A statistical physics approach to compressed sensing

Florent Krzakala
ESPCI, PCT and Gulliver CNRS

in collaboration with
Marc Mézard & **François Sausset** (LPTMS)
Yifan Sun (ESPCI) and **Lenka Zdeborová** (IPhT Saclay)

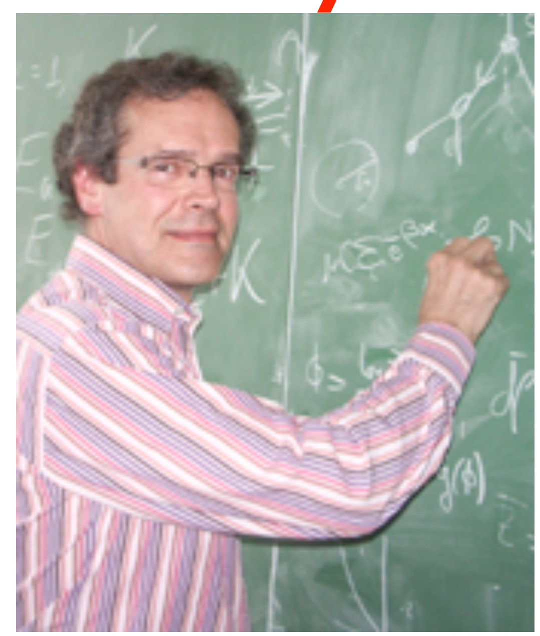
http://aspics.krzakala.org

Who are we? What do we do?

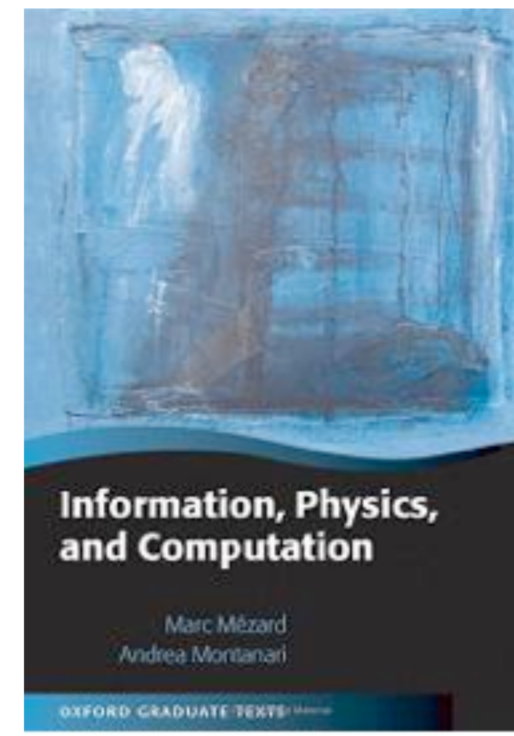
Interface between statistical physics, optimization, information theory and algorithms

Florent Krzakala
(ESPCI, Paris)

Lenka Zdeborová
(CNRS, Saclay)



Marc Mézard
(CNRS, Orsay)



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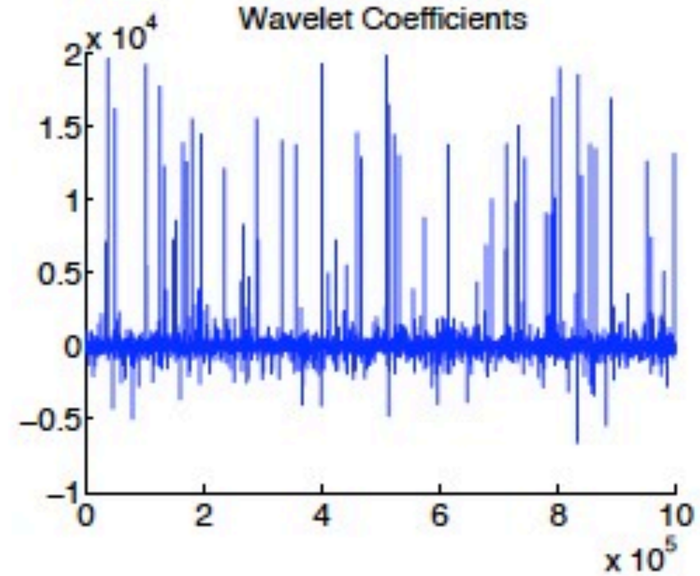
Sparse signals: what is compressed sensing?



From 10^6 wavelet coefficients, keep 25.000

Most signal of interest are sparse in an appropriated basis

⇒Exploited for data compression (JPEG2000).

Why do we record a huge amount of data, and then keep only the important bits?

Couldn't we record <u>only</u> the relevant information directly?

How does compressed sensing work?

How does compressed sensing work?

Image I

nxn pixels

vector of size N=n×n

$$\vec{I} = \begin{pmatrix} I^1 \\ . \\ . \\ . \\ . \\ I^N \end{pmatrix}$$

How does compressed sensing work?

M measurements

=

M linear operations on the vector

Image I

nxn pixels

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$$\vec{y} = \mathrm{G}\vec{\mathrm{I}}$$

G=M×N matrix

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vector of size M

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Problem: you know y and G, how to reconstruct I ?

If M=N ☞ easy, just use: I $=G^{-1}y$

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If M<N ☞ under-constrained system of equations

Many solutions are possible

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The idea of compressed sensing is to use the a-priori knowledge that the signal is sparse in <u>some</u> appropriate basis

Problem: you know y and G, how to reconstruct I ?

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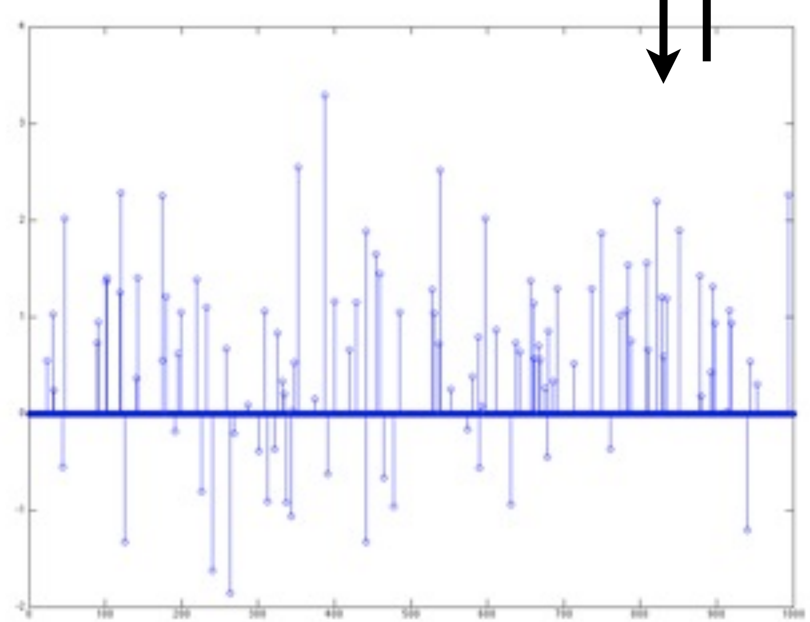
N×N matrix
Direct and inverse
Wavelet transforms

ψ^{-1} ψ



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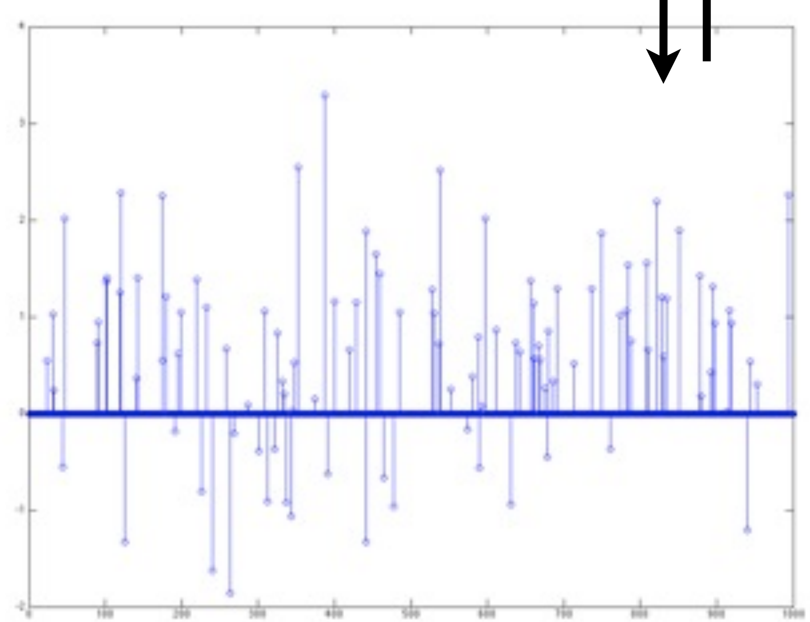

Sparse vector
of size N=n×n

$$\vec{x} = \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x^N \end{pmatrix}$$

How does compressed sensing work?

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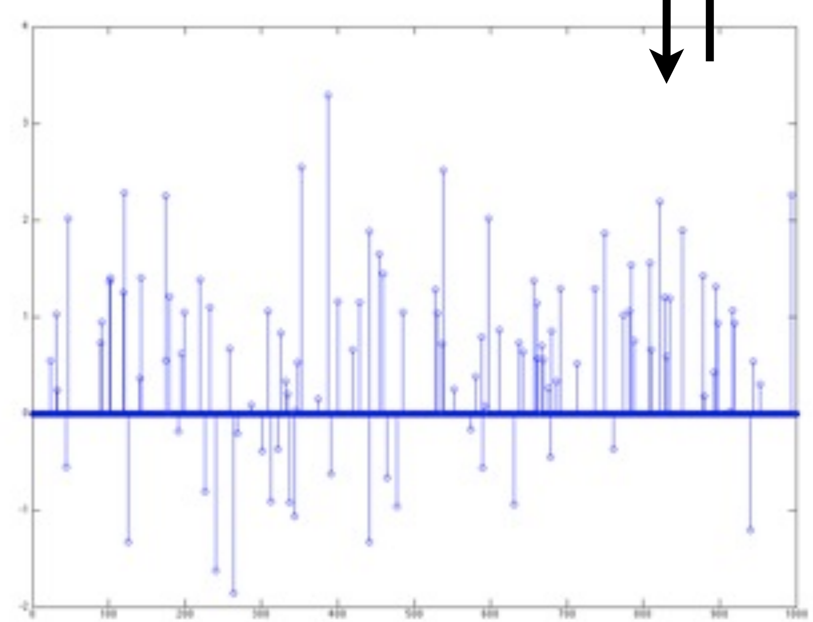

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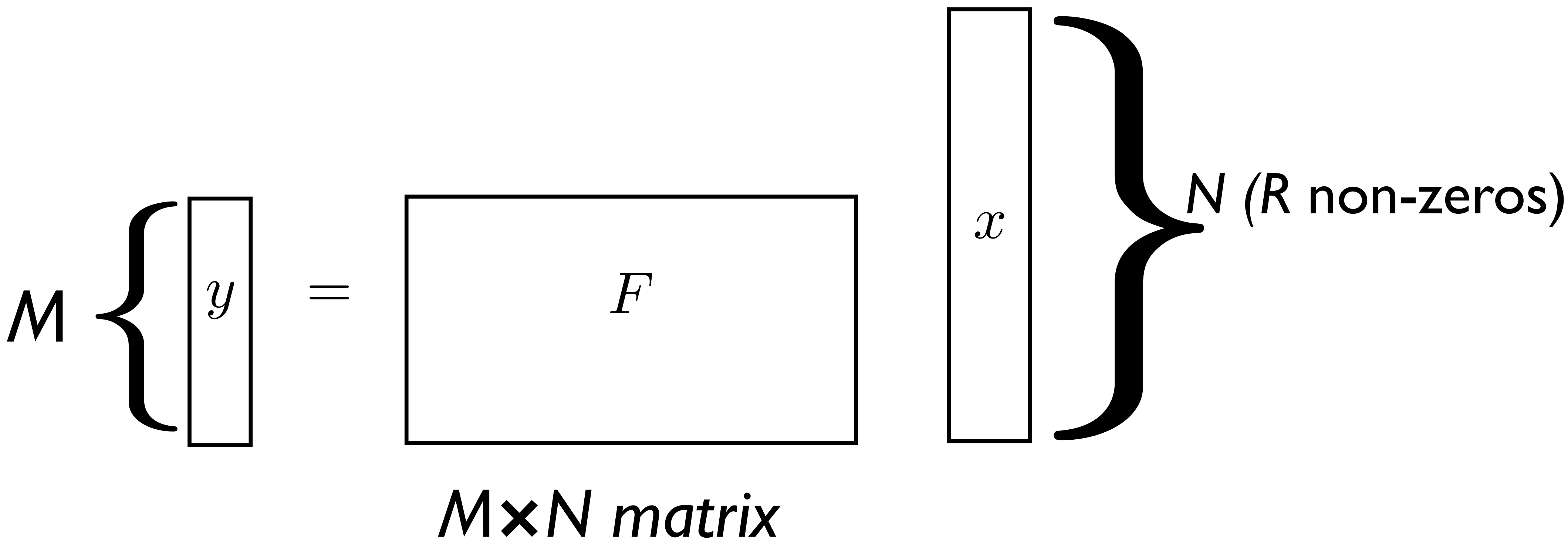
The problem to solve is now

$$\vec{y} = F\vec{x}$$

with $F = G\psi$

F=M×N matrix

How does compressed sensing work?



The problem to solve is now

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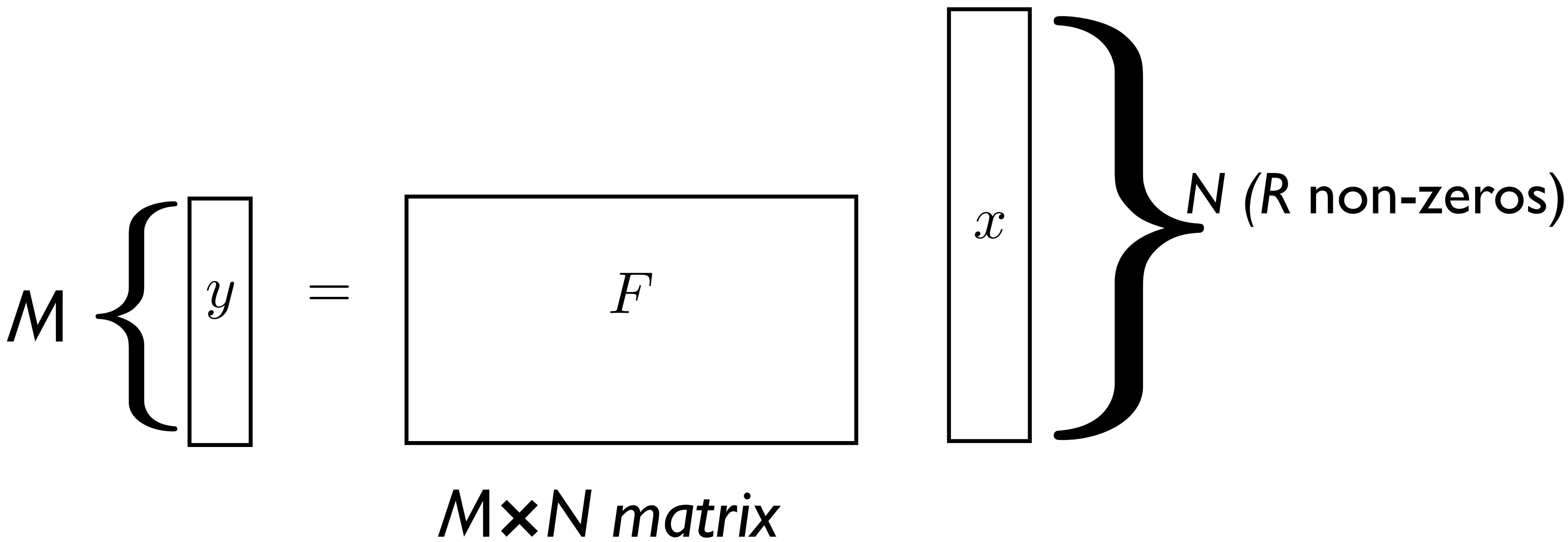
F=M×N matrix

- Needs for a solver that finds sparse solutions of an under-constrained set of equations
- Ideally works as long as M>R
- Robust to noise

State of the art in CS



- Incoherent samplings (i.e. a random matrix F)
- Reconstruction by minimizing the L1 norm $||\vec{x}||_{L1} = \sum_i |x_i|$

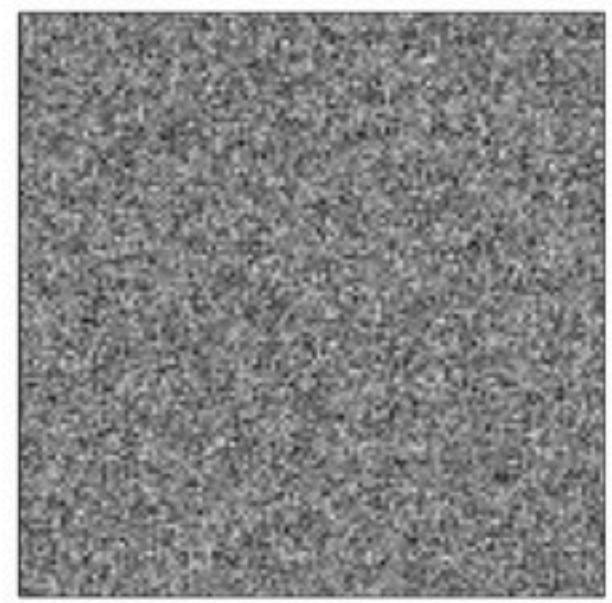
Candès & Tao (2005)
Donoho and Tanner (2005)

Example: measuring a picture

One measurement (scaling product with a random pattern)

$y_k = \langle$

,

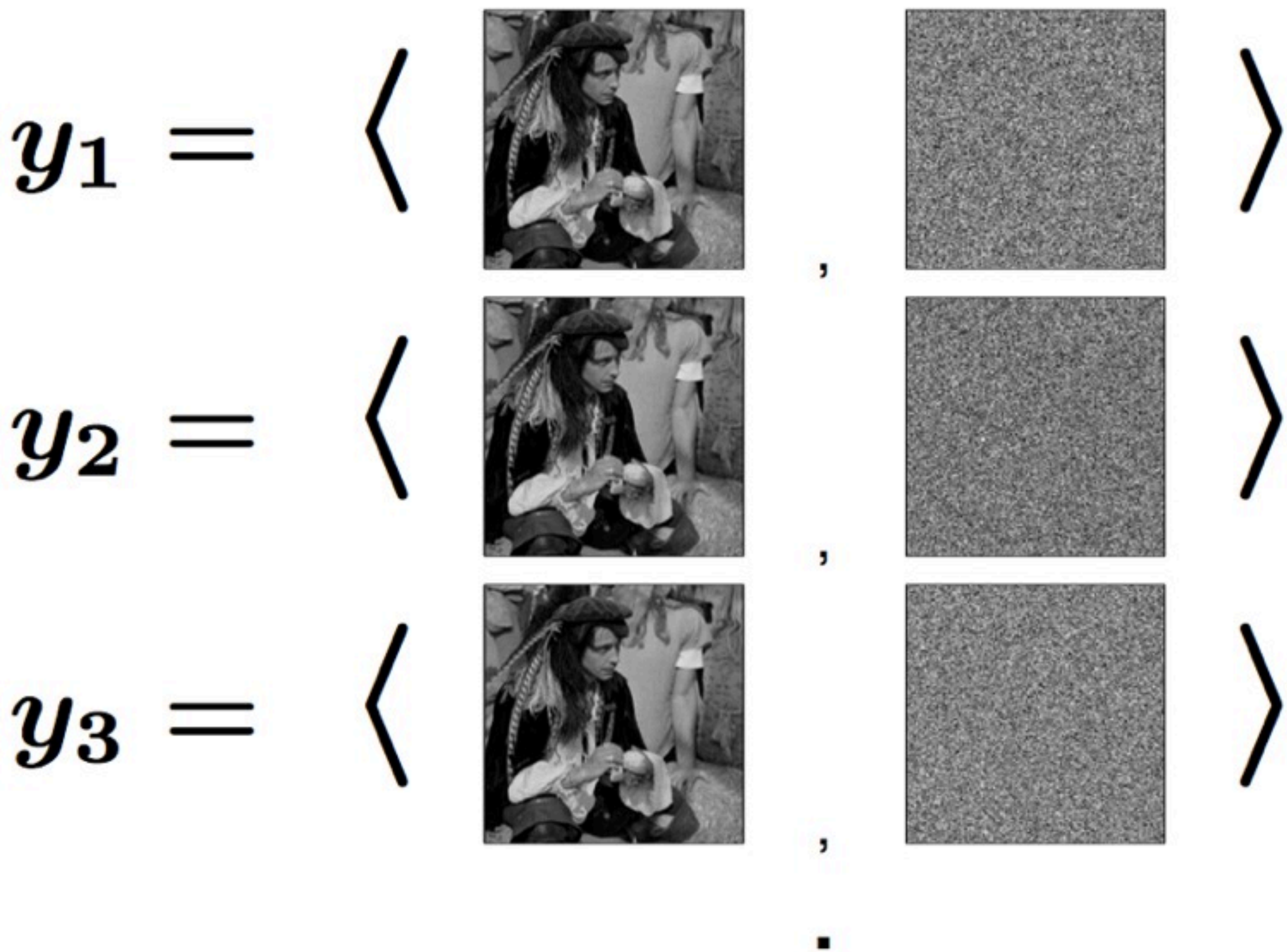


$\rangle$

- Each measurement touches every part of the underlying signal/image

Example: measuring a picture

Many measurements (scaling product with many random patterns)



