Jamming and hard spheres

Ludovic Berthier

Laboratoire Charles Coulomb
Université de Montpellier 2 & CNRS

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Outline

1 - Introduction: What is jamming?
2 - Early results
3 - Statistical mechanics approach
4 - Glassy phase diagram
5 - Microstructure of jammed packings
6 - Vibrational dynamics
7 - Rheology
I-Introduction: What is jamming?
A geometric transition

- Athermal packing of soft repulsive spheres, e.g. \( V(r < \sigma) = \epsilon(1 - r/\sigma)^2 \).

\[ \phi \]

Low \( \phi \): no overlap, fluid

Large \( \phi \): overlaps, solid

- Clearly useful for: non-Brownian suspensions (below), hard grains (at), foams and emulsions of large droplets (above).

- Since \( T = 0 \), this is a purely geometric problem, possibly a nonequilibrium phase transition. A finite dimensional version of the ideal glass transition?
1.1 Spin glass perspective
Constraint satisfaction problem

- Packing as a random constraint satisfaction problem $\rightarrow$ connection to spin glasses: Pack hard objects with no overlap. [Krzakala & Kurchan, PRE ’07]

- A mean-field version of hard sphere fluid: each particle interact with $z \ll (N - 1)$ neighbors.

- Interactions specified by quenched random graph.

- Aim: Understanding jamming within RFOT. [Mari et al., PRL ’08]

- This model can then be solved as other random constraint satisfaction problems, e.g. cavity method. [Mézard et al., JSTAT ’11]

("This paper is meant to be read by specialists in the field, so we did not make much attempt to explain...")
Connection to Giulio’s lecture

- Evolution of free energy landscape in the context of $q$-coloring problem on random graphs.


- “Condensation”: Ideal glass (Kauzmann) transition in finite $d$.

- “Uncol”: No solution found which satisfies all constraints. This is a jamming transition.
1.2 – Broader perspective
Disordered solid states

- Dense granular materials are disordered solids.

- Atomic glasses (window glasses, plastics) are solid materials frozen in an amorphous (non-crystalline, metastable) structure.

- Two possible pictures: Force chains and geometry of contact network versus complex energy landscape characteristic of disordered materials.
Jamming rheology

• Observed by compressing soft or hard macroscopic particles.

• Example: hard grain suspension. ‘Diverging’ athermal viscosity $\eta_0(\varphi)$.

Simple rheology: no time scale competes with shear rate

$\eta(\varphi, \dot{\gamma}) = \eta_0(\varphi) \sim |\varphi_J - \varphi|^{-\Delta}$. 

[Boyer & Pouliquen, PRL ’10]
More ‘jamming’ transitions

Vibrated grains [Philippe & Bideau, EPL ’02]

- Dense assemblies of grains, (large) colloids and bubbles stop flowing.

Air fluidized granular bed [Daniels et al., PRL ’12]

Sheared foam [Langer, Liu, EPL ’00]
Athermal rheology of soft particles

- Overdamped \((T = 0)\) simulations of sheared harmonic spheres. Diverging viscosity, emergence of yield stress. \(T = 0\) glass transition?

```
\begin{align*}
\eta^{-1} & \quad 10^{-3} \quad 10^{-2} \quad 10^{-1} \\
\sigma & \quad 10^{-5} \quad 10^{-4} \quad 10^{-3} \quad 10^{-2}
\end{align*}
```

- Scaling law: \(\eta(\phi, \dot{\gamma}) = |\phi_J - \phi|^{-\Delta} F(\dot{\gamma}|\phi - \phi_J|^{\beta}).\) Theoretical basis?

- Similar behaviour (and scaling laws?) observed in emulsion.

[Olsson & Teitel, PRL '07] [Paredes et al., PRL '13]
Athermal rheology of soft particles

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[Olsson & Teitel, PRL ’07]

[Olsson & Teitel, PRL ’07]

[Paredes et al., PRL ’13]
Bernal’s insight

“This theory treats liquids as homogeneous, coherent and irregular assemblages of molecules containing no crystalline regions or holes.”

