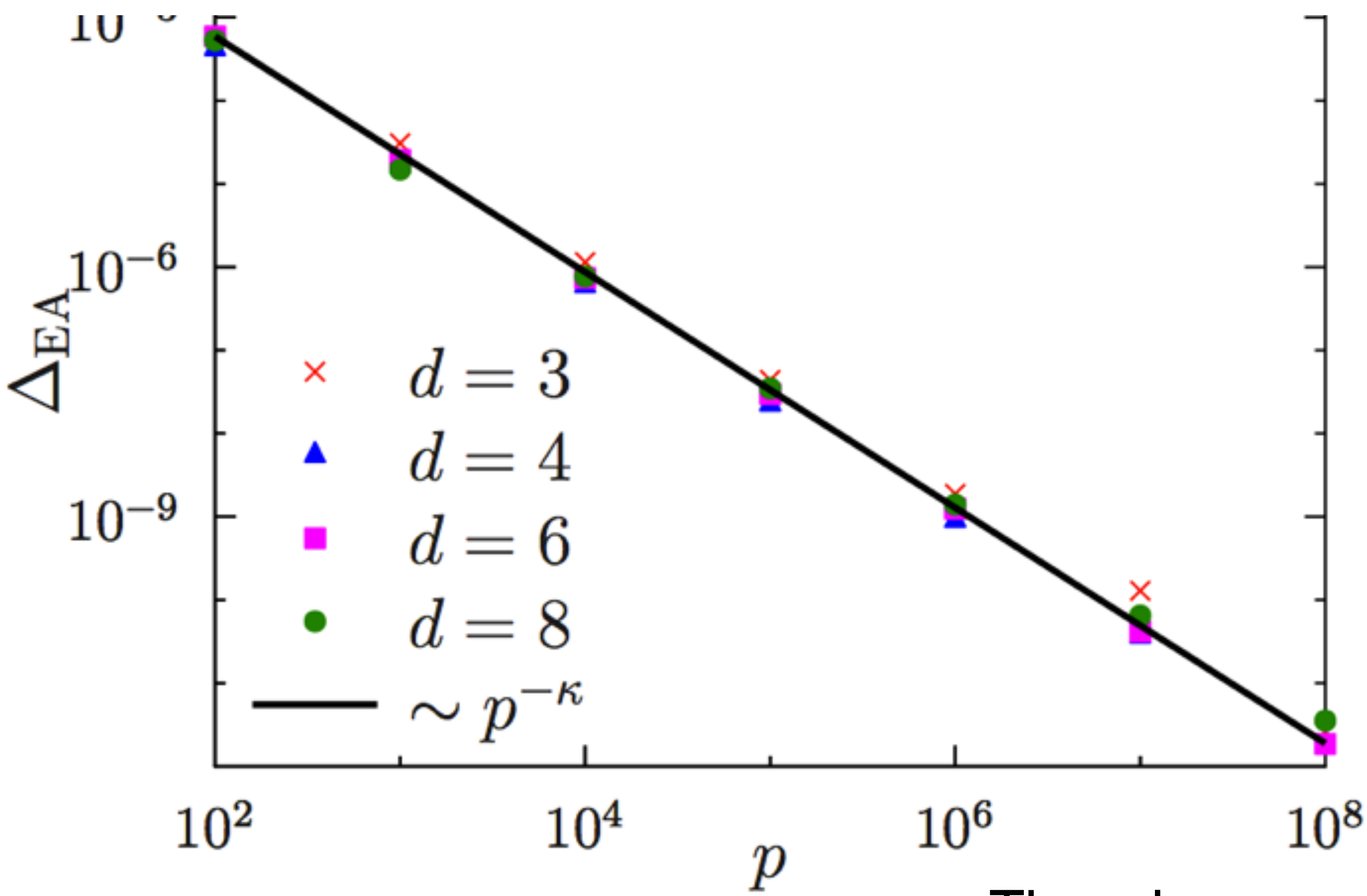
$$r = \frac{1}{N} \sum_{i=1}^{N} m_i^4$$

In the Hard Spheres case

$$\boxed{1 = \frac{\widehat{\varphi}}{2} \int_{-\infty}^{\infty} \mathrm{d}h \, \mathrm{e}^h \widehat{P}(y,h) \tilde{f}''(y,h)^2}$$

- <u>Crucial to obtain the values of the exponents (thus, also the microscopic marginal stability condition)</u>

- <u>Crucial to obtain that the fullRSB jammed packings are isostatic</u>



The value of k seems not to depend on the dimension

The value extracted fullRSB solution seem well the dat

A Gardner transition is present in the phase diagram and affects
jamming limit of the glass phase

- fullRSB effects must be included to describe the statistics of pack
- The asymptotic fullRSB equations in the jamming limit give the ex
jamming
- The microscopic marginal stability is directly linked to the fullRSE
stability
- The infinite dimensional calculation seems to predict well the bel
finite dimensions.

- Confirm the numerical analysis at finite pressure.
- Try to derive the statistics of avalanches using the results on the
(compute the power law exponent of the avalanches size distribu
- Try to detect the Gardner point (insights from the state followin
calculation)
- Try to extend the formalism to soft spheres.
- Try to understand why mean field theory works so well

Th