Example: measuring a picture

Random matrix

$$[\,y\,] = [\,\mathbf{G}\,]\,[\,\mathbf{I}\,]$$

Measurements

signal

Example: measuring a picture

From 10^6 points, but only, 25.000 non zero

- Take $K = 96000$ incoherent measurements $y = \mathbf{G}\mathbf{I}$

- Solve

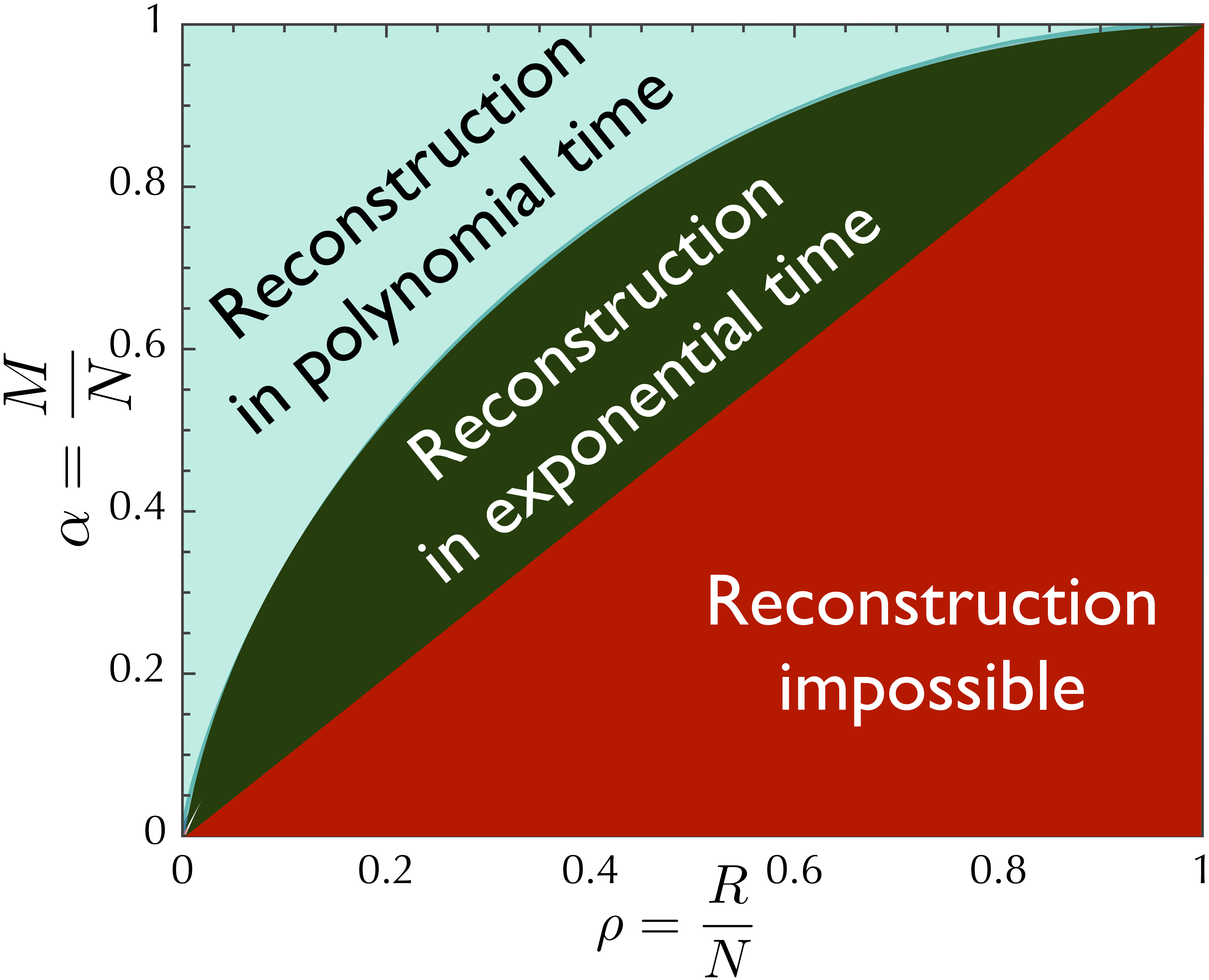
$$\min \|\mathbf{x}\|_{\ell_1} \quad \text{subject to} \quad \mathbf{G}\Psi\mathbf{x} = y$$

Ψ = wavelet transform

original (25k wavelets)

perfect recovery

State of the art in CS

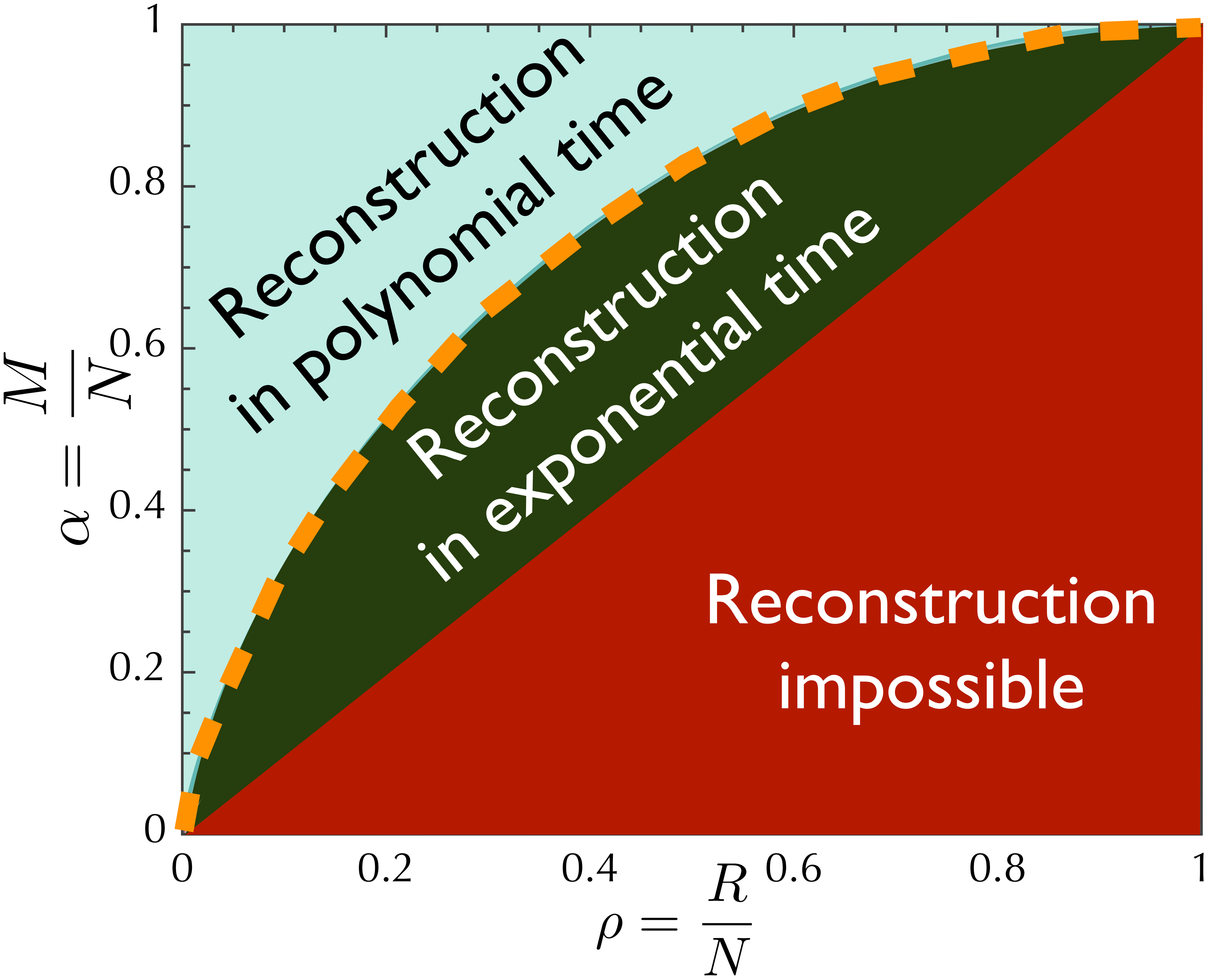


For a signal with

- (1-ρ)N zeros
- R=ρN non zeros

and a random iid matrix with M =α N

State of the art in CS



For a signal with
- (1-ρ)N zeros
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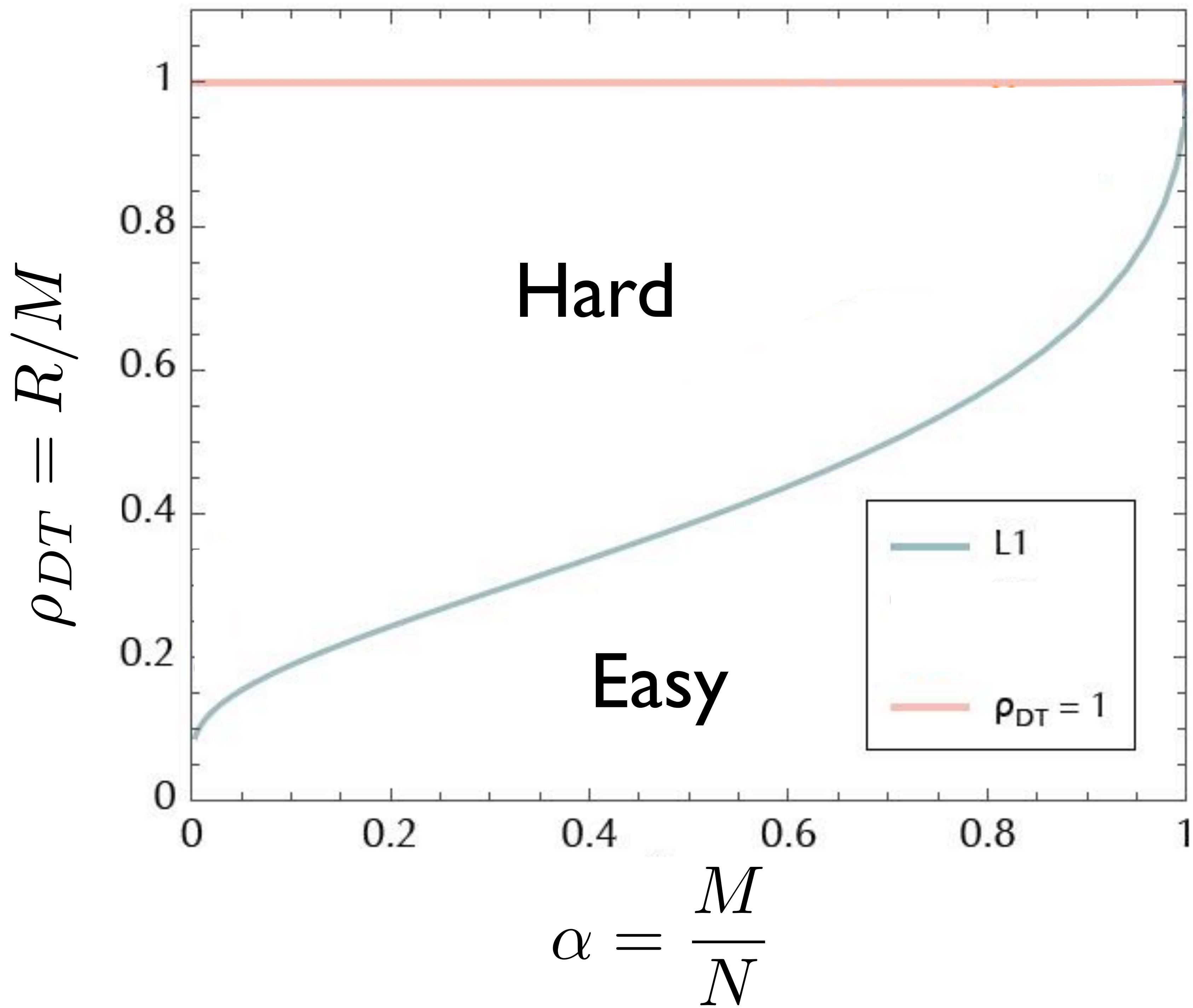
Reconstruction limited by the Donoho-Tanner transition for the L_1 norm minimization

State of the art in CS



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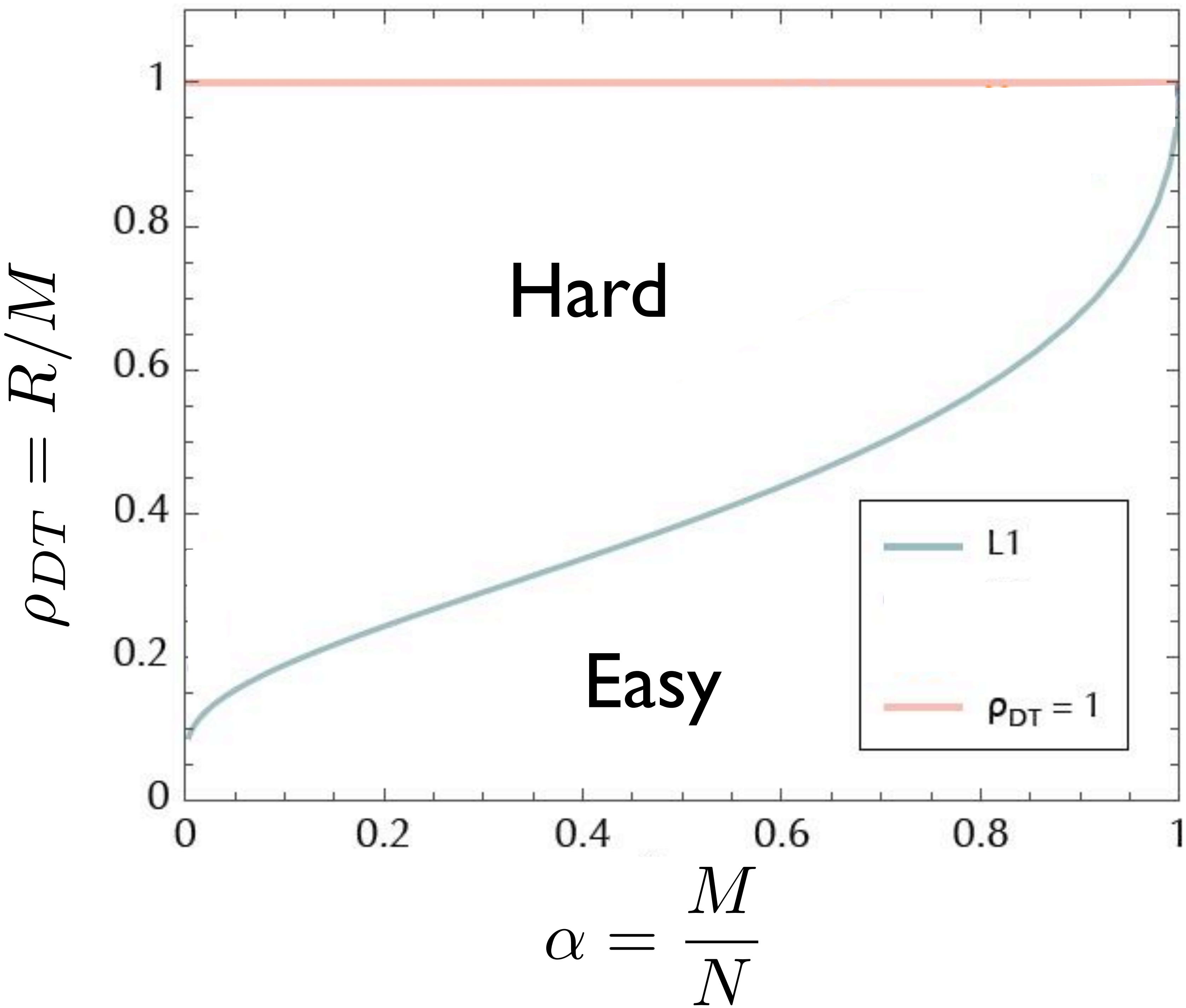

For a signal with

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A different representation of the same transition

Our work

A statistical physics approach to compressed sensing

- A probabilistic approach to reconstruction
- The Belief Propagation algorithm
- Seeded measurements matrices

Our work

A statistical physics approach to compressed sensing

- **A probabilistic approach to reconstruction**
- The Belief Propagation algorithm
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A probabilistic approach to compressed sensing

We want to sample from this distribution:

$$P(\vec{x}|\vec{y}) = \frac{1}{Z} \prod_{i=1}^{N} \left[(1-\rho)\,\delta(x_i) + \rho\phi(x_i)\right] \prod_{\mu=1}^{M} \delta\left(y_\mu - \sum_{i=1}^{N} F_{\mu i} x_i\right)$$

A probabilistic approach to compressed sensing

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Solution of
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$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Sparse vector}}\qquad\underbrace{\qquad\qquad\qquad\qquad}_{\text{Solution of the linear system}}$$

Sparse vector

Solution of
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<u>Theorem</u>: sampling from P(x|y) gives the correct solution in the large N limit as long as $\alpha > \rho_0$ if: a) $\Phi(x) > 0\ \forall x$ and b) $1 > \rho > 0$

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Sampling from P(x|y) is optimal,

(even if we do not know the correct $\Phi(x)$ or the correct ρ)

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Sampling from P(x|y) is optimal,

(even if we do not know the correct $\Phi(x)$ or the correct ρ)

In practice, we use a Gaussian distribution for $\Phi(x)$, with mean m and variance σ^2, and "learn" the best value for ρ, σ and m.

A sketch of the proof

Consider the system constrained to be at distances larger than D with respect to the solution

$$Y(D,\epsilon) = \int \prod_{i=1}^{N} (dx_i \; [(1-\rho)\delta(x_i) + \rho\phi(x_i)]) \prod_{\mu=1}^{M} \delta_\epsilon \left(\sum_i F_{\mu i}(x_i - s_i) \right) \mathbb{I} \left(\sum_{i=1}^{N} (x_i - s_i)^2 > ND \right)$$

1) Y(0) is infinite if $\alpha > \rho_0$ (equivalently if M>R)

(just count the delta functions! N-R+M deltas versus N integrals...)

2) Y(D) is finite for any D>0

(bound by a first moment method, or "annealed" computation)

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If $\alpha>\rho_0$, the measure is always dominated by the solution

A sketch of the proof

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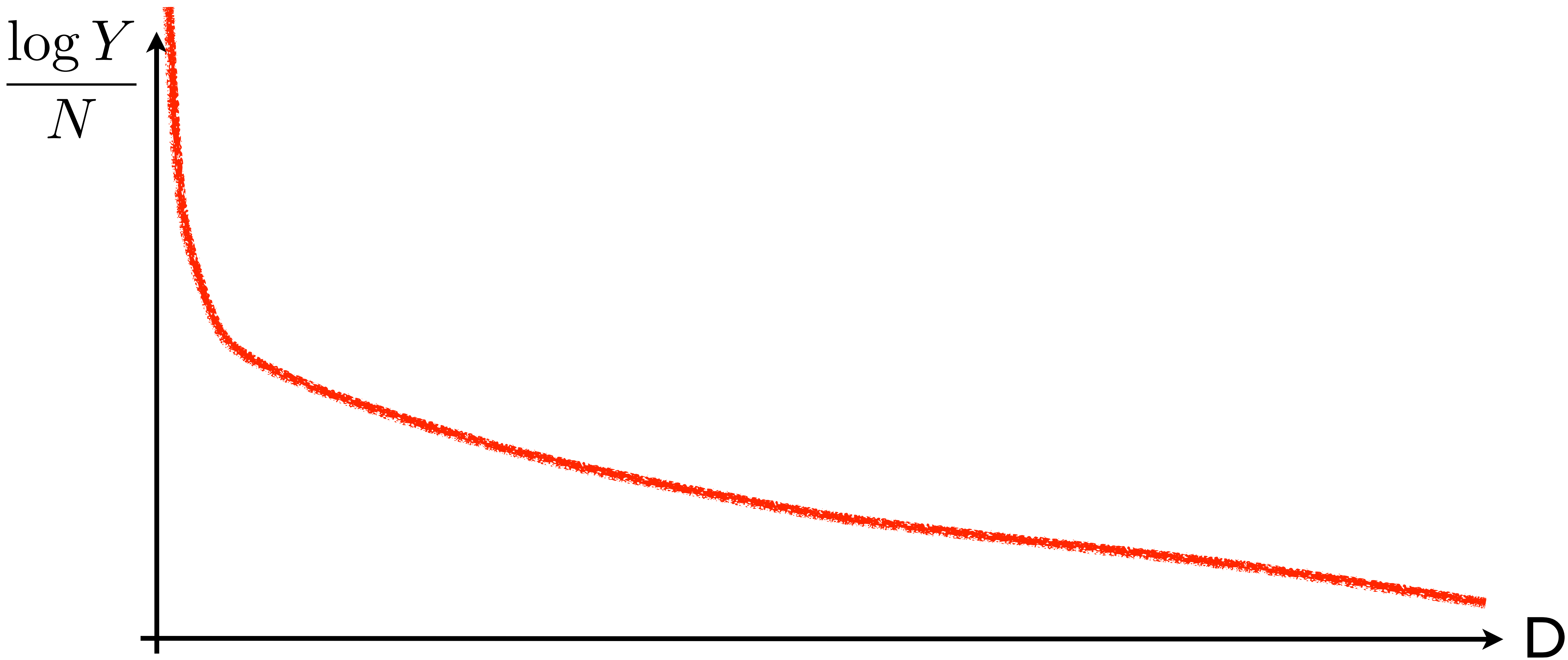

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A probabilistic approach to compressed sensing

Probabilistic reconstruction using:

$$P(\vec{x}|\vec{y}) = \frac{1}{Z} \prod_{i=1}^{N} \left[(1-\rho)\,\delta(x_i) + \rho\phi(x_i)\right] \prod_{\mu=1}^{M} \delta\left(y_\mu - \sum_{i=1}^{N} F_{\mu i} x_i\right)$$

Sampling from P(x|y) is optimal,

(even if we do not know the correct Φ(x) or the correct ρ)

Statistical physics and information theory tools can be readily used for

- Sampling
- Computing phase diagram
- etc etc...

The link with statistical physics and spin glasses

$$P(\vec{x}|\vec{y}) = \frac{1}{Z}\prod_{i=1}^{N} P(x_i) \prod_{\mu=1}^{M} \delta\left(y_\mu - \sum_{i=1}^{N} F_{\mu i} x_i\right)$$ with $$P(x_i) = (1-\rho)\delta(x_i) + \rho\phi(x_i)$$

$$P(\vec{x}|\vec{y}) = \frac{1}{Z} e^{-\sum_{i=1}^{N} \log P(x_i) - \frac{1}{2\Delta}\sum_{\mu=1}^{M}(y_\mu - \sum_{i=1}^{N} F_{\mu i} x_i)^2}$$

In physics, this is called a spin glass problem
This is studied since the early 80's

The link with statistical physics and spin glasses

$$P(\vec{x}|\vec{y}) = \frac{1}{Z}\prod_{i=1}^{N} P(x_i)\prod_{\mu=1}^{M}\delta\left(y_\mu - \sum_{i=1}^{N} F_{\mu i} x_i\right)$$ **with** $$P(x_i) = (1-\rho)\delta(x_i) + \rho\phi(x_i)$$

Hamiltonian

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Hamiltonian

Partition sum

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Disordered interaction

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Mean-field
long-range interactions

$$P(\vec{x}|\vec{y}) = \frac{1}{Z} e^{-\sum_{i=1}^{N} \log P(x_i) - \frac{1}{2\Delta}\sum_{\mu=1}^{M} (y_\mu - \sum_{i=1}^{N} F_{\mu i} x_i)^2}$$

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Statistical Physics approach (FK et al.)
+rigorous (Donoho, Montanari et al.)

How to sample?

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Solution number 1:using Markov-Chain Monte-Carlo

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Solution number 1:using Markov-Chain Monte-Carlo

Used in the litterature for ultrasound imaging (cf : Quinsac et al, 2011...)

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Solution number 2: estimate the marginal probabilities with a message passing algorithm

If we do it correctly, then the solution is given by $a_i = \int dx_i P_i(x_i) x_i$

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Solution number 2: estimate the marginal probabilities with a message passing algorithm

If we do it correctly, then the solution is given by $a_i = \int dx_i P_i(x_i) x_i$

In this model, this can be done <u>exactly</u> (for large N,M) using an approach known as:

1. Thouless-Anderson-Parlmer, or Cavity method in physics
   Bethe-Peierls, Onsager ('35) Parisi and Mezard ('02)
2. Belief propagation in artificial intelligence (Pearl, '82)
3. Sum-product in coding theory (Gallager, '60)
3. Approximate Message Passing in compressed sensing
   Rangan, Montanari...

How does BP works?

Gibbs free energy approach: $\log Z = \max_{\{\mathcal{P}(\vec{x})\}} f_{Gibbs}\left(\{\mathcal{P}(\vec{x})\}\right)$

With $f_{Gibbs}\left(\{\mathcal{P}(\vec{x})\}\right) = -\langle \log P(\vec{x}|\vec{y}) \rangle_{\mathcal{P}(\vec{x})} - \int d\vec{x}\mathcal{P}(\vec{x}) \log \mathcal{P}(\vec{x})$

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Mean-Field $\Rightarrow$ $\mathcal{P}(\vec{x}) = \prod_i \mathcal{P}_i(\vec{x}_i)$

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+Convergence problems

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(asymptotically)
exact in CS
with random matrices

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How does BP works?