- Theory of liquids as a random packing problem.
- Experiments with grains, computer simulation.

[J. D. Bernal, “A geometrical approach to the structure of liquids”, Nature (1959)]
Dynamics in colloidal hard spheres

- Glass ‘transition’: Dramatic increase in viscosity when $\phi$ decreases.

- Viscosity measurements difficult $\rightarrow$ light scattering.

- $\tau_T = \sigma^2/D_0 \sim 1$ ms.

- $T$ value irrelevant, but $T > 0$ for thermal equilibrium $\neq$ jamming?

- Mode-Coupling Theory fit?
  $\tau_\alpha \sim (\phi_c - \phi)^{-\gamma}, \phi_c \approx 0.58$.

- ‘Free volume’ $\rightarrow 0$ at $\phi_J$.
  $\tau_\alpha \approx \exp(P/T) \sim \exp(A/|\phi_J - \phi|)$.

Jamming’s back!

[van Megen et al., PRE '98]

- No neat experimental answer from early work ('86 - '05).
Activated colloidal dynamics

- Most recent set of light scattering data to date.

- Mode-coupling prediction fails, no algebraic divergence at $\varphi_c$.

- Best fit to ‘activated’ dynamics:
  $\tau_\alpha \approx \exp\left(\frac{A}{(\varphi_0 - \varphi)}\right)^\delta$, $\delta \approx 2$.

- Simulations suggest $\varphi_0$ distinct from jamming point, but rely on extrapolations.

[Brambilla et al., PRL '09]
Colloidal rheology

- Non-linear rheological study: Transition from viscous fluid to yield stress amorphous solid, in the presence of thermal fluctuations, non-linear rheology: \( \eta = \eta(\varphi, \dot{\gamma}) \).

![Graph showing rheological properties]

- Linear viscosity: \( \eta_T(\varphi) = \eta(\dot{\gamma} \rightarrow 0, \varphi) \). Similar to \( \eta_0(\varphi) \) for non-Brownian suspensions, or activated dynamics as found for \( \tau_\alpha \).
The glass ‘transition’

- Many molecular materials become glasses at low temperature.

- Glass $\equiv$ liquid “too viscous” to flow. Glass formation is a gradual process with activated dynamics (not algebraic) in thermal equilibrium.

- ‘Ideal’ glass transition at equilibrium? Jamming point for molecules?

- Existence of many metastable states: glasses are many-body “complex” systems, due to disorder and geometric frustration.
Molecular glassy liquids rheology

- Flow curves at finite shear rate $\dot{\gamma}$ in simple shear flow: $\sigma = \sigma(\dot{\gamma}) = \eta(\dot{\gamma})\dot{\gamma}$, in binary LJ mixture.

- Transition from viscous fluid to solid material (finite yield stress), with non-linear behaviour (shear-thinning).

- Rheological behaviour (again) very similar to colloidal hard spheres and non-Brownian soft particles. Viscosity shows activated dynamics.

[Berthier & Barrat JCP '01]
Jamming phase diagram

- Suggested by similarity of rheological behaviour observed in non-Brownian & Brownian suspensions, and molecular liquids.


- Has given rise to a whole field of ‘jamming’ studies, many experimental measurements in connection to jamming phase diagram.
II-Early results
Harmonic spheres

- “Bubble model” introduced by Durian in ’95 to study wet foams: 
  \[ V(r < \sigma) = \epsilon(1 - r/\sigma)^2. \] 
  Can be used to explore the complete \((T, \varphi, \sigma)\) “jamming phase diagram” at once.

- Two ways of going athermal, \(\epsilon/T \to \infty\).
  
  (i) \(T = \text{const.}\) and \(\epsilon \to \infty\): Brownian hard spheres (e.g. colloids).
  
  (ii) \(\epsilon = \text{const.},\) \(T \to 0\): athermal soft suspensions (e.g. foams), equivalent to hard spheres below jamming.

- For thermal systems, \(T/\epsilon\) quantifies the particle softness: large \(T/\epsilon = \) soft particles. Useful for emulsions, microgels, simple liquids.
Durian’s bubble model: ’95

- Nice early work:
  [Bolton & Weaire, PRL 65, 3449 (1990)]

- Durian: Rheological study at $T = 0$ using computer simulations.

- Transition from fluid to solid behaviour at $\varphi_J$ with scaling properties.

- Scaling laws for emergence of shear modulus $G$, packing pressure $P$ and number of contacts per particle $z$ at critical density $\varphi_J$.

[Durian, PRL '95]
Point $J$: ’02

- $J$-point from random packing properties of soft repulsive particles at $T = 0$. [O’Hern et al. PRL ’02, PRE ’03]

- Scaling laws and structure of packings near jamming [vanHecke JPCM’10]

Energy: $E = 0$ for $\varphi < \varphi_J$; $E \sim (\varphi - \varphi_J)^\alpha$ for $\varphi > \varphi_J$.

Contact number: $z = 0 \rightarrow z = z_c + a(\varphi - \varphi_J)^{1/2}$ with $z_c = 2d$: isostaticity.

- Major numerical and experimental effort over the last decade; considered as a new nonequilibrium phase transition with relevant physical consequences.
Density of states

- Density of state above jamming transition reveals the presence of ‘soft modes’, with $\omega^* \sim (\varphi - \varphi_J)^{1/2}$.

- Deep link with isostaticity: $z \rightarrow z_c = 2d$.

- $z = z_c$: each particle has just enough contacts to maintain mechanical stability (i.e. $N_c \equiv z_c N/2 = Nd$).

- Isostatic packings have flat density of states; compressed packings are flat down to $\omega^* \sim (z - z_c) \sim (\varphi - \varphi_J)^{1/2}$.


[Wyart et al., EPL ’05]
Structure of packings
**Structure of packings**

- **Pair correlation function:**
  \[ g(r) = (\rho N)^{-1} \sum_{ij} \delta(r - |r_i - r_j|) \]
  and structure factor \( S(q) \) quantify two-point density correlation functions.

- **Multiple singularities** revealed by \( g(r) \) analysis. The “structure” is not boring!

- **Physics near contact** \( r \sim \sigma \) is crucial and reveals 3 different power laws—described later.

- **Suppressed density fluctuations** at large scale: \( S(q) \sim q \) at low \( q \). ‘Hyperuniform’ state with \( V \langle \delta \varphi^2 \rangle_V \rightarrow 0 \). [Donev et al., PRL '05]
III-Statistical mechanics approach
Geometric problem

- **Equilibrium statistical mechanics** approach to jamming transition: Introduce a Hamiltonian penalizing overlaps (e.g. harmonic spheres) and a temperature $T$.

- Take the $T \to 0$ limit of the constructed $(T, \varphi)$ phase diagram.
Start with the fluid

- Consider the fluid, $V(r) = (1-r)^2$, at equilibrium at $(T > 0, \varphi)$.

- Liquid state theory solves the structure, $g(r)$, thus the thermodynamics using integral equations. We use (for instance) HNC.

- $e(T, \varphi) \sim T^{3/2}$.

- Pressure is finite at $T = 0$ and continuous function of $\varphi$.

- No jamming (or glass) transition.

- Liquid state theory is essentially blind to relevant phase transitions. Useful?

[Jacquin & Berthier, Soft Matter '10]
Nonmonotonic structure

- Anomalous structural evolution at constant $T = 1.2 \cdot 10^{-3}$. The system orders, $\varphi = 0.55 \rightarrow 0.78$, then disorders $\varphi = 0.78 \rightarrow 0.96$.