Simplification thanks to the large connectivity limit:
Projection on first two moments is enough :

$$f\left(\{\mathcal{P}_i(x_i), \mathcal{P}_{ij}(x_i, x_j)\}\right) \Rightarrow f\left(\{\langle x_i\rangle, \langle x_i^2\rangle\}\right)$$

Belief-Propagation equations $\Rightarrow$

$$\begin{cases} \langle x_i\rangle^{t+1} = \langle x_i\rangle^t + \dfrac{\partial f}{\partial\langle x_i\rangle} \\ \langle x_i^2\rangle^{t+1} = \langle x_i^2\rangle^t + \dfrac{\partial f}{\partial\langle x_i^2\rangle} \end{cases}$$

The Belief-Propagation algorithm

Iterate these variables

$$U_i^{(t+1)} = \frac{\alpha}{M} \sum_\mu \frac{1}{\Delta_\mu + \gamma^{(t)}}$$

$$V_i^{(t+1)} = \sum_\mu F_{\mu i} \frac{(y_\mu - \alpha_\mu^{(t)})}{\Delta_\mu + \gamma_\mu^{(t)}} + f_a\left(U_i^{(t)}, V_i^{(t)}\right) \frac{\alpha}{M} \sum_\mu \frac{1}{\Delta_\mu + \gamma^{(t)}}$$

$$\alpha_\mu^{(t+1)} = \sum_i F_{\mu i} f_a(U_i^{(t+1)}, V_i^{(t+1)}) - \frac{(y_\mu - \alpha_\mu^{(t)})}{\Delta_\mu + \gamma^{(t)}} \frac{1}{N} \sum_i \frac{\partial f_a}{\partial Y}\left(U_i^{(t+1)}, V_i^{(t+1)}\right)$$

$$\gamma^{(t+1)} = \frac{1}{N} \sum_i f_c(U_i^{(t+1)}, V_i^{(t+1)})$$

Using these functions:

$$f_a(X,Y) = \left[\frac{\rho Y}{(1+X)^{3/2}} e^{Y^2/(2(1+X))}\right] \left[1 - \rho + \frac{\rho}{(1+X)^{1/2}} e^{Y^2/(2(1+X))}\right]^{-1}$$

$$f_c(X,Y) = \left[\frac{\rho}{(1+X)^{3/2}} e^{Y^2/(2(1+X))} \left(1 + \frac{Y^2}{1+X}\right)\right] \left[1 - \rho + \frac{\rho}{(1+X)^{1/2}} e^{Y^2/(2(1+X))}\right]^{-1} - f_a(X,Y)^2$$

And finally at the end:

$$\langle x_i \rangle = f_a\left(U_i, V_i\right)$$

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Simple algebraic operations!

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Complexity is O(N^2×convergence time)

The Belief-Propagation algorithm: How to learn the parameter in the Prior?

Three parameters $\rho, \overline{x}, \sigma$ $\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\overline{x})^2/(2\sigma^2)}$

Compute the Bethe free-entropy $F = \log Z$ using the BP messages.

Compute the gradient $\left(\frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \overline{x}}, \frac{\partial F}{\partial \sigma} \right)$

and update parameters to maximize F at each step of the BP iteration

Learning makes the algorithm faster
(equivalent to Expectation-Maximization)

Analysis of the algorithm

The performance of the algorithm for a given distribution of signals can be analyzed using a method knows as density evolution (coding theory) or replica method (physics)

$$Z(y) = \int \prod_{i=1}^{N} dx_i P(x|y) \qquad F(\vec{y}) = -\log Z(\vec{y})$$

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Averaging over a signal distributiion (ex: Gauss Bernoulli)

$F_{\mu i}$ iid Gaussian, variance $1/N$

$y_\mu = \sum_{i=1}^{N} F_{\mu i} x_i^0$ where x_i^0 are iid distributed from $(1-\rho_0)\delta(x_i^0) + \rho_0 \phi_0(x_i)$

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Replica method ⇒ $\overline{\log Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}$

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$$\Phi(Q,q,m,\hat{Q},\hat{q},\hat{m}) = -\frac{1}{2N}\sum_\mu \frac{q-2m+\rho+\Delta_\mu}{\Delta_\mu+Q-q} - \frac{1}{2N}\sum_\mu \log\left(\Delta_\mu+Q-q\right) + \frac{Q\hat{Q}}{2} - m\hat{m} + \frac{q\hat{q}}{2}$$

$$+\int \mathcal{D}z \int \mathrm{d}x_0 \left[(1-\rho_0)\delta(x_0)+\rho_0\phi_0(x_0)\right] \log\left\{\int \mathrm{d}x\, e^{-\frac{\hat{Q}+\hat{q}}{2}x^2 + \hat{m}x x_0 + z\sqrt{\hat{q}}x} \left[(1-\rho)\delta(x)+\rho\phi(x)\right]\right\}$$

Order parameters:

$$Q = \frac{1}{N}\sum_i \langle x_i^2\rangle \qquad q = \frac{1}{N}\sum_i \langle x_i\rangle^2 \qquad m = \frac{1}{N}\sum_i x_i^0\langle x_i\rangle$$

Mean square error:

$$E = \frac{1}{N}\sum_i \left(\langle x_i\rangle - x_i^0\right)^2 = q - 2m + \langle (x_i^0)^2\rangle_0$$

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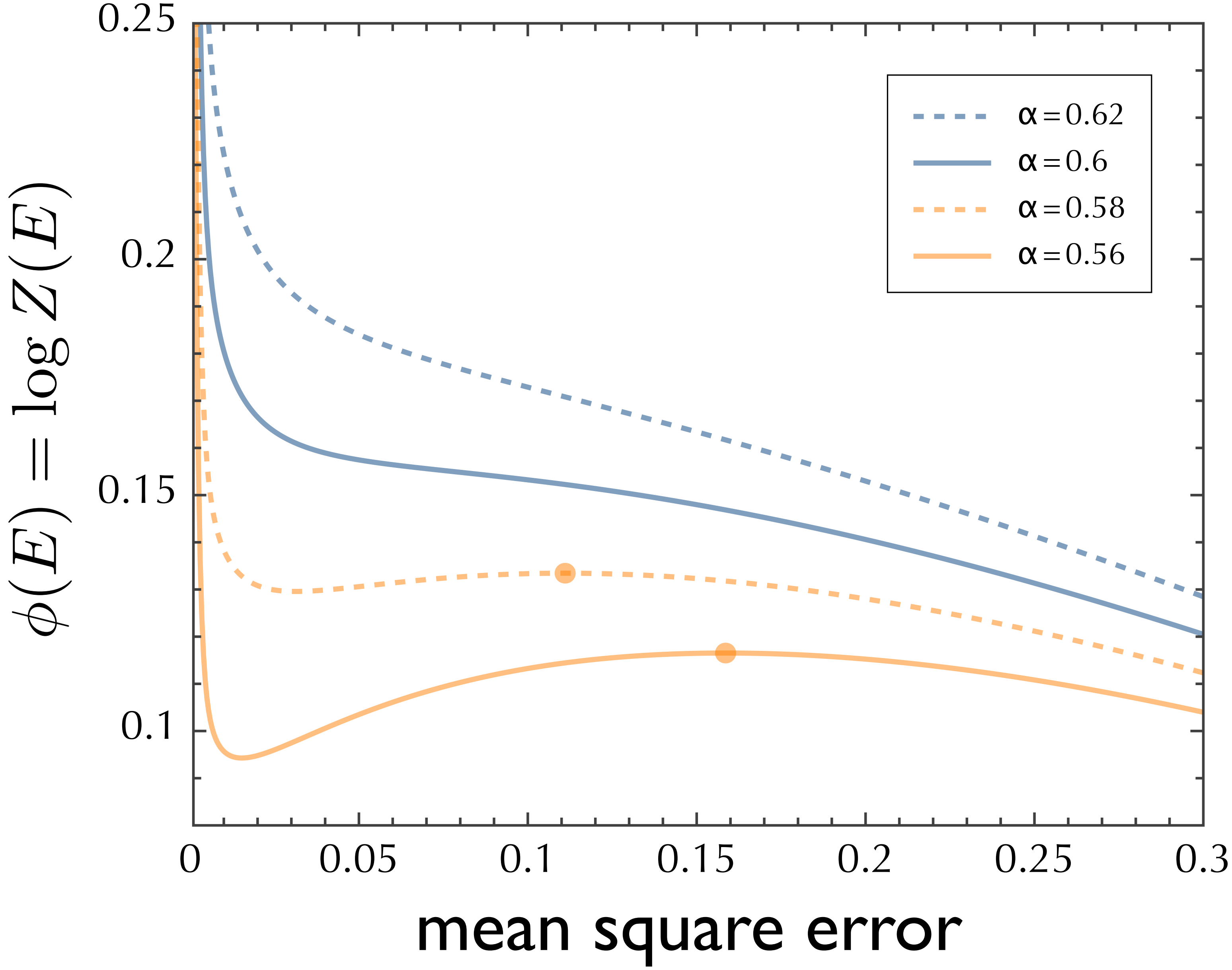
Computing the free entropy

Example with ρ_0=0.4, and Φ_0 a Gaussian distribution with zero mean and unit variance



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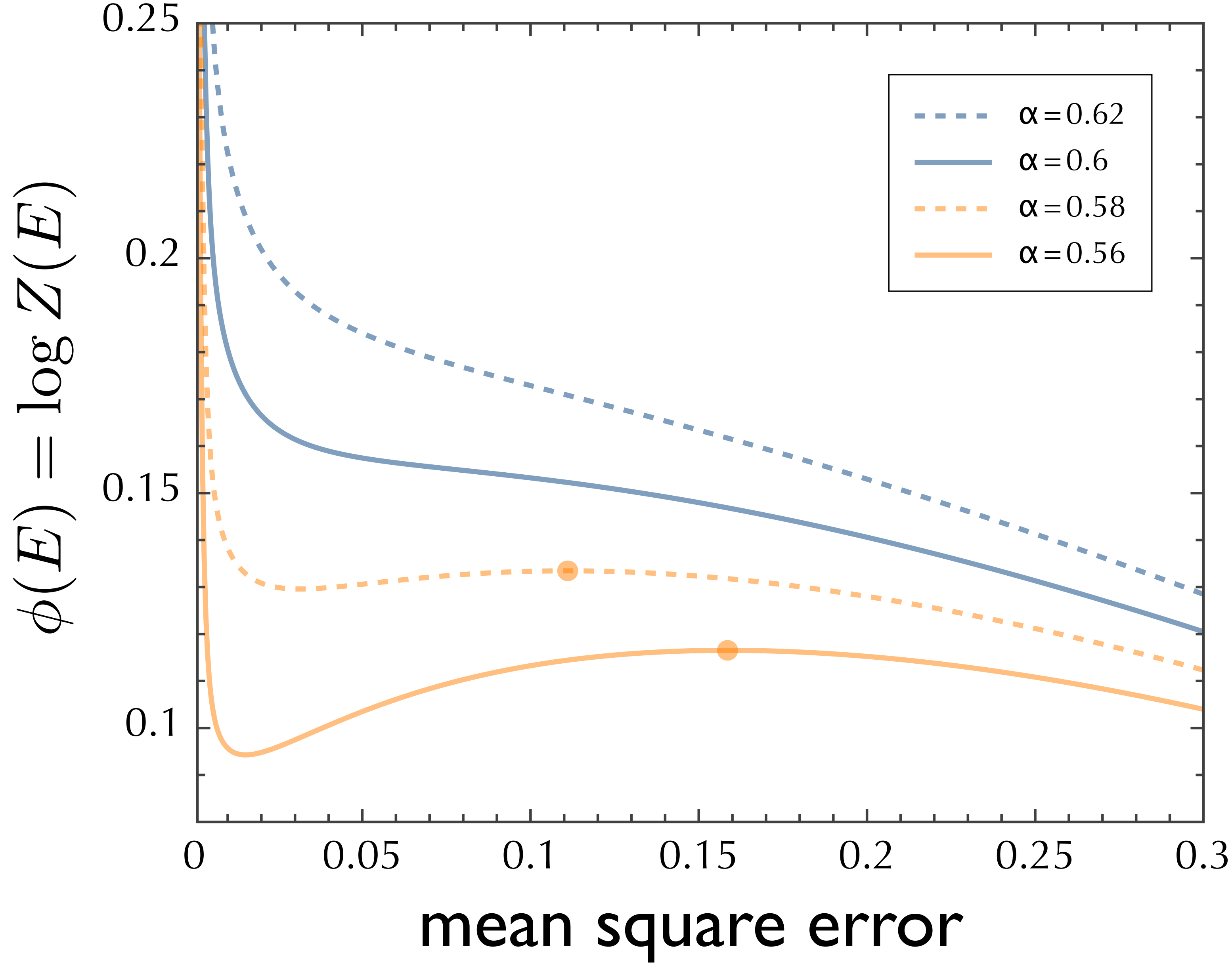

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- Maximum is at E=0 (as long as $\alpha > \rho_0$): Equilibrium behavior dominated by the original signal

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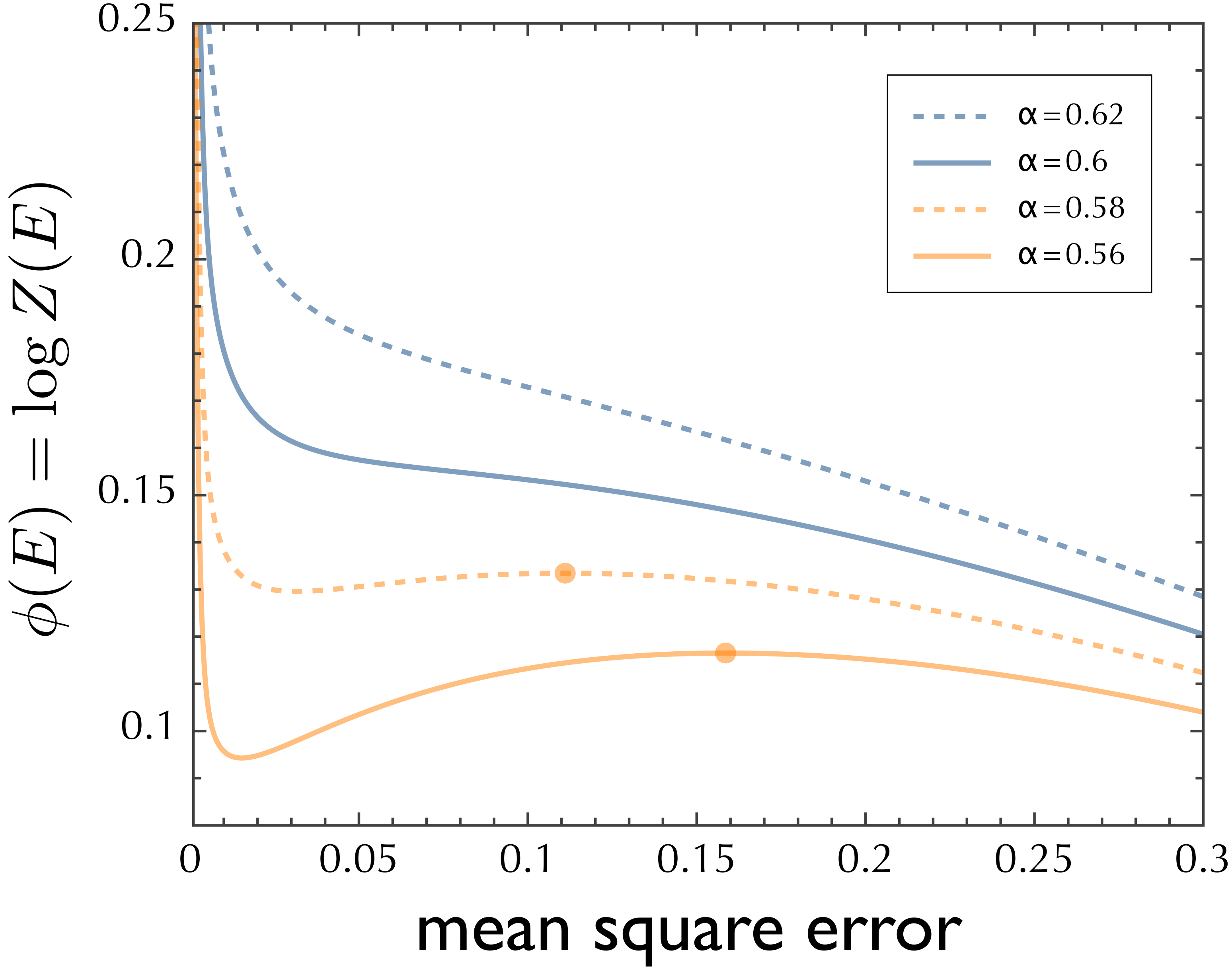

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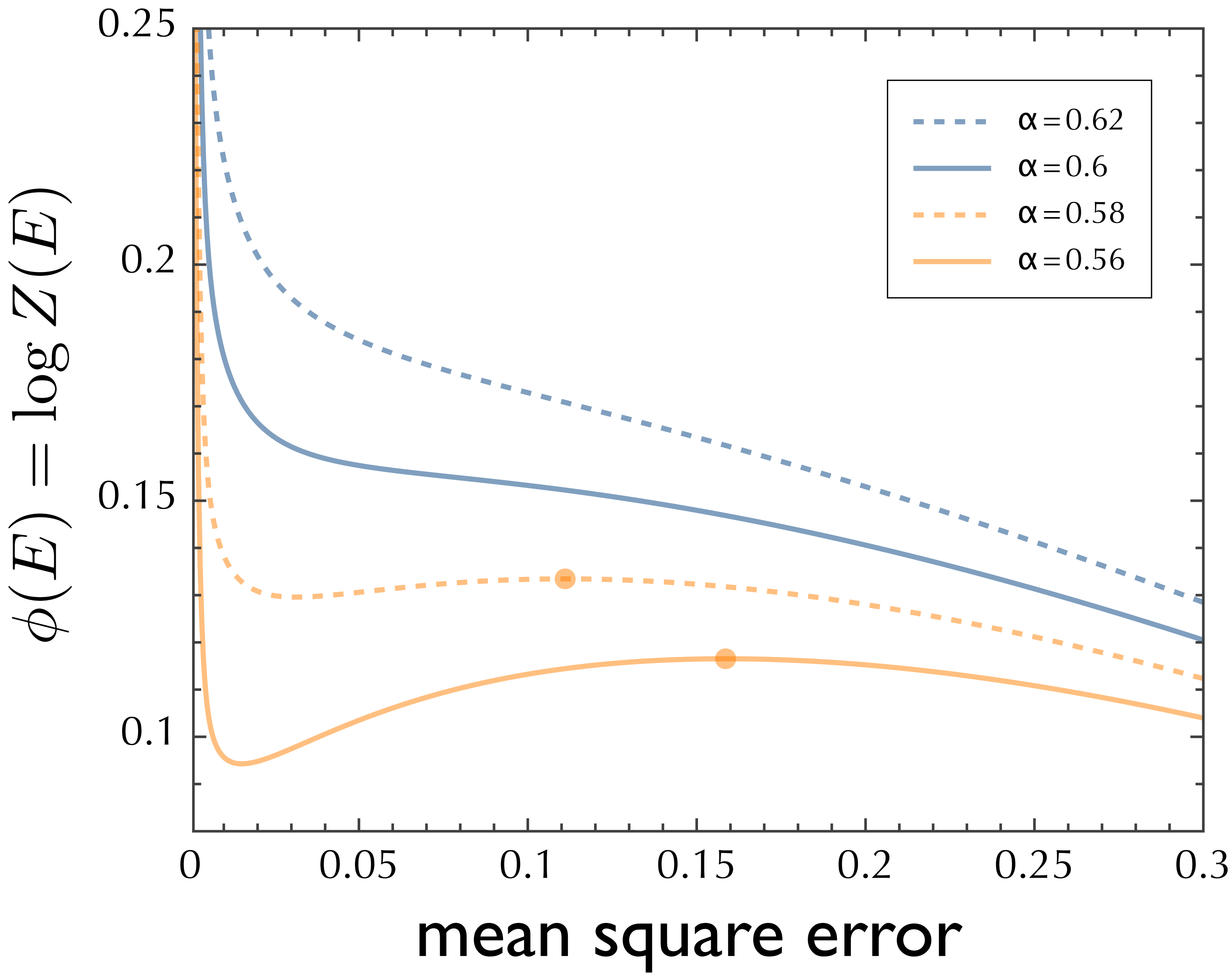

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- A steepest ascent dynamics starting from large E would reach the signal for α>0.58, but would stay block in the meta-stable state for α<0.58, even if the true equilibrium is at E=0.