$F = E - TS$: Avoiding overlaps to reduce energy becomes difficult (entropically disfavoured) at large $\varphi$. Solution: increase overlap using particle softness (lose energy) to make space and gain entropy.

- Smooth version of jamming (‘thermal vestige’) arising at equilibrium.
The structural anomaly seen in jammed colloids is naturally explained by equilibrium concepts—in fact seen in many ultrasoft materials.
Why liquid state theory fails

- Numerical phase diagram of soft harmonic spheres
  [Berthier & Witten, EPL, PRE ‘09]

- Glassy ‘phase’ is found. Glasses jam, fluids do not.

- Statmech needs to first handle the complexity of the glass phase before addressing the jamming transition.
IV-Glassy phase diagram
Statistical mechanics of glasses

• Assuming exponential number of metastable states exists:

\[
\begin{align*}
\langle L \rangle_{\ast} & = \frac{1}{(L',L_{\ast})} \int_{\mathcal{L}} f(T,V) = s_{\text{glass}} \text{ and } 0 = \langle L, L \gtrsim L \rangle_T^{\text{conf}}(f', T) \\
\end{align*}
\]

• In the ideal glass: 

\[
\begin{align*}
\text{Monasson, PRL ~95} & \quad \text{[cf all lectures on Tuesday]} \\
\end{align*}
\]

\[
\begin{align*}
\langle L', f \rangle_{\text{conf}} N & + \frac{L}{N} f \int_{\mathcal{L}} \exp \left[ -N f' T + N s_{\text{conf}}(f', T) \right] dx \exp \int_{\mathcal{L}} \log \frac{A}{L} = (L, f) \\
\end{align*}
\]

• In practice, take $m$ replicas (and minimize the replicated free energy).

\[
\begin{align*}
\text{Monasson, PRL ~95} & \quad \text{[cf all lectures on Tuesday]} \\
\end{align*}
\]

\[
\begin{align*}
\langle L', f \rangle_{\text{conf}} N & + \frac{L}{N} f \int_{\mathcal{L}} \exp \left[ -N f' m T + N s_{\text{conf}}(f', T) \right] dx \exp \int_{\mathcal{L}} \log \frac{A}{L} = (L, f) \\
\end{align*}
\]

• In the 'ideal' glass:

\[
\begin{align*}
s_{\text{conf}}(T \leq T_K) = 0 \quad \text{and} \quad f_{\text{glass}} = f(m \ast (T), T) / m \ast (T) \quad \text{[Mézard & Parisi, PRL ~99].} \\
\end{align*}
\]

\[
\begin{align*}
\text{One-step replica symmetry breaking.} \\
\text{with } m \ast (T) > (L)_{\ast} \quad \text{[Mézard & Parisi, PRL ~99].} \\
\end{align*}
\]

Physical idea:

\[
\begin{align*}
- T s_{\text{conf}} \quad \text{represents the free energy cost of 'forcing' two} \\
\text{replicas within the same state.} \\
\end{align*}
\]

[cf all lectures on Tuesday]

• Physical idea: $- L s_{\text{conf}}$ represents the free energy cost of forcing two metastable states.

• One-step replica symmetry breaking.

• Replicas are then introduced to recognize multiple amorphous metastable states.

\[
\begin{align*}
\text{Assuming exponential number of metastable states exists:} \\
\end{align*}
\]

\[
\begin{align*}
\langle L', f \rangle_{\text{conf}} N & + \frac{L}{N} f \int_{\mathcal{L}} \exp \left[ -N f' m T + N s_{\text{conf}}(f', T) \right] dx \exp \int_{\mathcal{L}} \log \frac{A}{L} = (L, f) \\
\end{align*}
\]
Replicated liquid state theory

• Liquid state theory of replicated liquid.

• Mézard-Parisi’s ‘small cage’ (harmonic) expansion breaks down near jamming.