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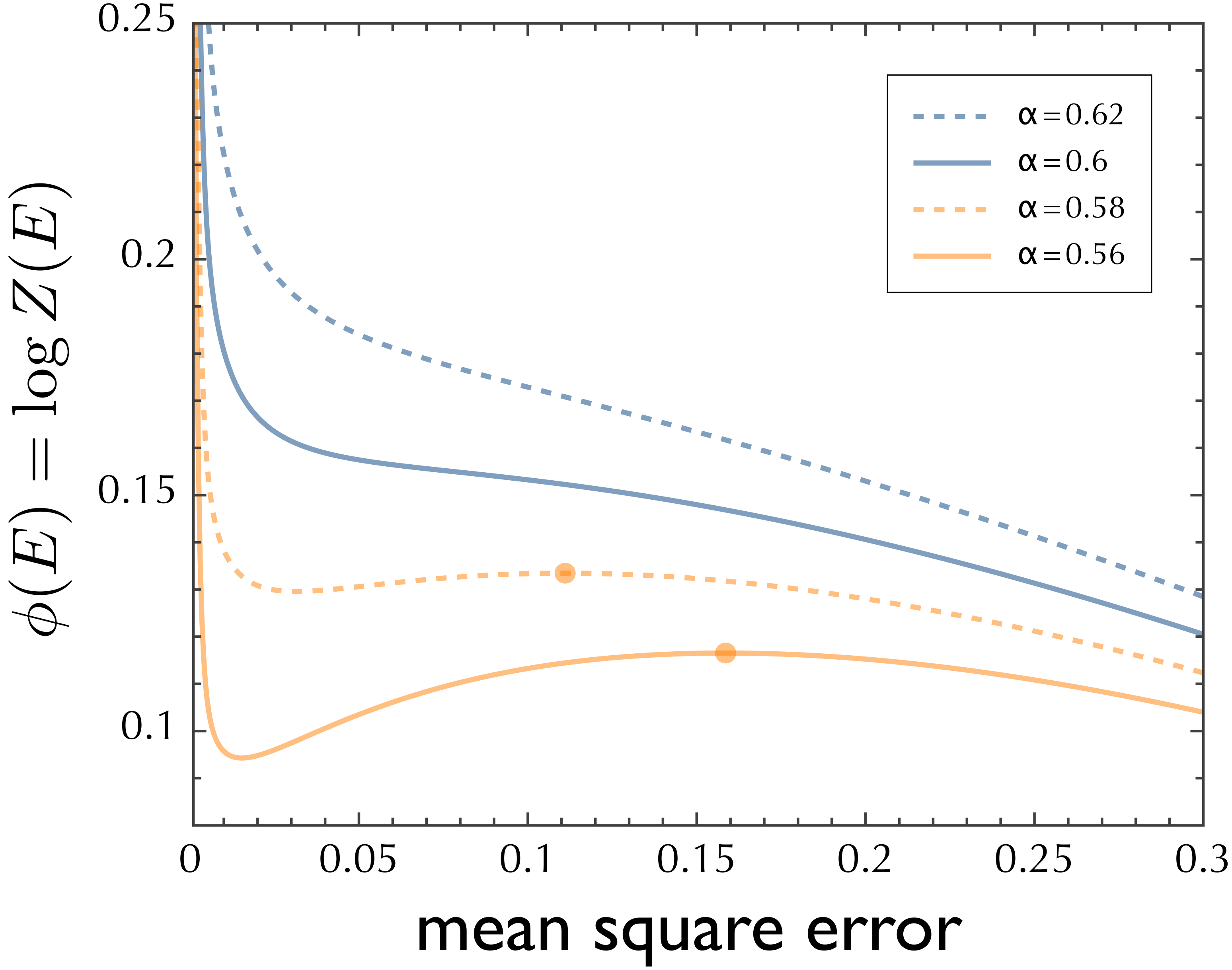

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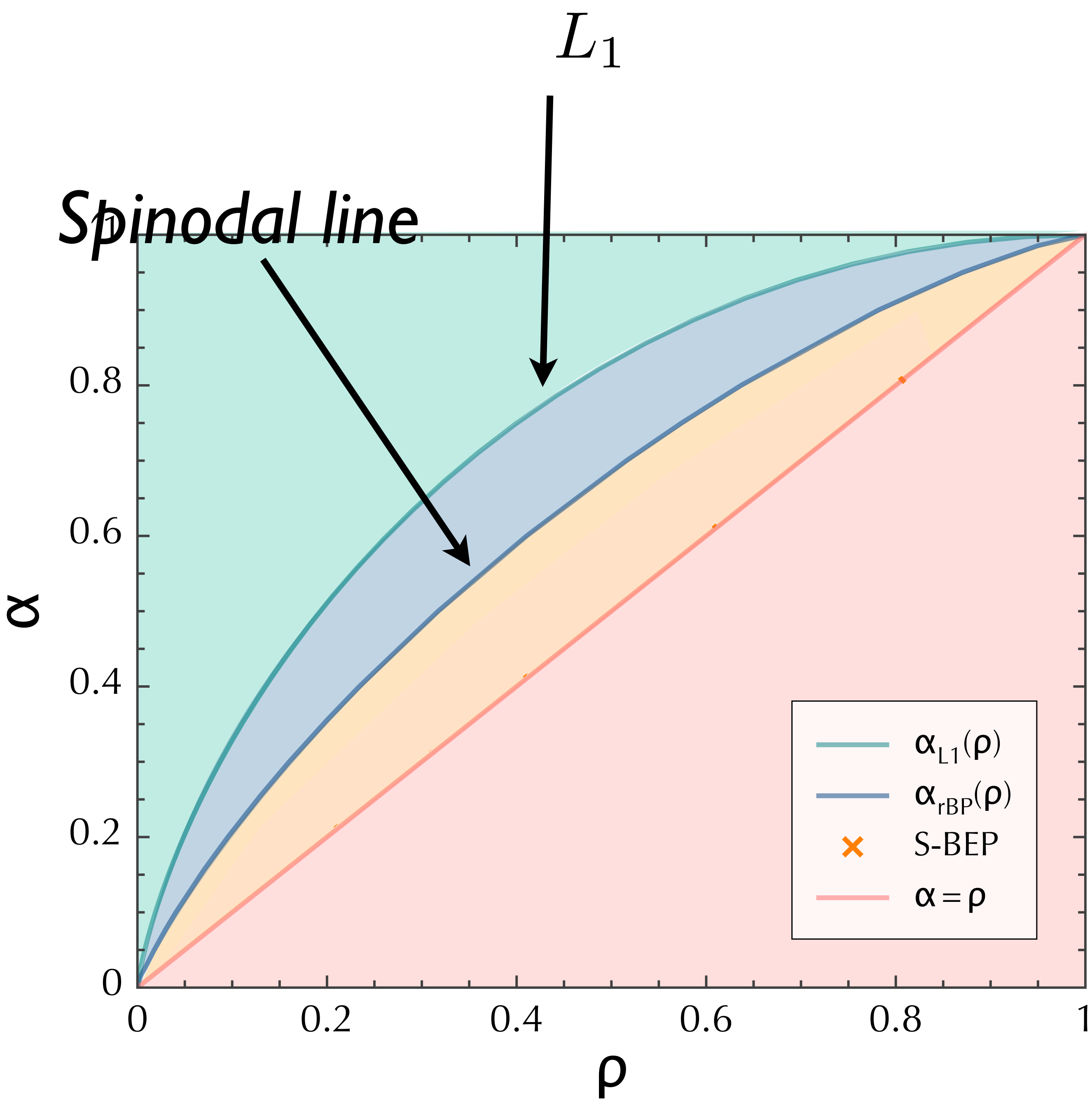
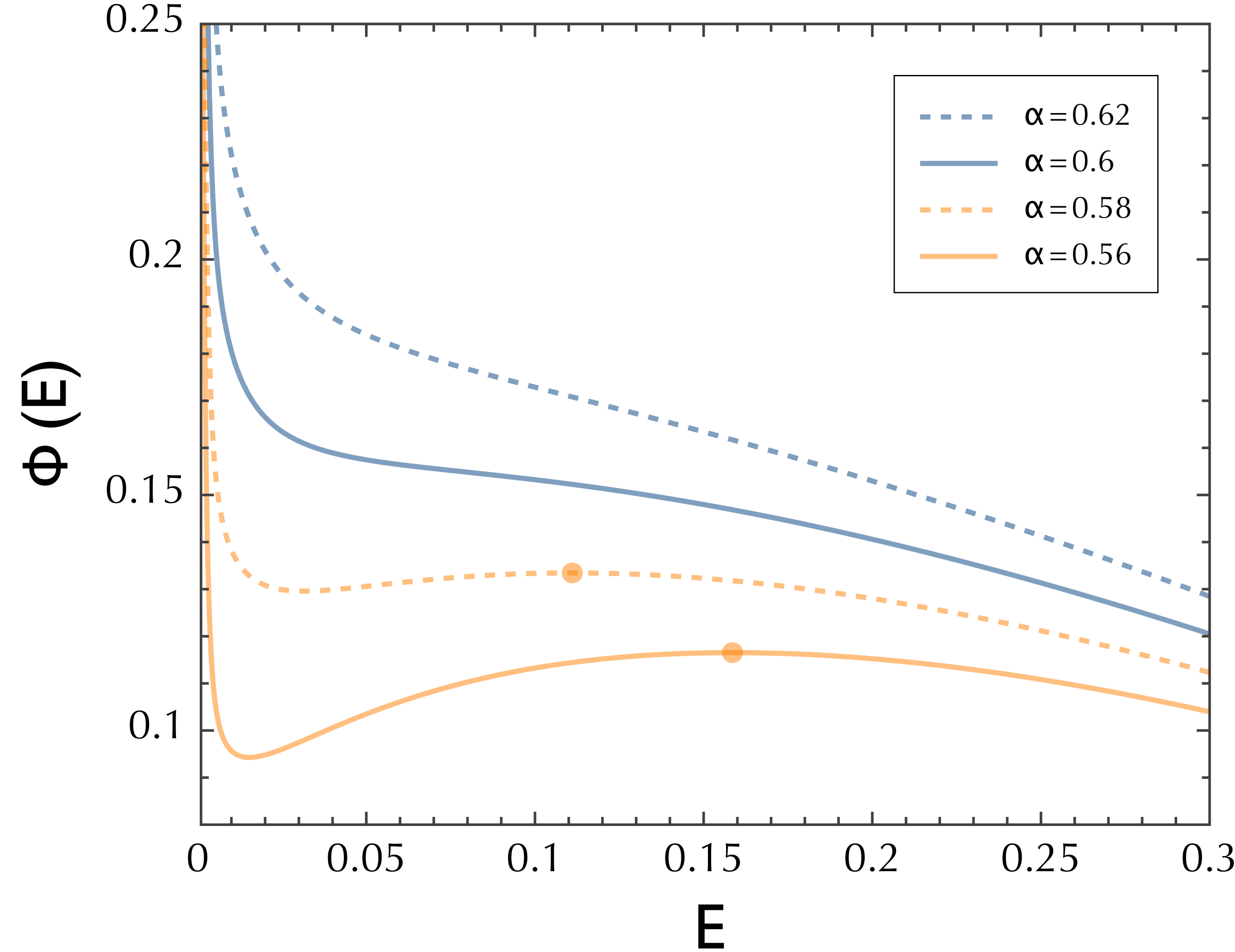
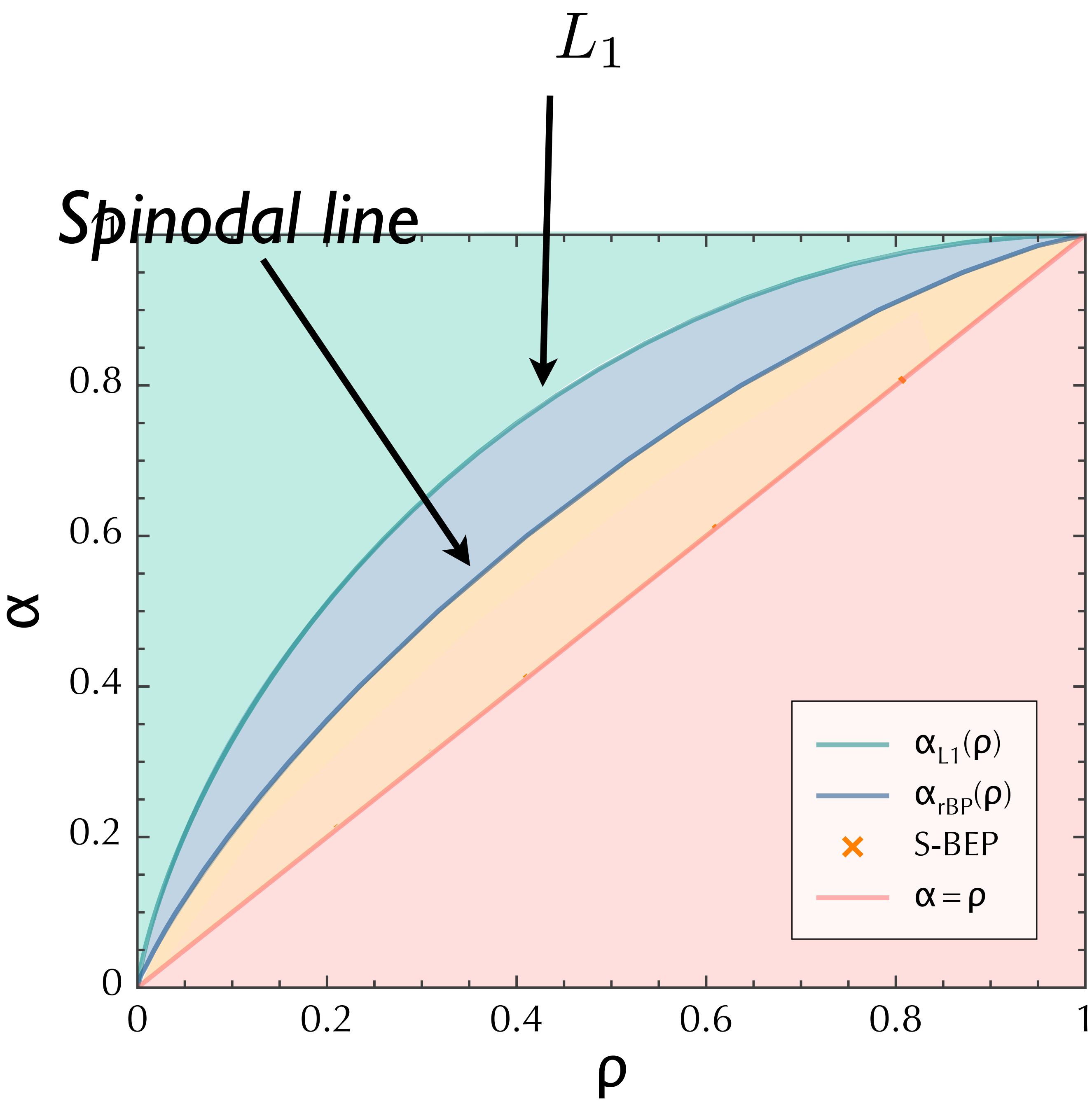
- A steepest ascent dynamics starting from large E would reach the signal for $\alpha>0.58$, but would stay block in the meta-stable state for $\alpha<0.58$, even if the true equilibrium is at E=0.
- Similarity with supercooled liquids

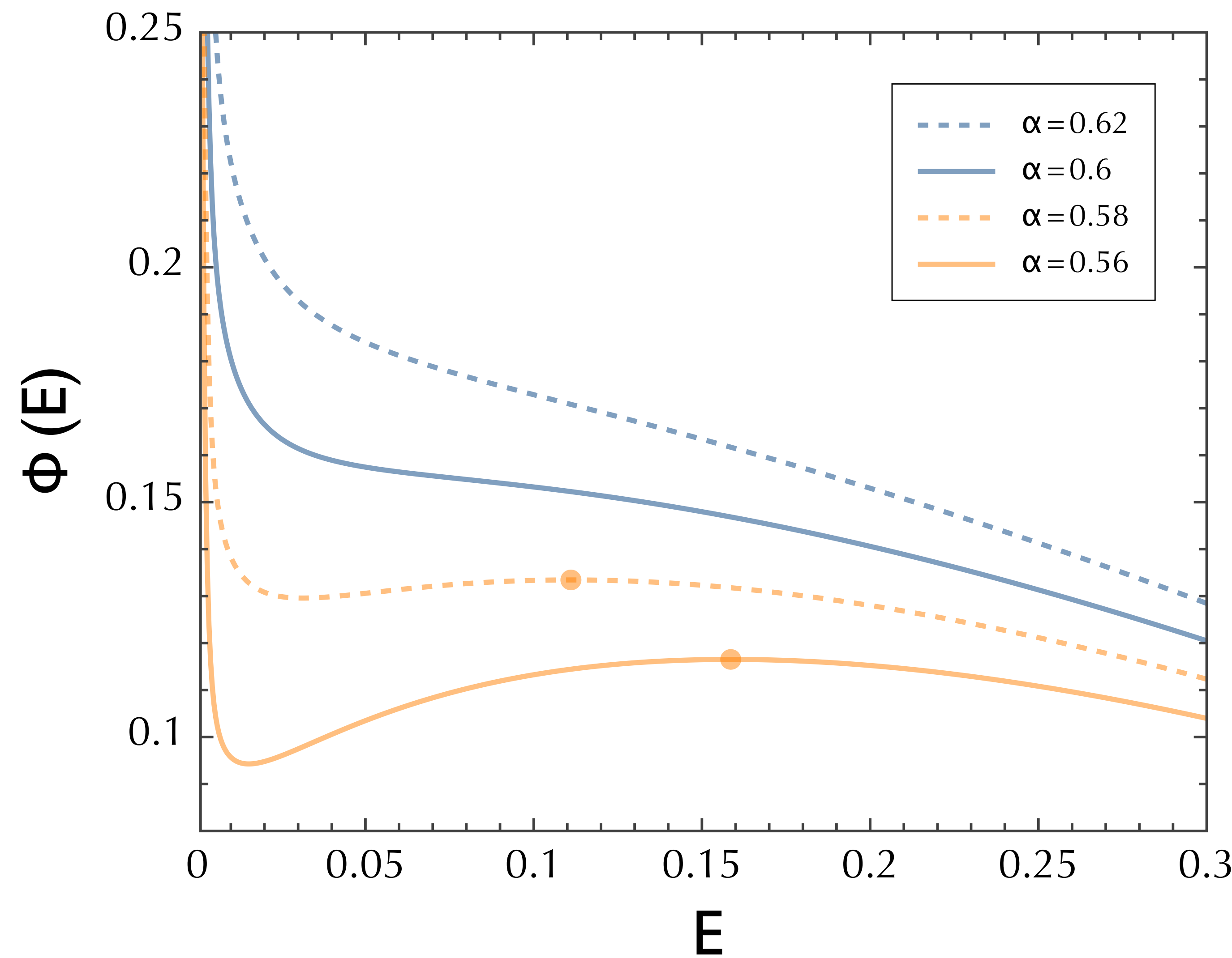
Computing the Phase Diagram





Computing the Phase Diagram

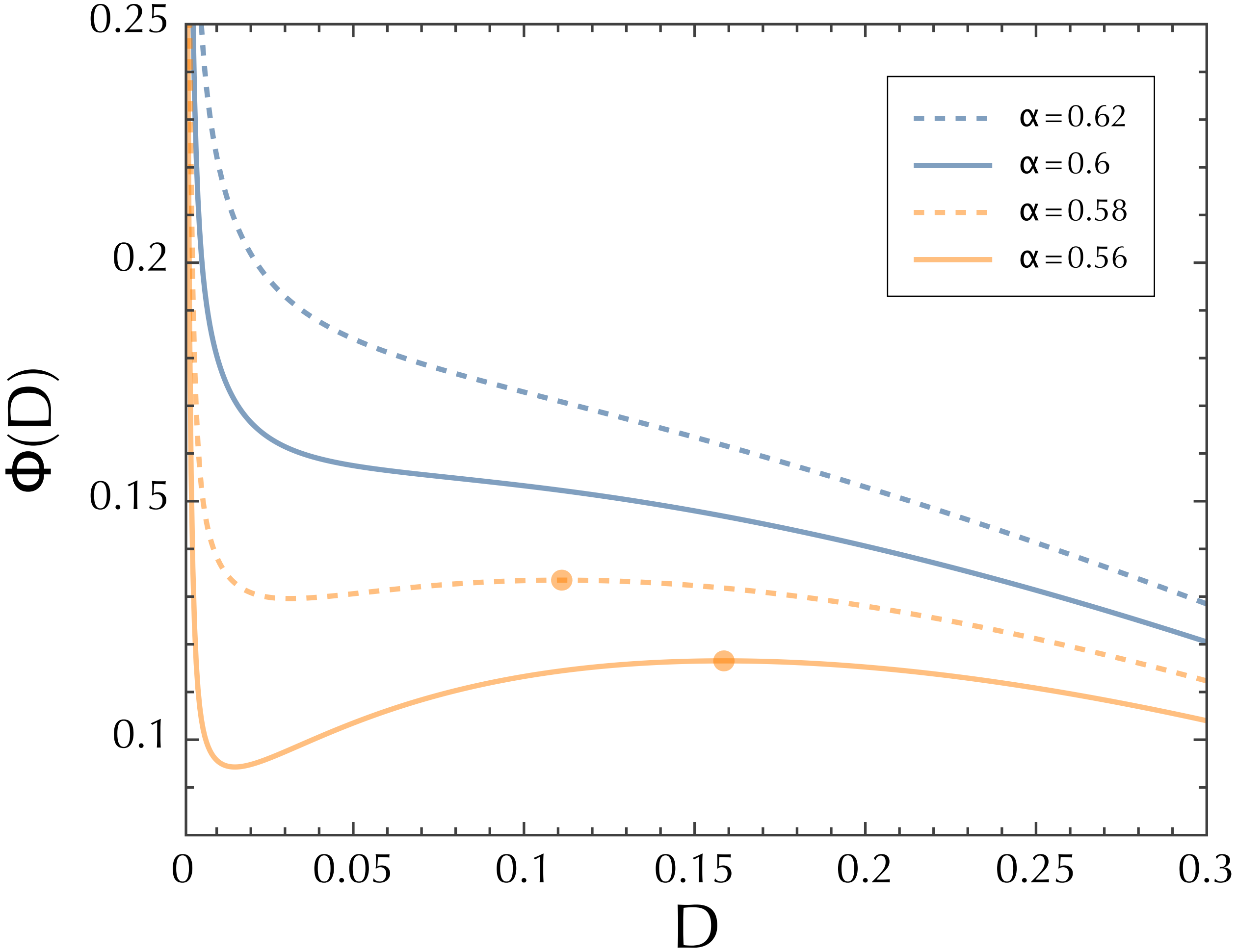




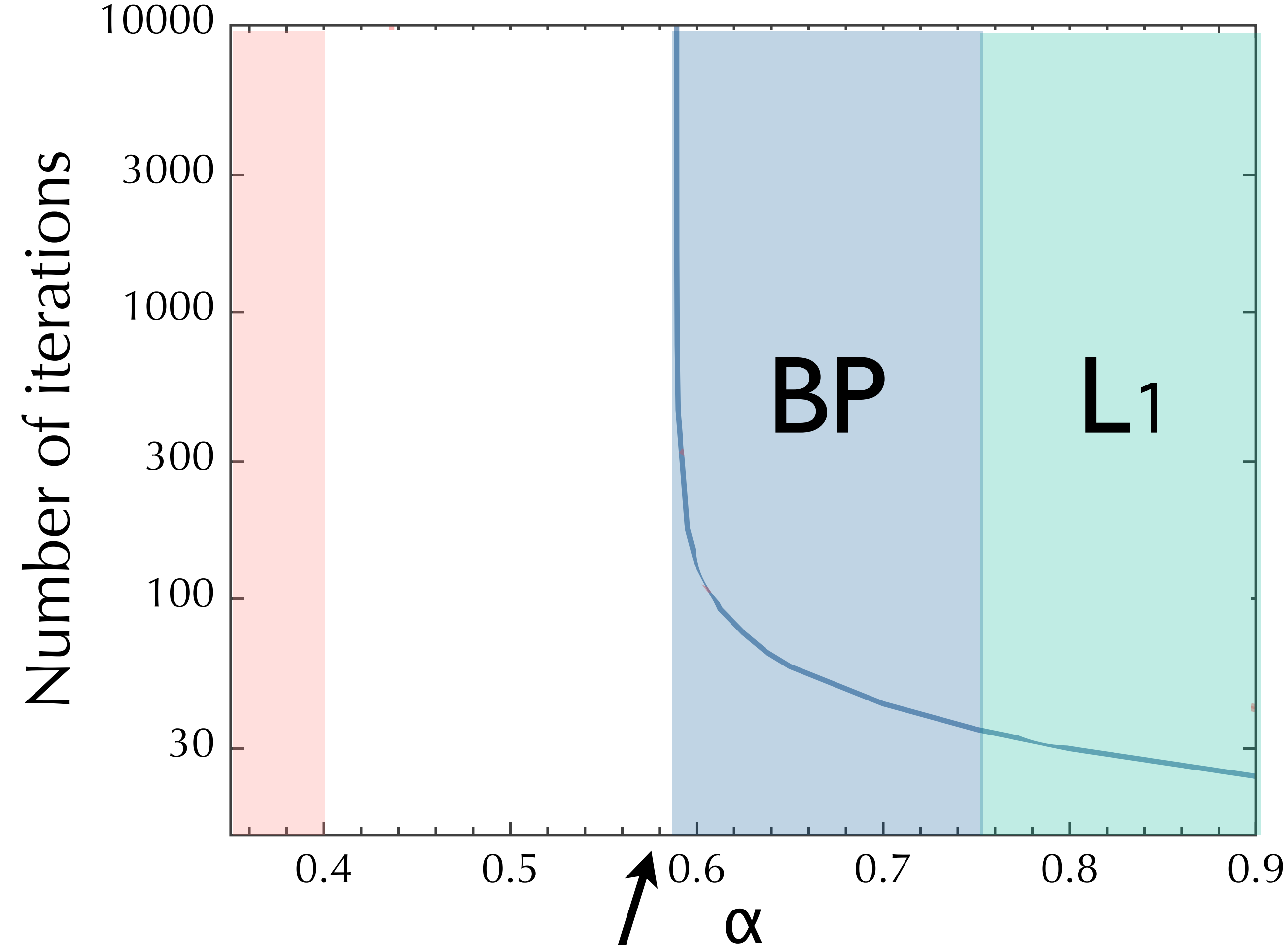
A steepest ascent of the free entropy allows
a perfect reconstruction until the spinodal line.
This is more efficient than L_1-minimization

Thermodynamic potential

BP convergence time



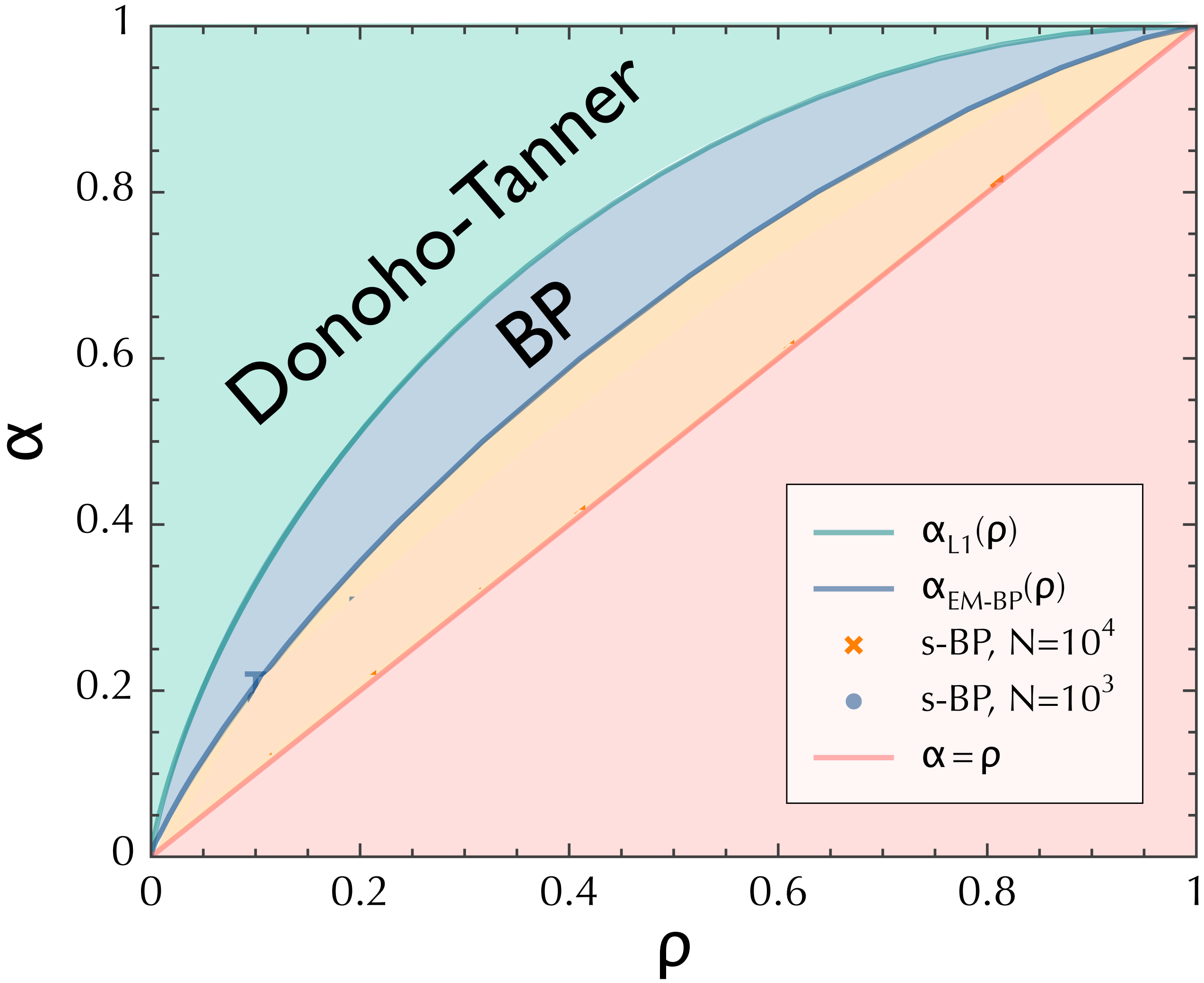
distance to native state



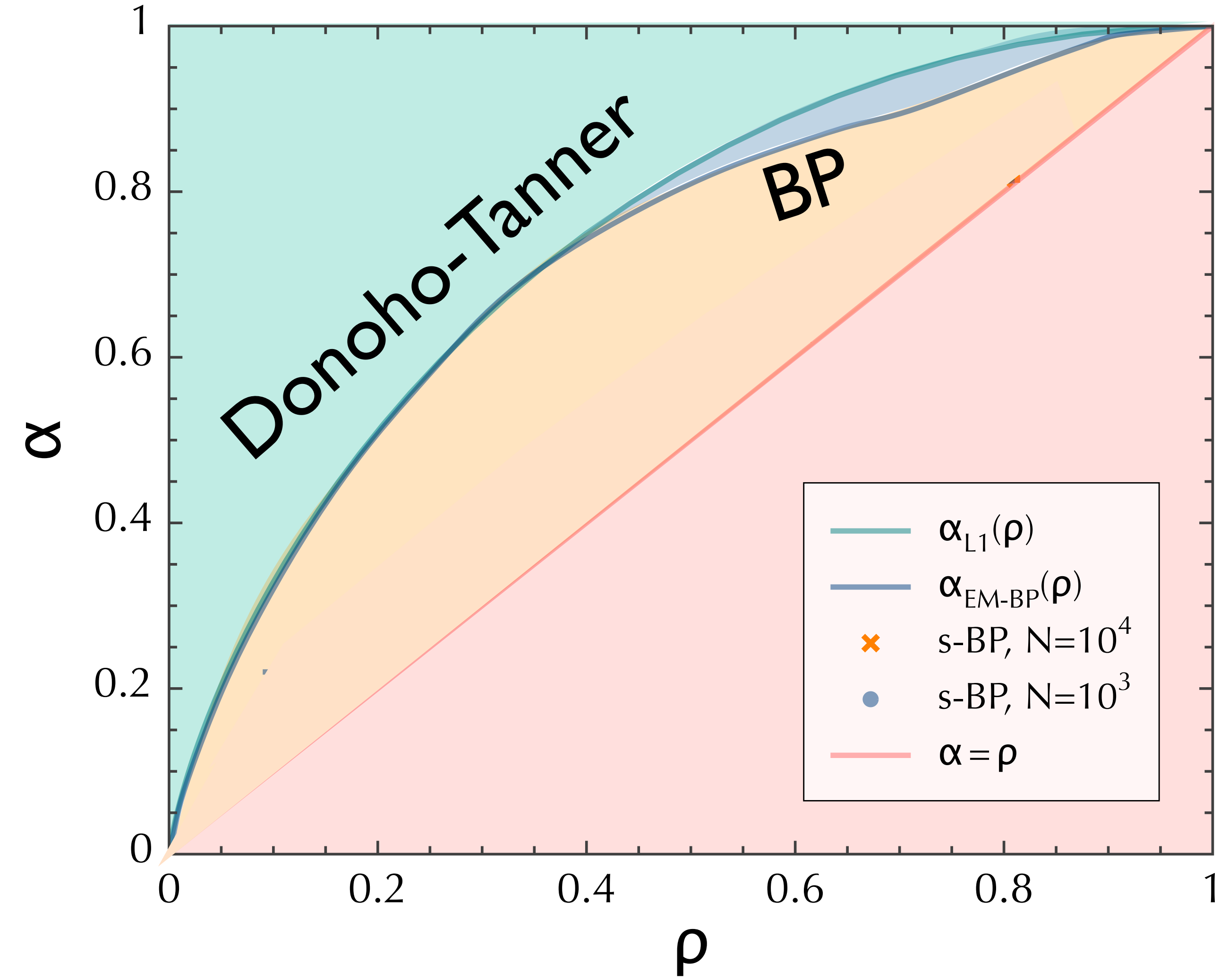
Spinodal transition
(supercooled limit)

Trying different type of signals

The limit depends on the type of signal
(while the Donoho-Tanner is universal)



Gauss-Bernoulli signal



Binary signals

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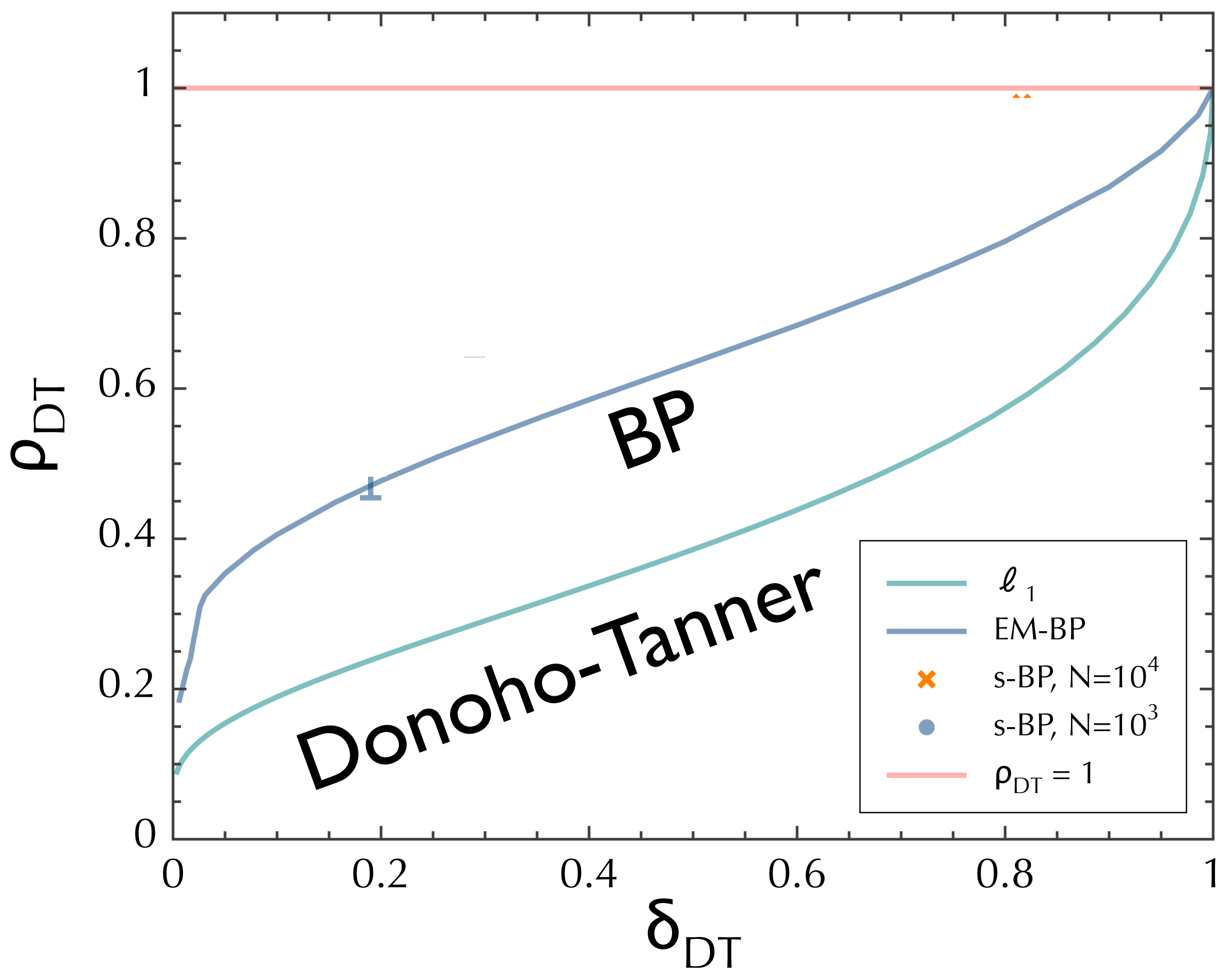

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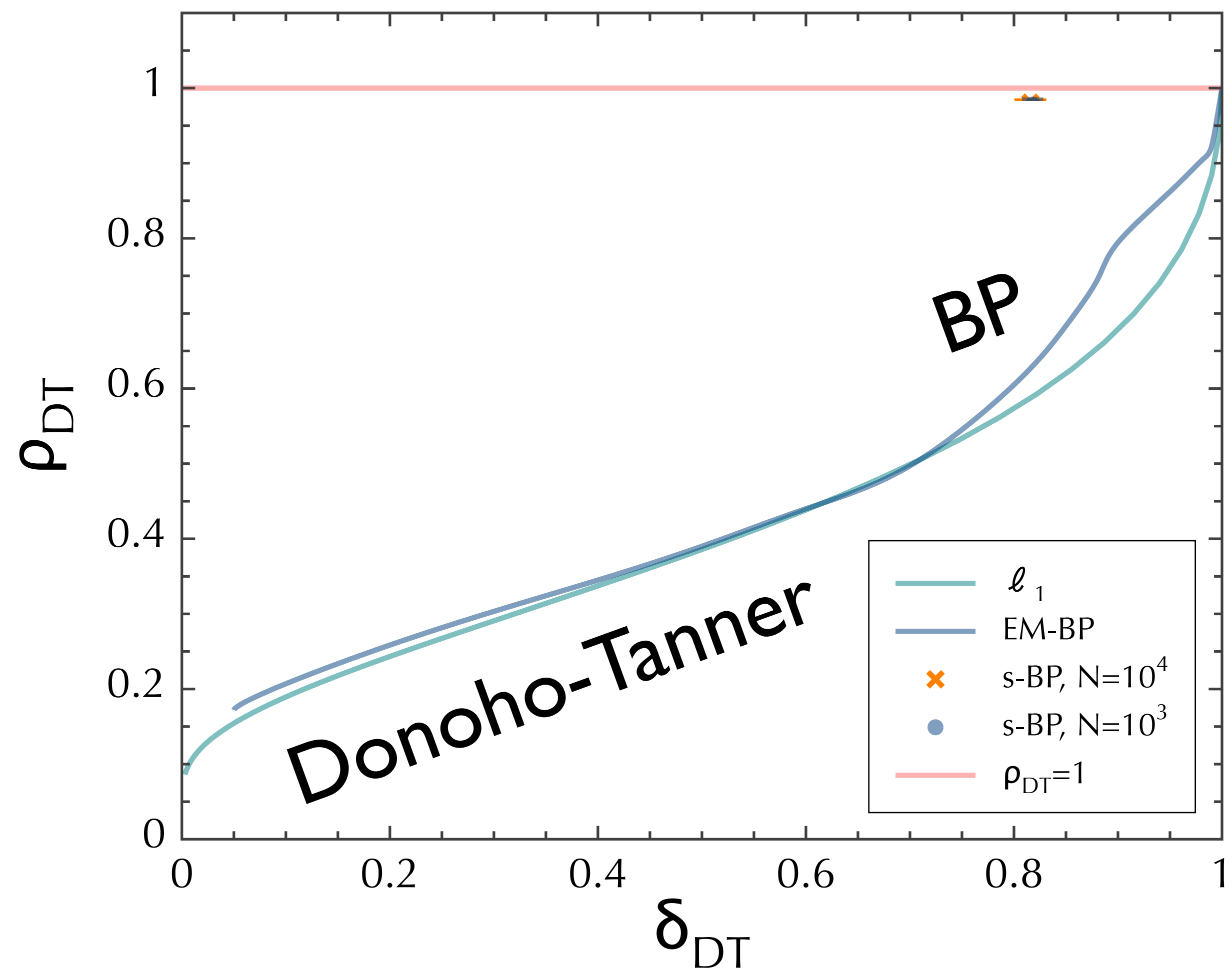
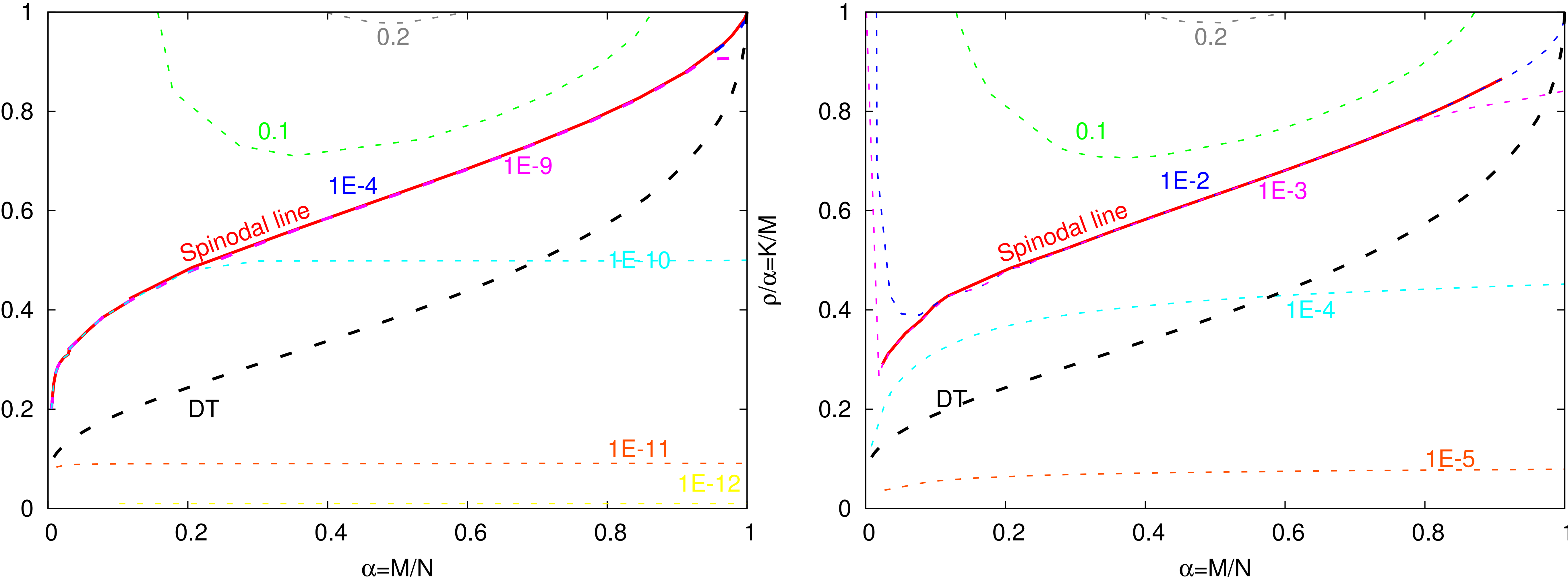


Binary signals

BP is Robust to noise



Noise with $\sigma = 10^{-5}$

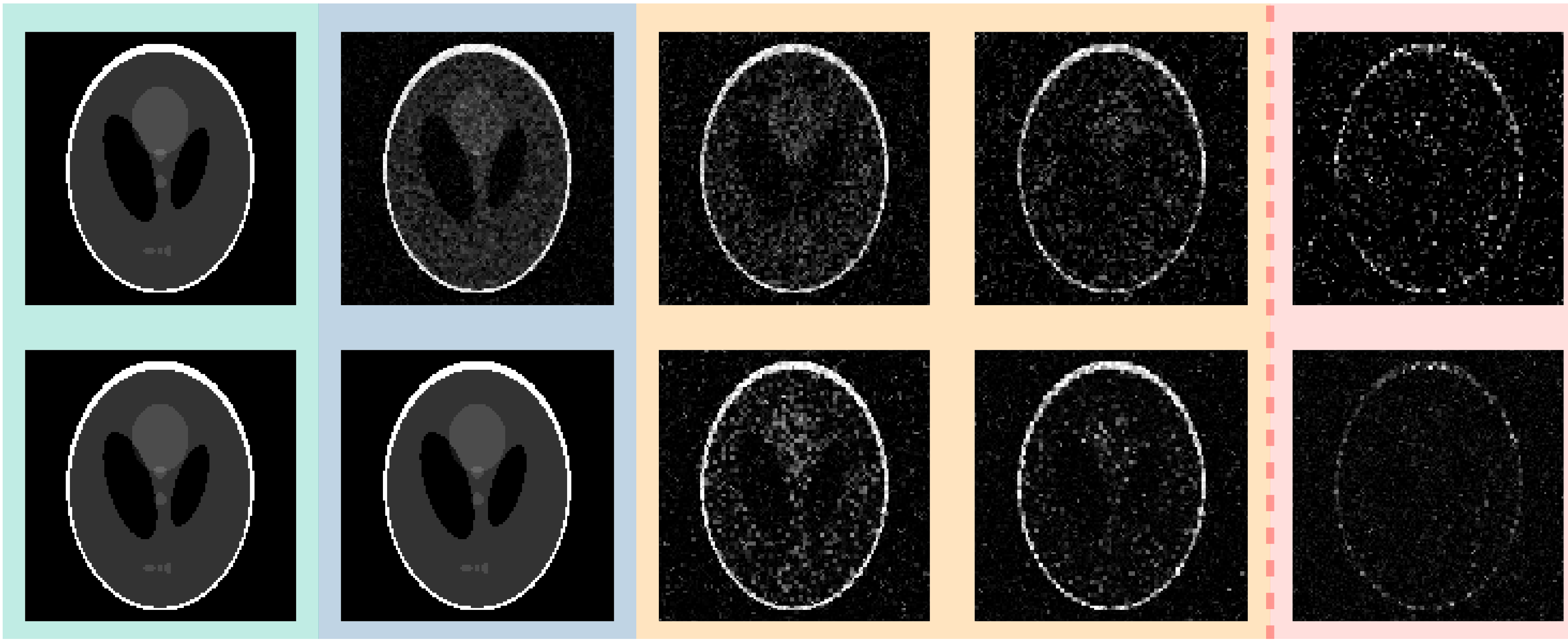
Noise $\sigma = 10^{-2}$

A more complex signal

$\alpha = \rho \approx 0.15$

L_1

BP



Shepp-Logan phantom, in the Haar-wavelet representation

$\alpha = 0.5$ $\alpha = 0.4$ $\alpha = 0.3$ $\alpha = 0.2$ $\alpha = 0.1$

BP + probabilistic approach

- Efficient and fast
- Robust to noise
- Very flexible (more information can be put in the prior)