• Zamponi-Parisi ‘small (but hard) cage’ approximation is correct for hard spheres. [Zamponi & Parisi, RMP ’10].

• An effective potential valid for both hard spheres ($T \to 0$ small $\varphi$) and soft glasses ($T \to 0$ large $\varphi$)

$$f(m, A, \varphi, T) = f_{\text{harm}}(m, A) + f_{\text{liquid}}(\varphi, T/m) - \frac{\rho}{2} \int dr g(r) [e^{-\beta(V_{\text{eff}}(r) - mV(r))} - 1]$$

• Low-$T$ approximation to treat analytically the glass & jamming transitions of harmonic spheres.
The ‘ideal’ glass transition

- High $T$ fluid: $m = 1$, $s_{\text{conf}}(T) > 0$ and simple liquid theory is enough.

- $s_{\text{conf}}(T)$ vanishes at $T_K(\varphi) > 0$ for $\varphi > \varphi_K \equiv$ hard sphere glass transition.

- Low-$T$ scaling: $T_K \sim (\varphi - \varphi_K)^2$ (robust scaling) with $\varphi_K \approx 0.577$ (value depends on specific approx). Correct phase diagram.
The ‘ideal’ jamming transition

- Glass thermodynamics: energy, pressure, specific heat, fragility...

- Jamming at $T = 0 \Leftrightarrow$ Change in ground state glass properties.

- $\varphi_{GCP} = 0.633353...$ such that:
  
  $E_{GS} = 0$ below,

  $E_{GS} \simeq a(\varphi - \varphi_{GCP})^2$ above.

- Glass Close Packing: densest $T = 0$ glass with no overlap.
  [Zamponi & Parisi, RMP ’10]

- $P_{GS} \sim (\varphi - \varphi_{GCP})$.

- Existence, location(s), and scaling laws of jamming from ‘first principles’.
Constraint satisfaction problem

- Similar phase diagram for Potts glass model on random graphs, obtained using the cavity method ($q$-coloring problem as $T \to 0$).

[Krzakala & Zdeborova, EPL '08]
Locating the jamming transition

- Glass Close Packing buried deep inside glass phase.

- Locating it means solving a very hard computational problem.

- A standard algorithm will fall out of equilibrium before reaching glass transition and be trapped in one of many metastable state.

- Chosen state is followed down to $T = 0$, jams at $\varphi < \varphi_{GCP}$.

- Measured location of jamming point is therefore strongly algorithm-dependent. Point $J$ is one of the ‘worst’!
Numerical evidence: ‘\( J \)-line’

- Rapid compressions of hard sphere fluid configurations, starting from various packing fractions reveals continuous range of jamming densities: \( J \)-point \( \rightarrow \) \( J \)-line.

- The normalized pressure, \( Z = P/(\rho k_B T) \), diverges when \( \varphi \rightarrow \varphi_J \).

[Chaudhuri et al., PRL '11]

- Location of jamming transition is not uniquely defined but its physical & scaling properties seem ‘universal’, i.e. shared by all packings.
• Problems of 1-RSB solution revealed by comparison to numerical work.

• Suggests that 1-RSB might be inconsistent... full RSB needed below the Gardner transition. Hard sphere limit has been treated in $d = \infty$.

[cf. P. Urbani, this afternoon]
V-Microstructure of jammed packings
Thermodynamic properties

- Emergence of $T = 0$ discontinuity as $T \to 0$ in harmonic spheres.

- Scaling laws for several properties are predicted within 1-RSB solution.

  $$U(T, \varphi) = T \mathcal{F}_U(\frac{|\varphi - \varphi_J|}{\sqrt{T}}),$$

  $$P = \sqrt{T} \mathcal{F}_P(\frac{|\varphi - \varphi_J|}{\sqrt{T}}).$$

- Natural matching between jammed and unjammed regions.