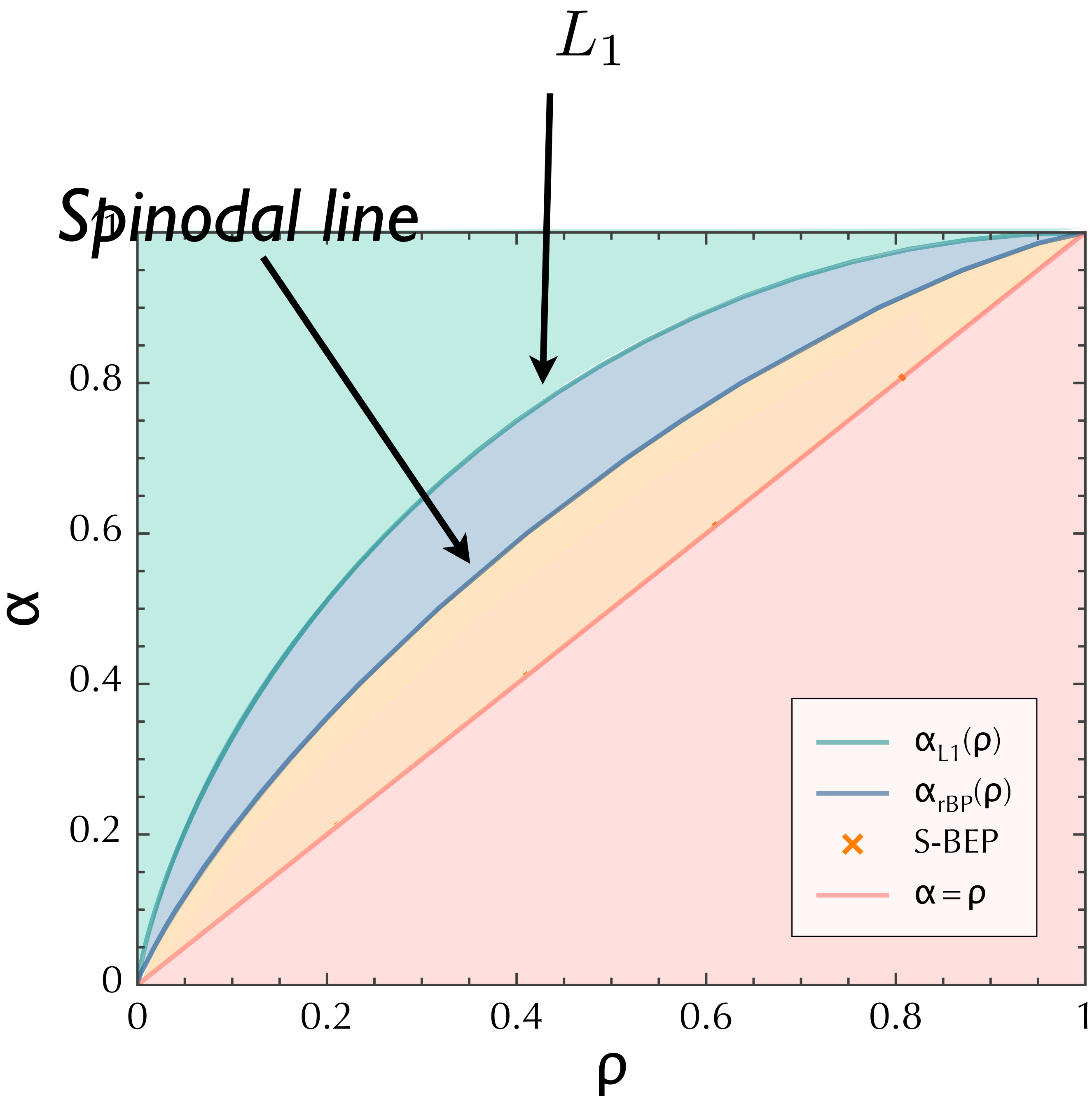
$$P(\vec{x}|\vec{y}) = \frac{1}{Z} \prod_{i=1}^{N} \left[(1-\rho)\,\delta(x_i) + \rho\phi(x_i)\right] \prod_{\mu=1}^{M} \delta\left(y_\mu - \sum_{i=1}^{N} F_{\mu i} x_i\right)$$

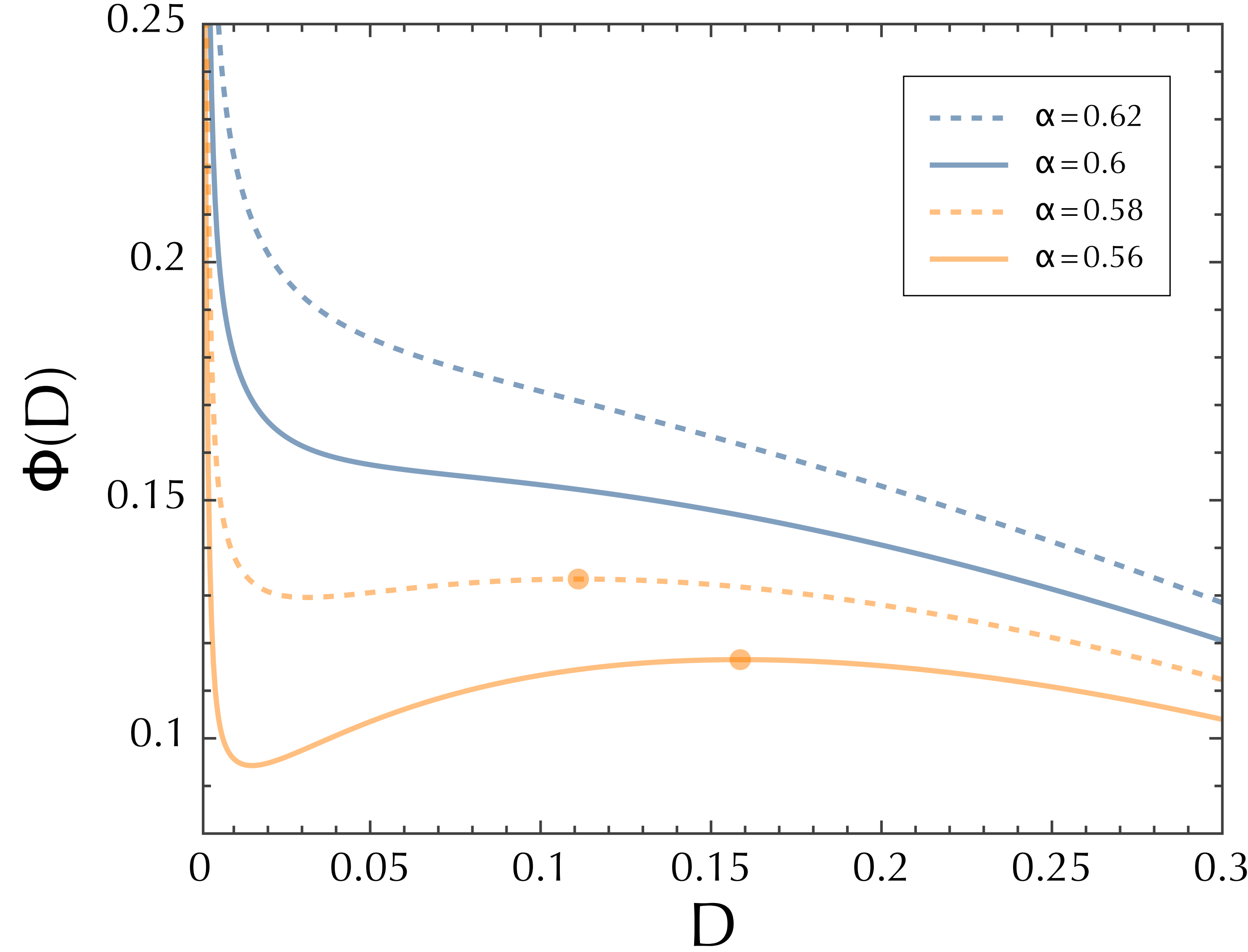
Our work

A statistical physics approach to compressed sensing

- A probabilistic approach to reconstruction
- The Belief Propagation algorithm
- **Seeded measurements matrices**

This is good, but not good enough





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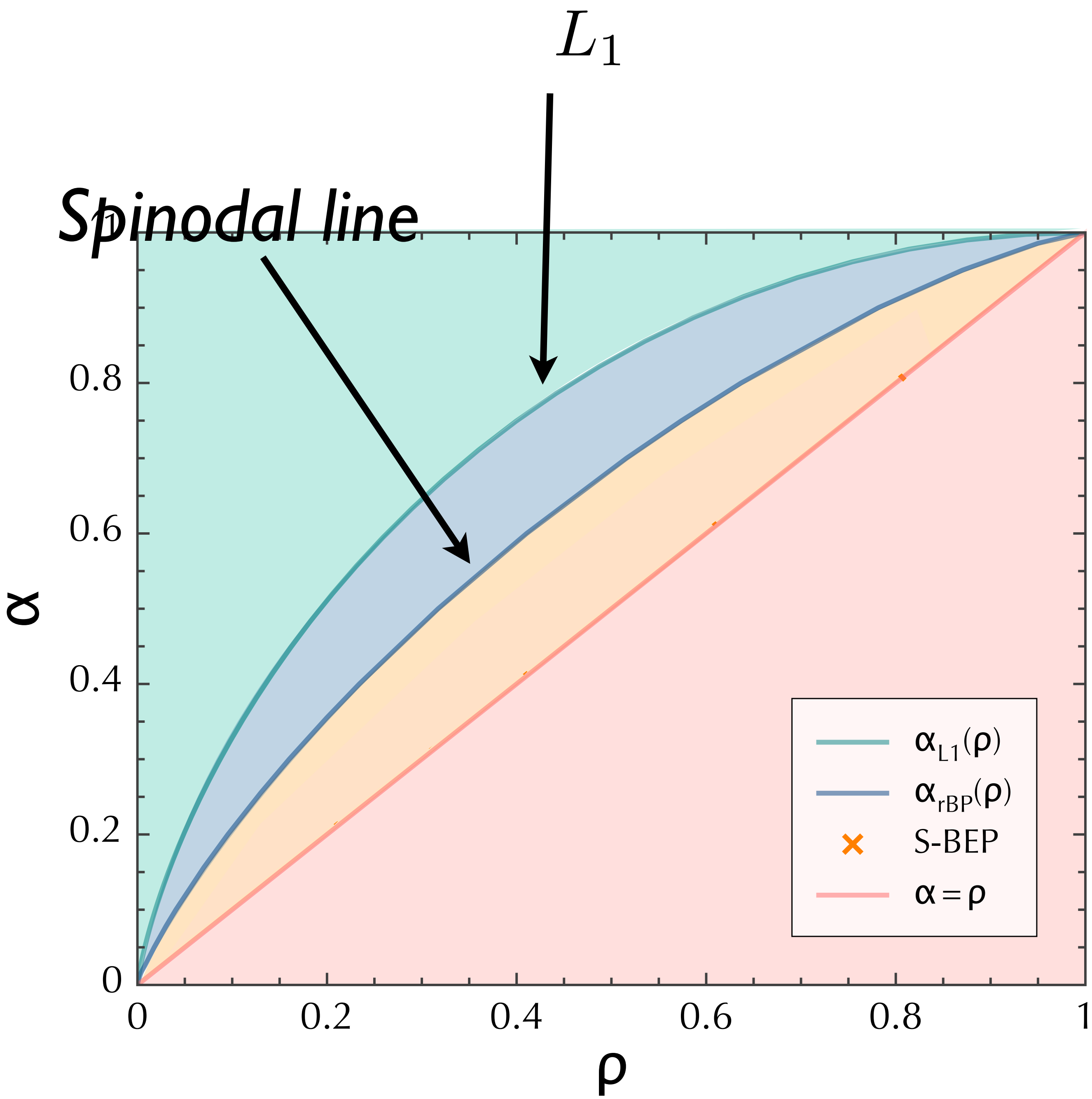
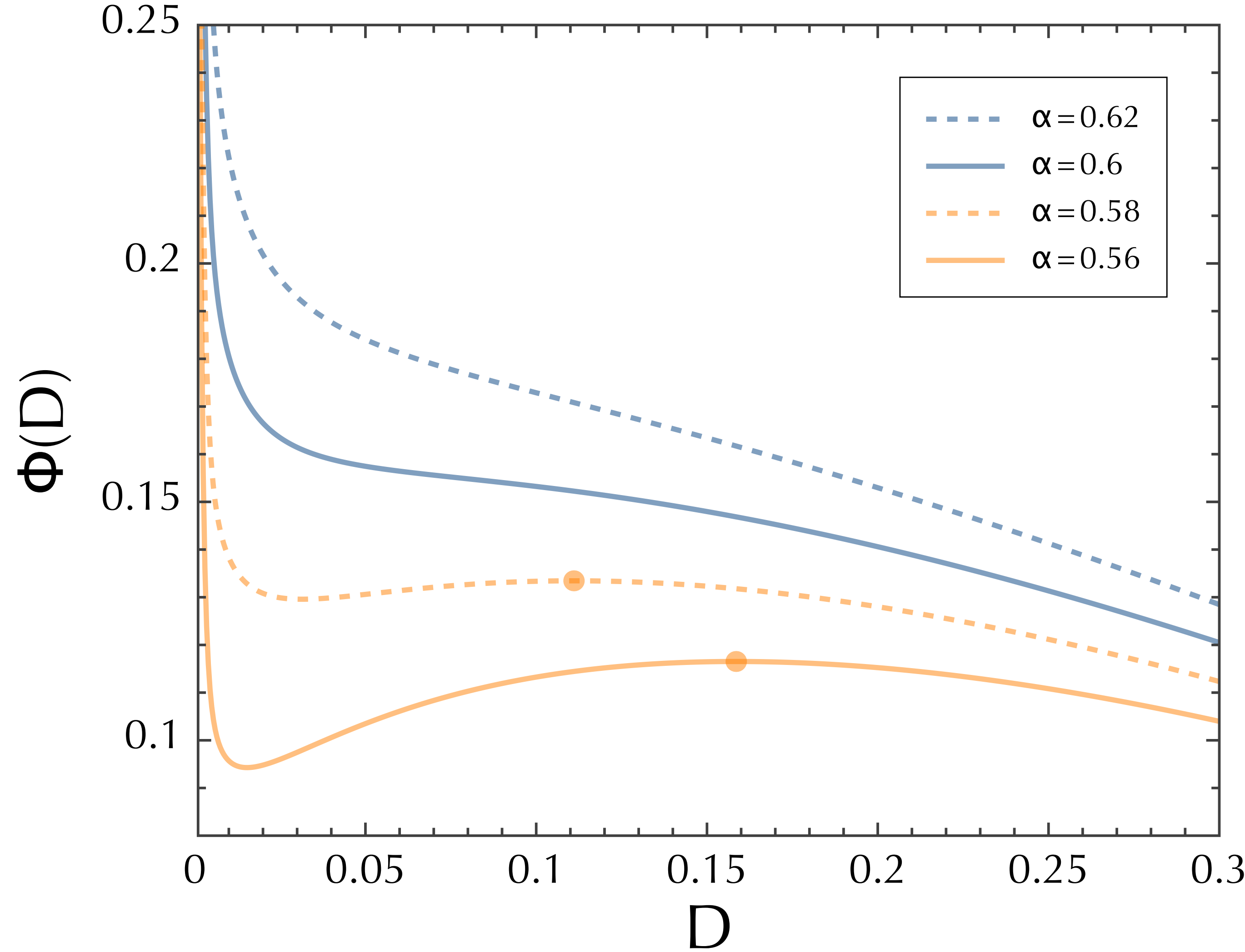
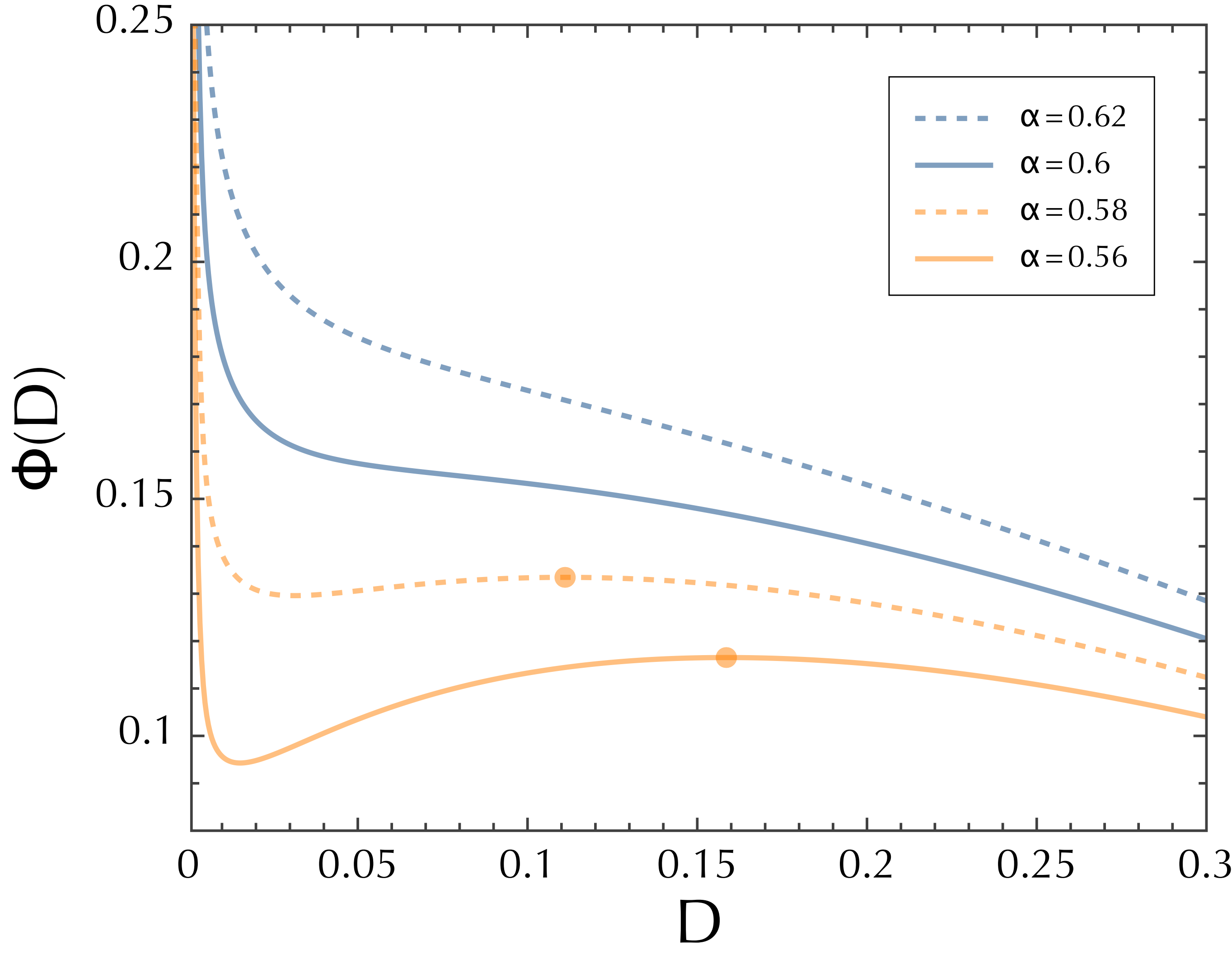




The dynamics is stuck in a metastable state, just as a liquid cooled too fast remains in a supercooled liquid state instead of crystalizing

This is good, but not good enough

How to pass the spinodal point?



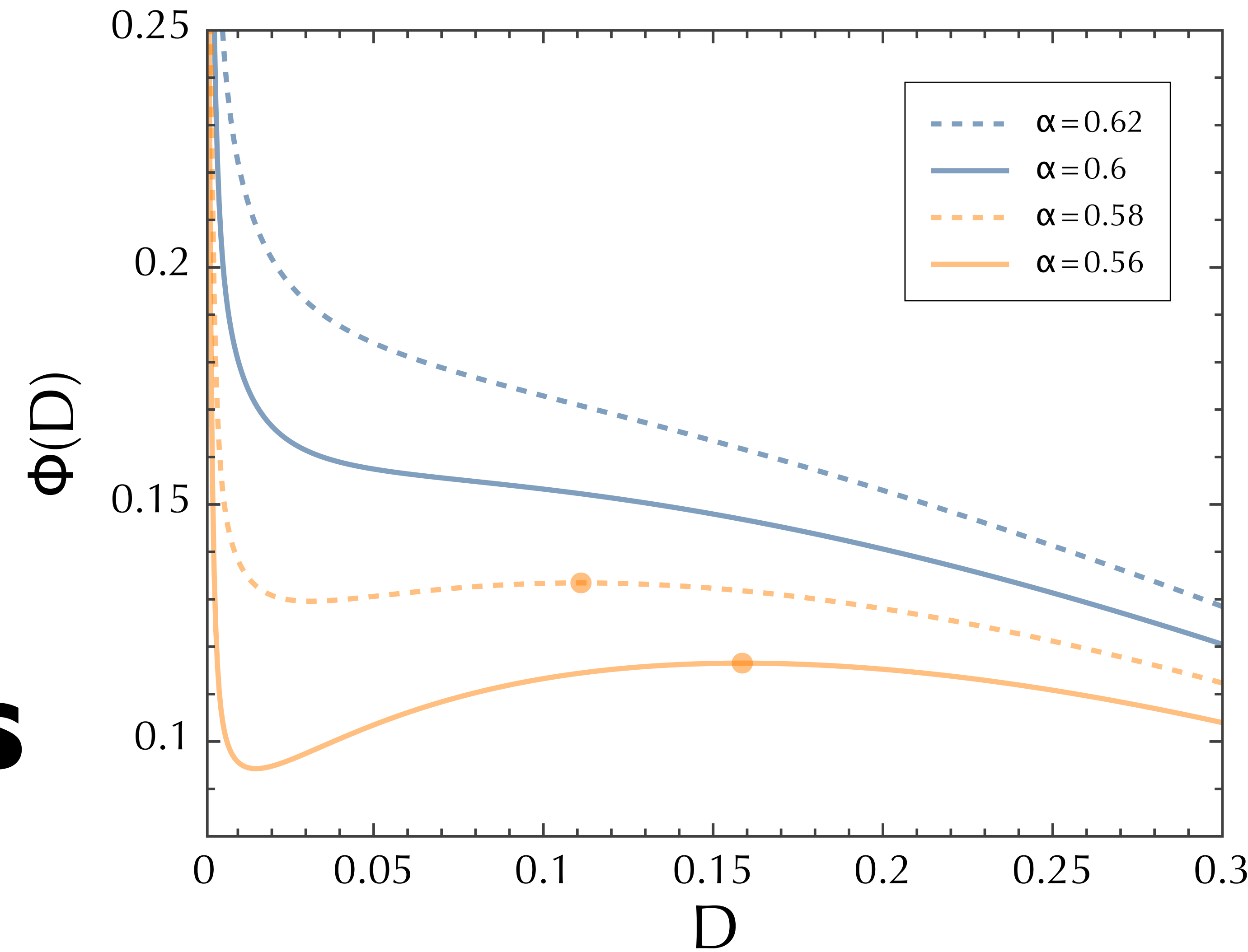
The dynamics is stuck in a metastable state, just as a liquid cooled too fast remains in a supercooled liquid state instead of crystalizing

This is good, but not good enough

How to pass the spinodal point?

By nucleation!