[Berthier, Jacquin, Zamponi, PRE '11]
Nonmonotonic glass structure

- Evolution of maximum of $g(r)$ at low $T$ has been measured numerically.

- $g(r)$ near contact is quite well described—other distances are harder to treat...

[Zhang et al. Nature ’09]
$T = 0$ pair correlation

- Predictions for $g(r)$ near contact at all $(T, \varphi)$.
- $g(r) \approx z_c \delta(r - 1)$ at $(T = 0, \varphi = \varphi_{GCP})$ with $z_c = 6$: isostaticity is predicted.
- Asymmetric scaling: $g(r) \approx |\delta\varphi|^{-1} F_{\pm} \left[ \frac{r - 1}{|\delta\varphi|} \right]$ at $(T = 0, \varphi = \varphi_{GCP} \pm \delta\varphi)$.

$T = 0$: Critical version of fluid ‘anomaly’. Occurs for the same physical, ‘equilibrium’, reason (energy/entropy competition).
Low temperature structure

- 1-RSB theory seems correct on hard sphere side, $\varphi < \varphi_J$.

\[ \delta \varphi = -0.00075 - T_h = 0 \]

\[ g(r) \]

<table>
<thead>
<tr>
<th>$T$</th>
<th>$T_e$</th>
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<tbody>
<tr>
<td>$10^{-7}$</td>
<td>$1.0012$</td>
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<tr>
<td>$10^{-8}$</td>
<td>$1.0008$</td>
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<tr>
<td>$10^{-9}$</td>
<td>$1.0004$</td>
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<tr>
<td>$10^{-10}$</td>
<td>$0.9996$</td>
</tr>
<tr>
<td>$10^{-11}$</td>
<td>$0.9992$</td>
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Numerics, $T = 10^{-7}$

Theory, $T = 10^{-7}$
Low temperature structure

- OK above jamming if $T < T_h(\varphi) \sim (\varphi - \varphi_J)^2$: Limit of 1-RSB validity?
1-RSB biggest failure

- \( z(\varphi, T) = \rho \int_0^1 dr 4\pi r^2 g(r) \) is the number of contacts.

- \( z(\varphi > \varphi_{GCP}, T = 0) \approx z_c + c(\varphi - \varphi_{GCP})^\alpha \) with incorrect exponent \( \alpha = 1 \).

- Large number of ‘near contacts’ below \( \varphi_{GCP} \) at \( T = 0 \) is missed by 1-RSB effective potential approach.

- This failure is fully cured by the full RSB solution.
Final (?) results–this afternoon

\[ z(r) = \rho \int_{0}^{r} 4\pi r^2 g(r)dr \]

- **Power law** approach to isostatic plateau, corresponds to weak inter-particle forces, \( P(f \ll \langle f \rangle) \sim f^{\theta} \).

- Abundance of **near contact**, \( g(r) \sim (r-1)^{-\gamma} \).

- Non-trivial **full-RSB** \( d = \infty \) predictions for \( \theta = 0.42311 \) and \( \gamma = 0.41269 \) [cf P. Urbani]

- Non-trivial scaling relations from stability requirements [cf M. Wyart]

- Precise numerical measurements in many \( d \) [cf P. Charbonneau]

[Charbonneau et al., PRL ’12]
Vibrational dynamics in glass

• First step towards dynamics: mean-squared displacements at finite $T$.

[Ikeda, Berthier, Biroli, JCP’13]

• Ballistic time: $\tau_0 \sim |\varphi - \varphi_J|/\sqrt{T}$ for $\varphi < \varphi_J$, $\tau_0 \sim \text{const.}$ for $\varphi > \varphi_J$.

• “Anomalous” timescale to reach plateau: $t^* \sim \frac{2\pi}{\omega^*}$, where $\omega^*$ is the frequency of soft modes if harmonic approximation works.
Amplitude of vibration much larger than naive guess $\ell_0^2 = T\tau_0^2$.