Special design of "seeded" matrices



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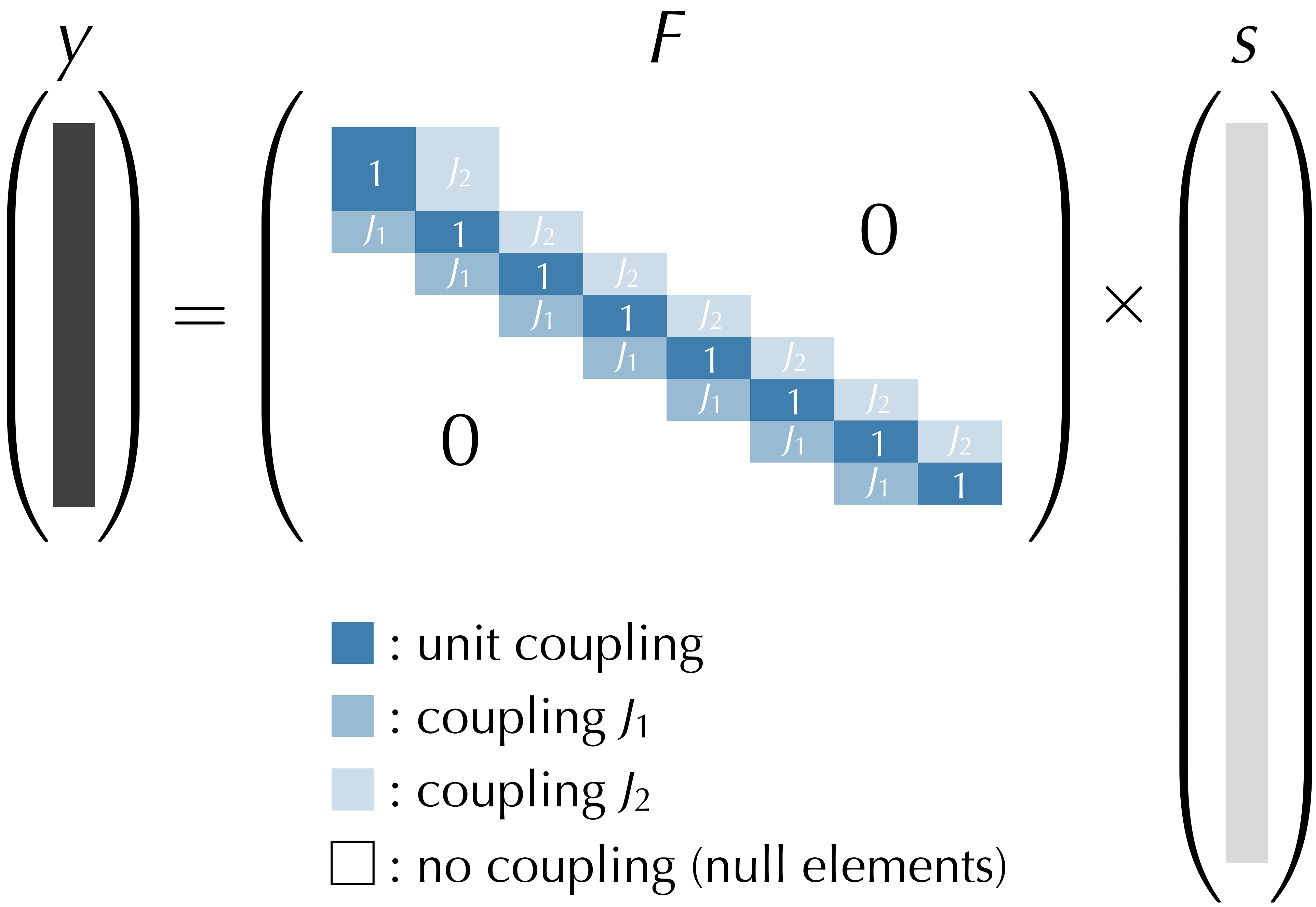


$L = 8$

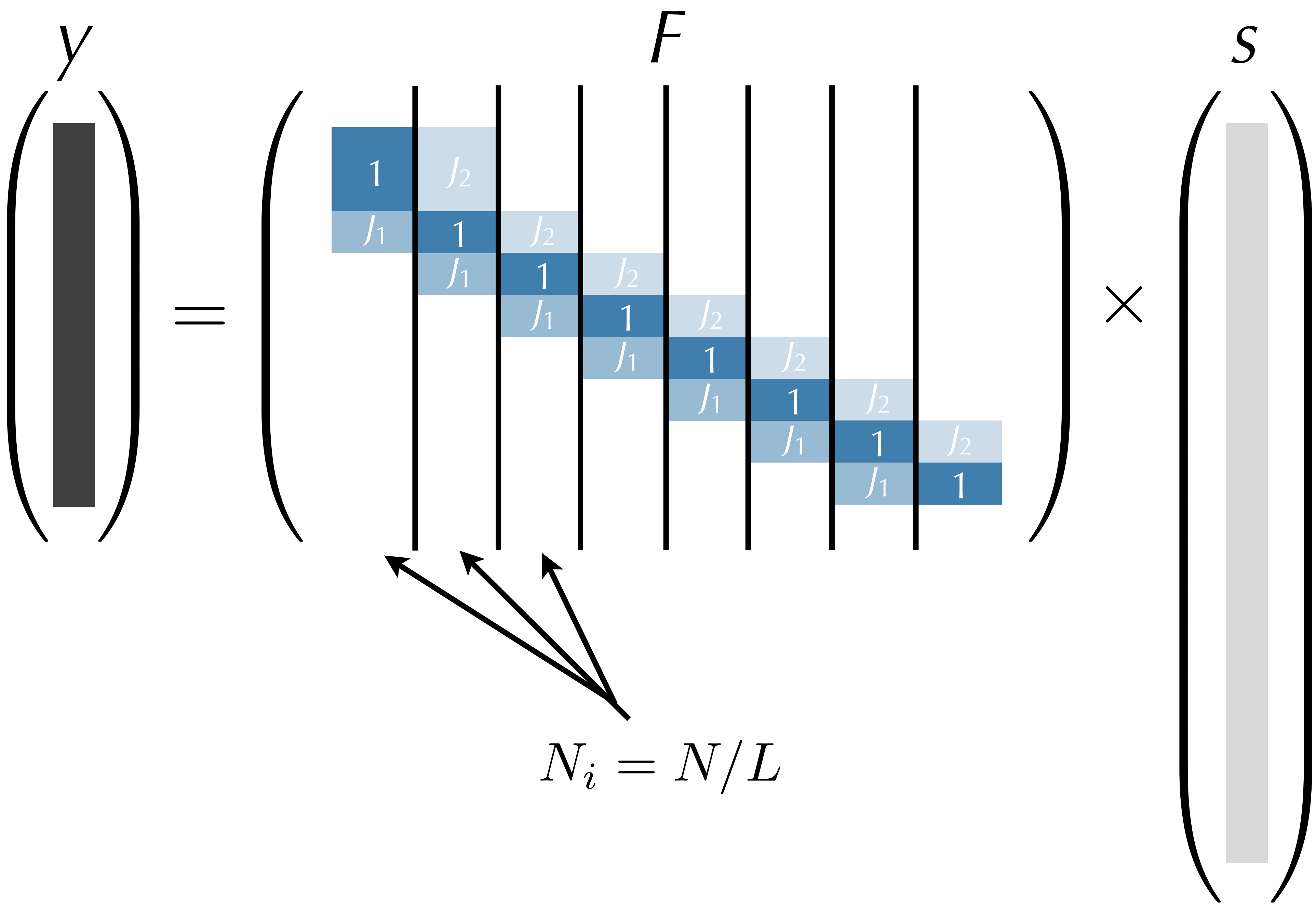
$N_i = N/L$

$M_i = \alpha_i N/L$

$\alpha_1 > \alpha_{BP}$

$\alpha_j = \alpha' < \alpha_{BP} \qquad j \geq 2$

$$\alpha = \frac{1}{L}\left(\alpha_1 + (L-1)\alpha'\right)$$



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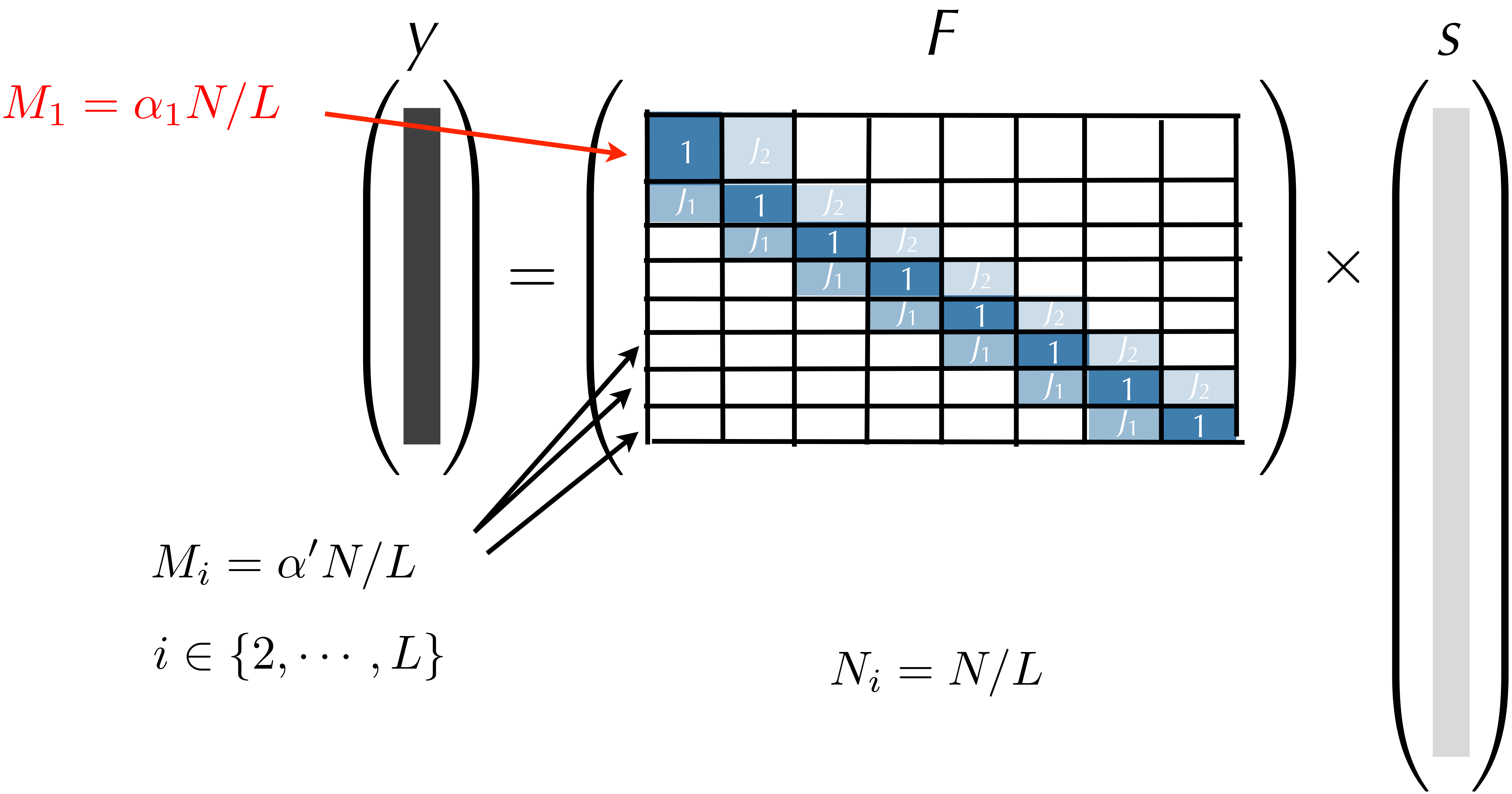

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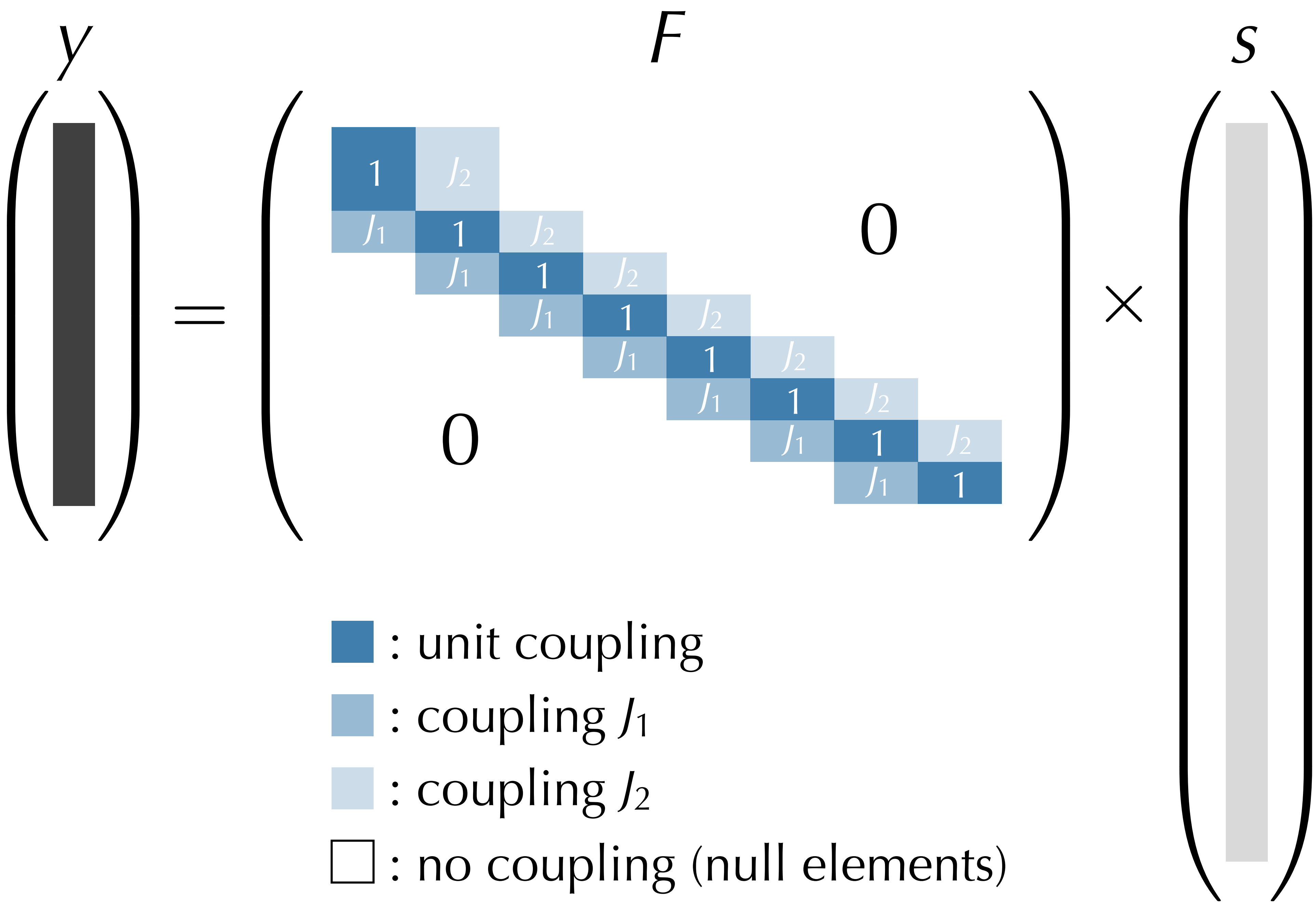

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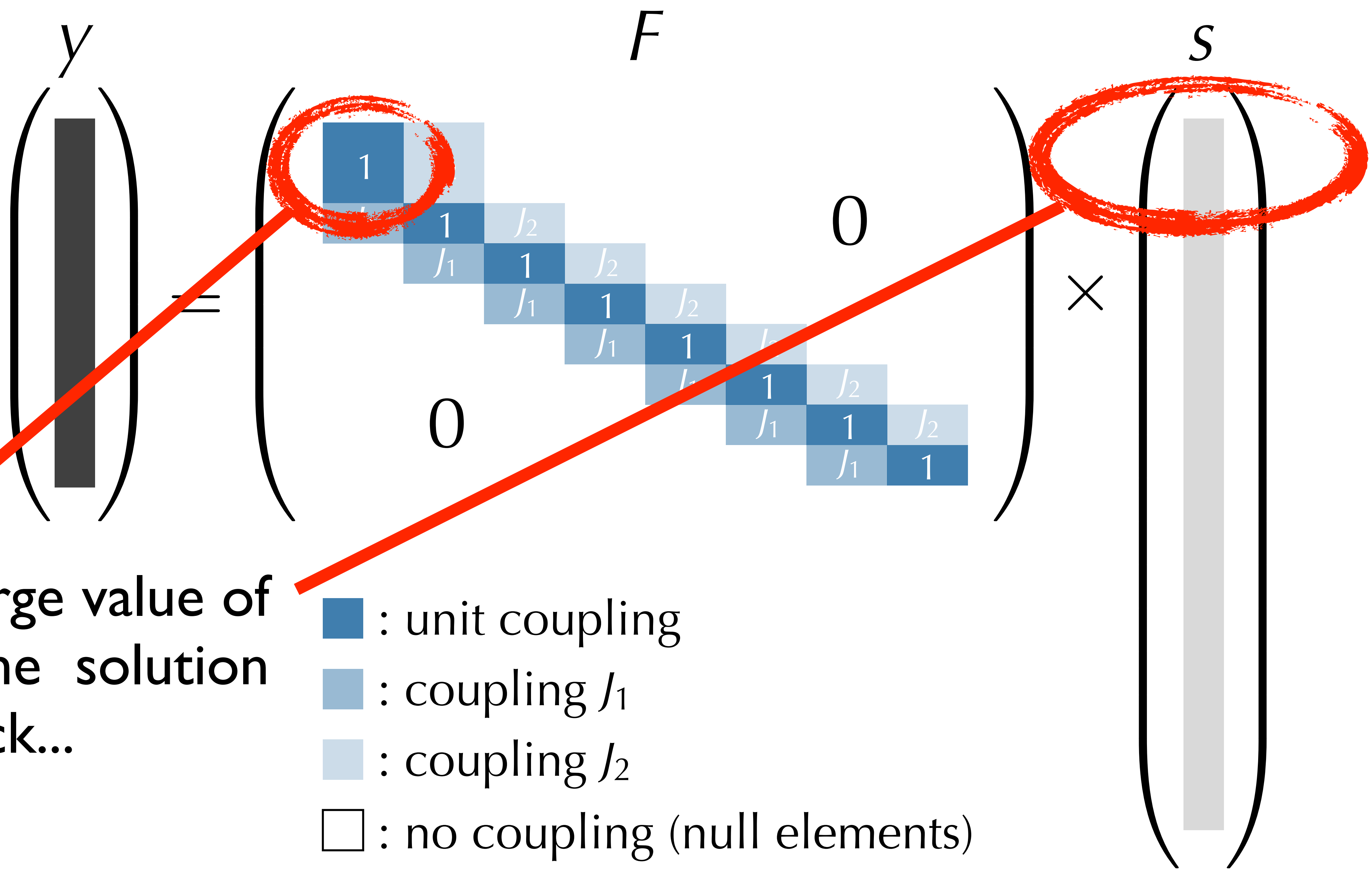

Block 1 has a large value of M such that the solution arise in this block...

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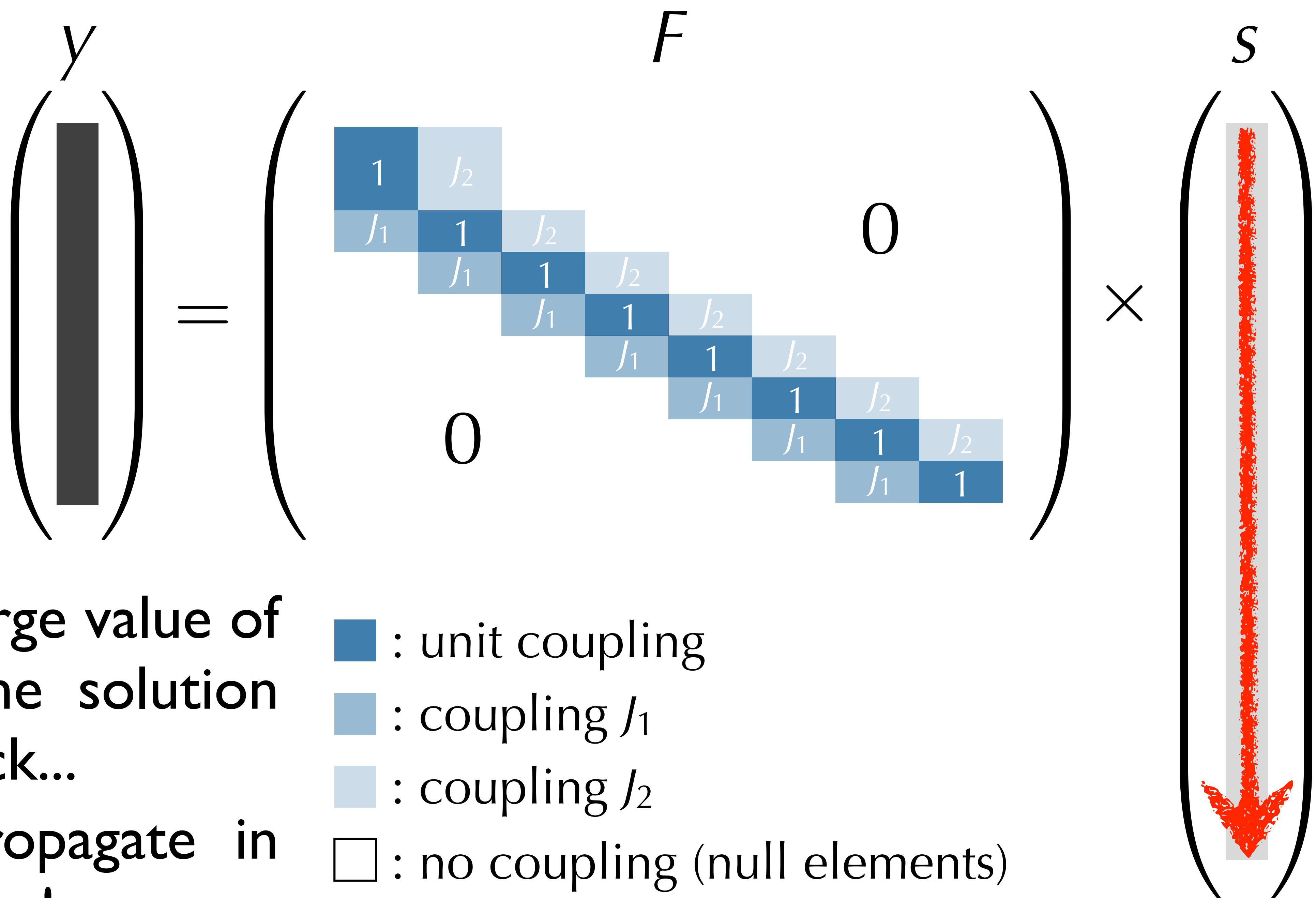


Block 1 has a large value of M such that the solution arise in this block...