Non-trivial power law scaling: $\Delta^2(\infty) \sim (\varphi_J - \varphi)^\kappa$ with $\kappa \approx 1.5$.

$\kappa = 1.41574$ directly predicted from full RSB $d = \infty$ hard sphere.

Density of states and MSD related to $P(f)$ and $\theta$ via scaling arguments.
Dynamic criticality near jamming

- Diverging (rescaled) timescale:
\[ \tau = \frac{t^*}{\tau_0} \sim |\phi - \phi_J|^{-1/2}. \]

- Diverging dynamic susceptibility:
\[ \chi_4 \sim |\phi - \phi_J|^{-1/2}. \]

- Diverging dynamic correlation length:
\[ \xi_4 \sim |\phi - \phi_J|^{-1/4}. \]

- Time and length scales associated to anomalous vibrational motion diverge near jamming.

- Criticality observed if \( T \) is very low (particles not too soft), and very near \( \phi_J \).
Connection to experiments?

- Many experiments have measured “soft modes” (hard spheres, microgels, grains...).
- What is the extension of the critical regime?
- When does the harmonic approximation work?

[cf O. Dauchot’s lecture]
Critical motion in grains

[O. Dauchot]
VII-Rheology
Revisit jamming phase diagram

• **Aim:** Study the full \((\sigma, T, \varphi)\) phase diagram of harmonic spheres.

• **Langevin dynamics** with shear and thermostat in \(d = 3\):
  \[
  \xi \left( \frac{dr_i}{dt} - \gamma y_i e_x \right) = - \sum_j dV(|r_i - r_j|) \frac{dr_i}{dr_i} + \eta_i,
  \]
  with \(\langle \eta_i(t)\eta_j(t') \rangle = 2k_B T\xi \delta(t - t')\).

• **Two important microscopic timescales:**
  (i) **dissipation:** \(\tau_0 = \xi a^2/\epsilon = 1\), our time unit.
  (ii) **thermal time:** \(\tau_T = \xi a^2/(k_B T) \rightarrow \infty\) when \(T \rightarrow 0\). \(Pe \equiv \dot{\gamma}\tau_T\) (Peclet).

• **Two stress scales:** \(\sigma_T = k_B T/a^3\) (entropy / thermal), \(\sigma_0 = \epsilon/a^3\) (energy / athermal).

• Study both finite and zero temperatures, both thermal \((Pe < 1)\) and athermal \((Pe > 1)\) rheologies at once.  

[Ikeda *et al.*, PRL '12]
Flow curves

- Steady state rheology at $T = 10^{-4}$, $T = 10^{-6}$, and $T = 0$.

- From glass to jamming rheology, with interesting crossover.

- Two types of Newtonian regimes, depending on the Péclet number and particle softness.

- Striking similarity between glass & jamming flow curves.
\( \eta_T(T > 0) \) does not converge to \( \eta_0(T = 0) \) when \( T \to 0 \). These are distinct divergences at distinct densities with distinct physics.

- Solidity (yield stress) emerges at the glass transition at any \( T > 0 \), and jamming transition seen in density dependence of \( \sigma_Y \).
‘Jamming phase diagram’
Non-Brownian hard spheres

- Nature of viscosity divergence at $T = 0$?

- Simulations suggest genuine power law divergence (i.e. no crossover to some activated dynamics).

- Analogy with (nearly isostatic) elastic network relates viscosity to structure. (cf Wyart this afternoon.) [Lerner et al. ’11]

- Structure of sheared suspensions seems difficult to analyse from first principles.

[T. Kawasaki]
Conclusion

- Nature of jamming transition has been greatly clarified over recent years, salient features identified, consequences and generalisations explored.

- Remarkable success of replica approach... in $d = \infty$.

- Remarkably compatible with geometrical approach using marginal stability and isostaticity.