... and then propagate in the whole system!

$$L = 8$$

$$N_i = N/L$$

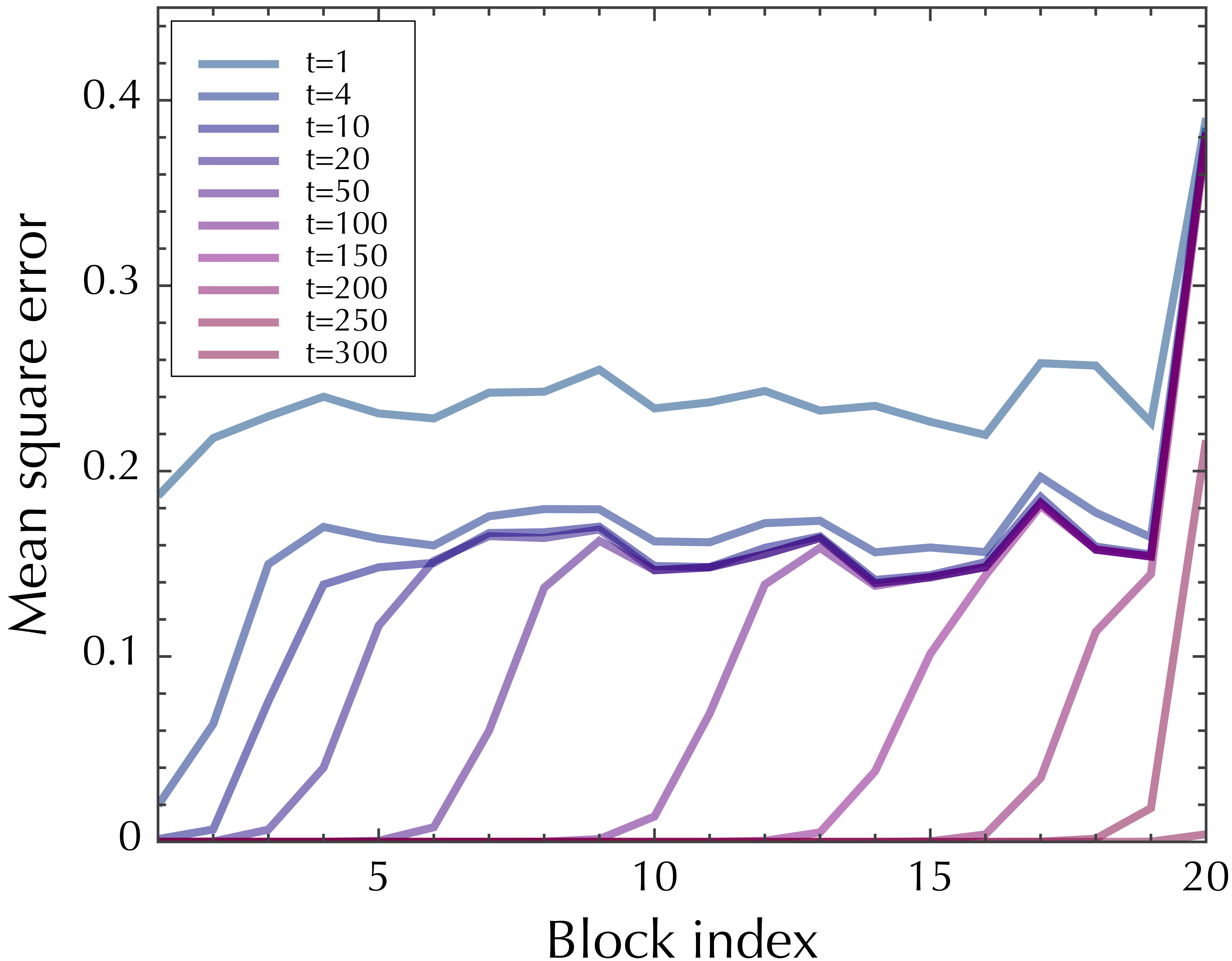
$$M_i = \alpha_i N/L$$

$$\alpha_1 > \alpha_{BP}$$

$$\alpha_j = \alpha' < \alpha_{BP} \qquad j \geq 2$$

$$\alpha = \frac{1}{L}\left(\alpha_1 + (L-1)\alpha'\right)$$

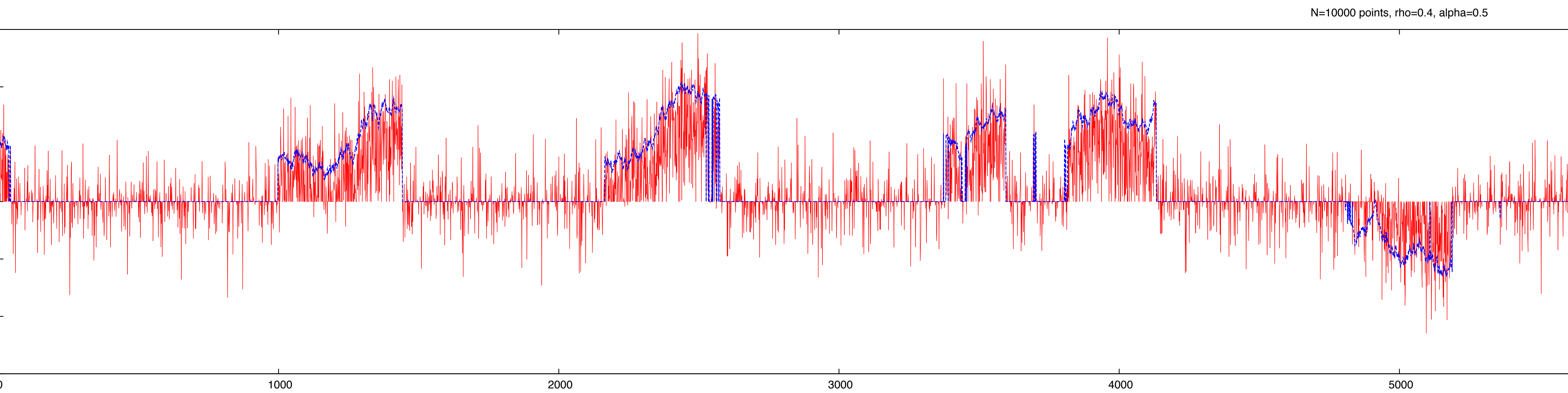
Example with ρ_0=0.4, and Φ_0 a Gaussian distribution with 0 mean and unit variance



$L = 20$ $\quad N = 50000$ $\quad \rho = .4$ $\quad J_1 = 20$ $\quad \alpha_1 = 1$

$J_2 = .2$ $\quad \alpha = .5$

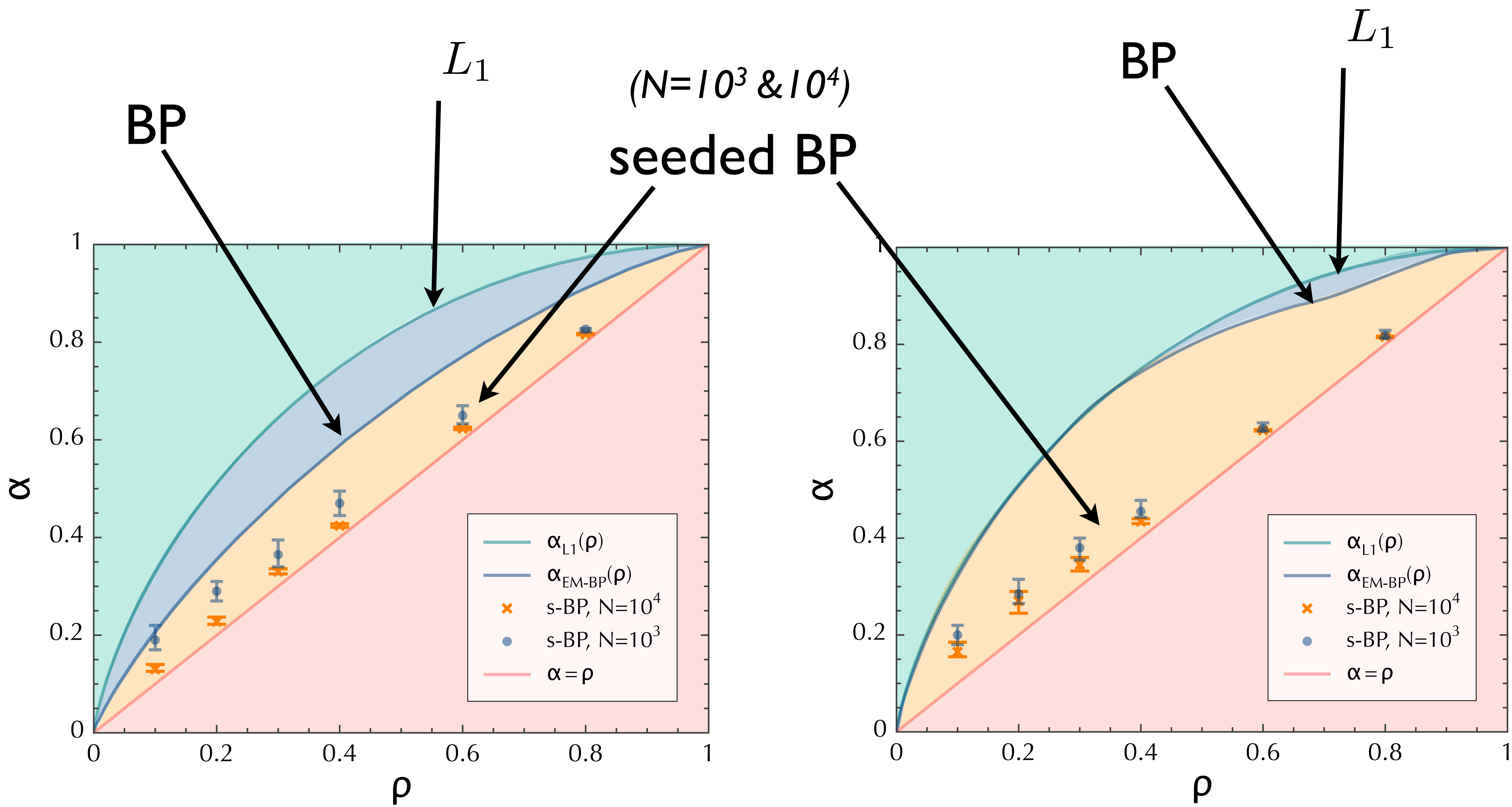
A signal with α=0.5 and ρ=0.4



Blue is the true signal reconstructed by s-BP

Red is the signal found by L_1

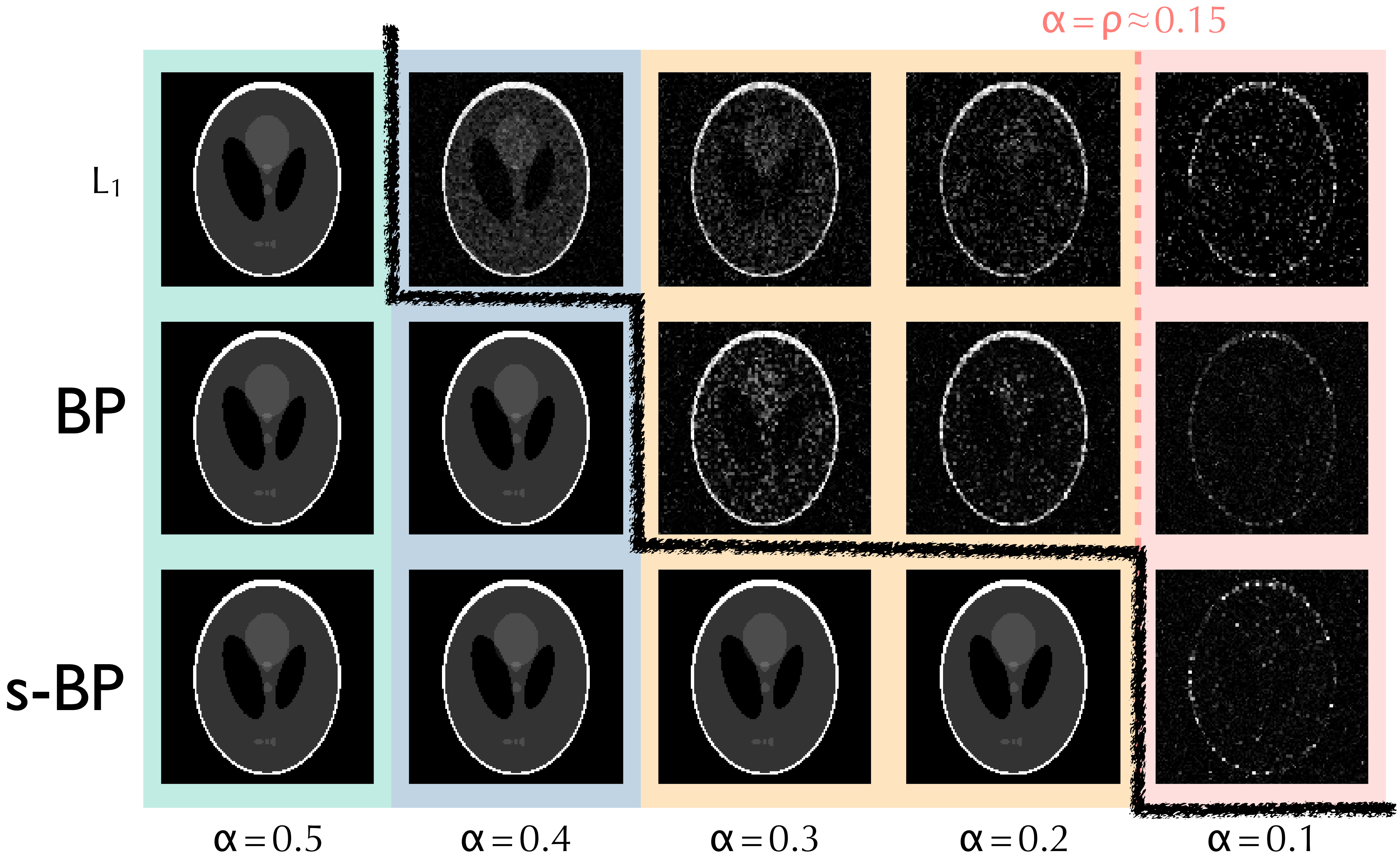
Phase Diagrams



Gauss-Bernoulli signal

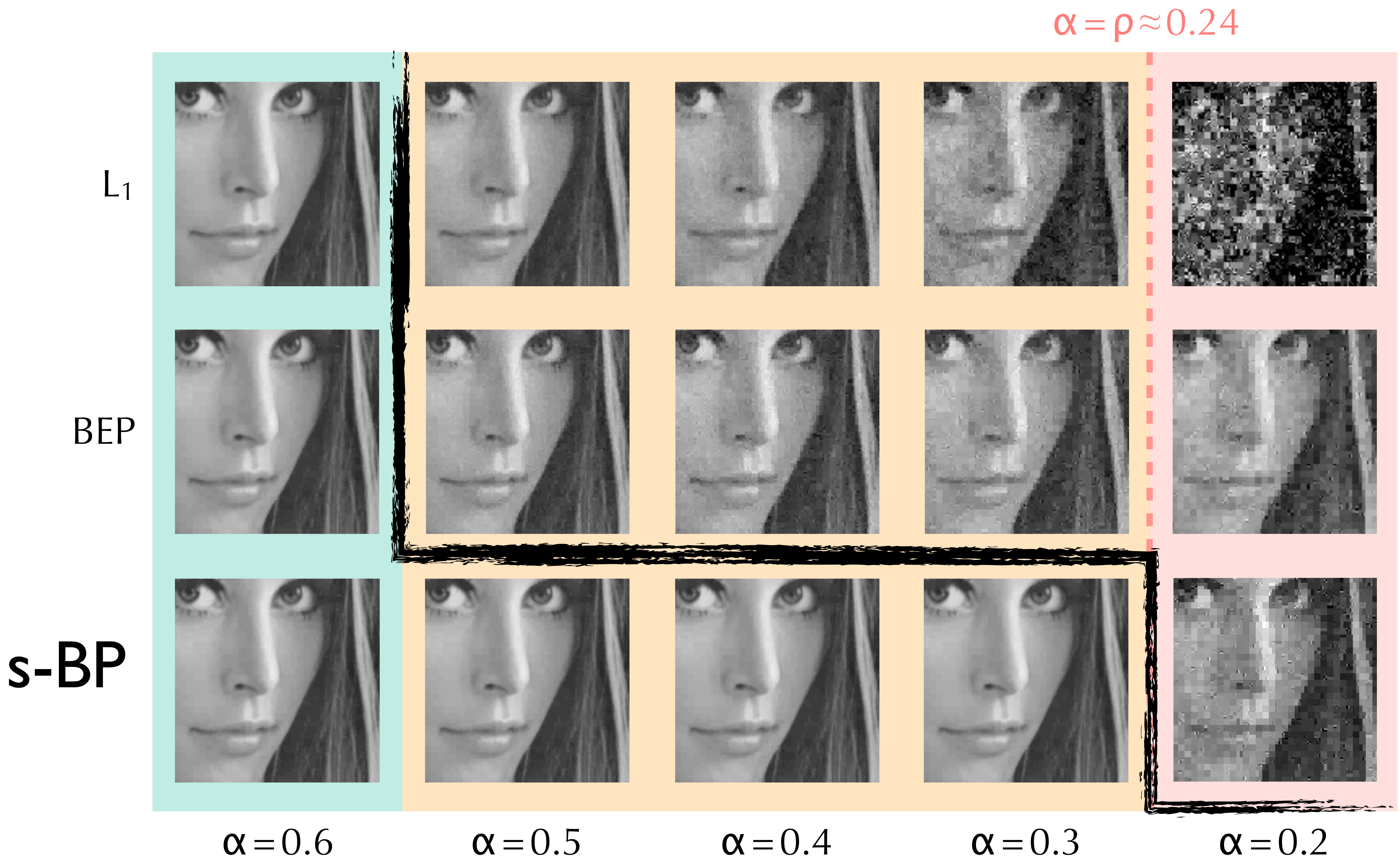
Binary signals

A more interesting example



Shepp-Logan phantom, in the Haar-wavelet representation

A EVEN more interesting example



The Lena picture in the Haar-wavelet representation

Analytical results for seeding matrices

- One can repeat the replica analysis for the seeded matrices, and the performance of the algorithm can be studied analytically, leading to $\alpha > \rho$ in the large N limit:

- These results have been recently confirmed by a rigorous analysis by Donoho, Montanari and Javanmard **(arXiv:1112.0708)**

- There is a lot of liberty in the design of the seeded matrices.

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- There is a lot of liberty in the design of the seeded matrices.

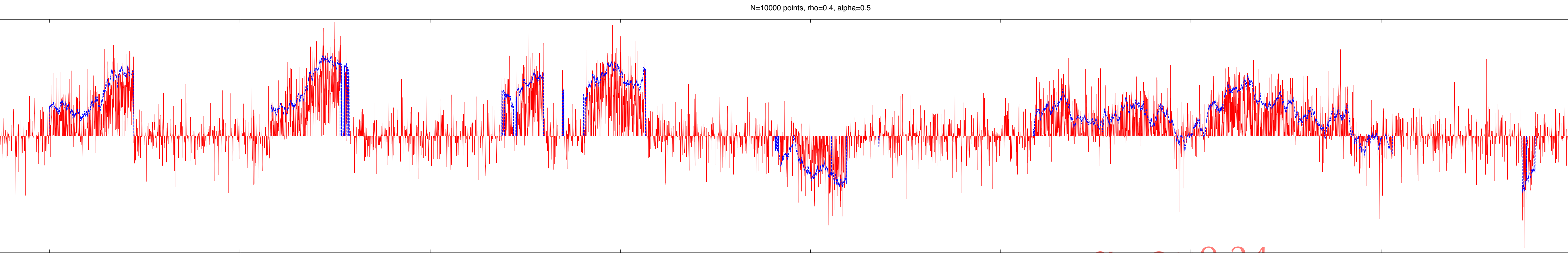
Asymptotically optimal measurements

Conclusions...

- A probabilistic approach to reconstruction
- The Belief Propagation algorithm
- Seeded measurements matrices

... and perspectives:

- More information in the prior?
- Other matrix with asymptotic measurements?
- Calibration noise, additive noise, quasi-sparsity, etc... ?
- Applications ?



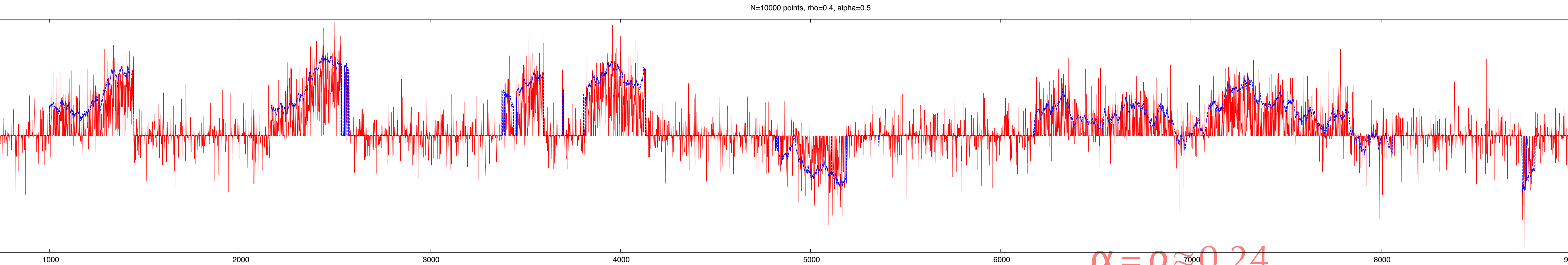
$\alpha = \rho \approx 0.24$

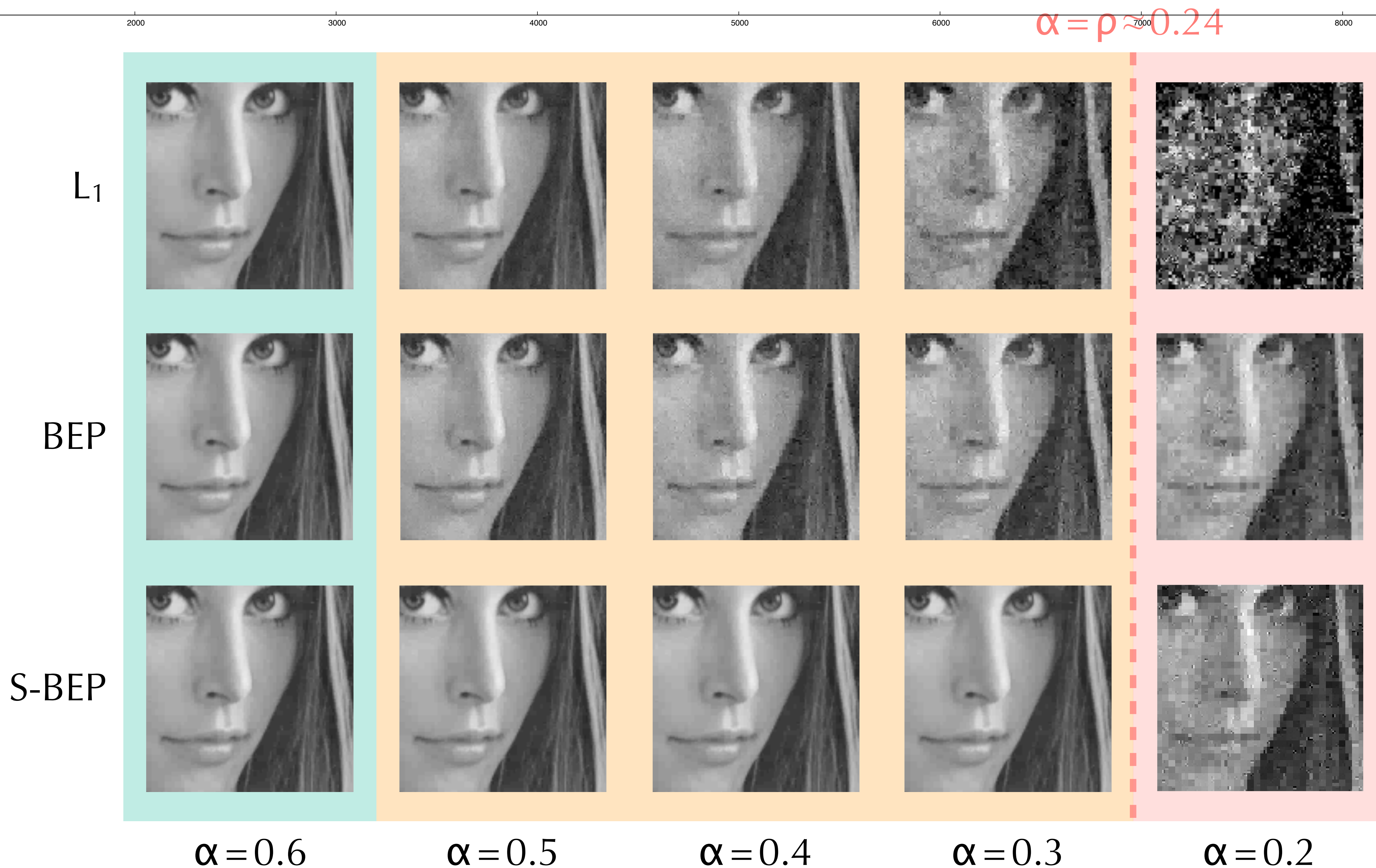
L_1

BEP

S-BEP

$\alpha = 0.6$ $\alpha = 0.5$ $\alpha = 0.4$ $\alpha = 0.3$ $\alpha = 0.2$

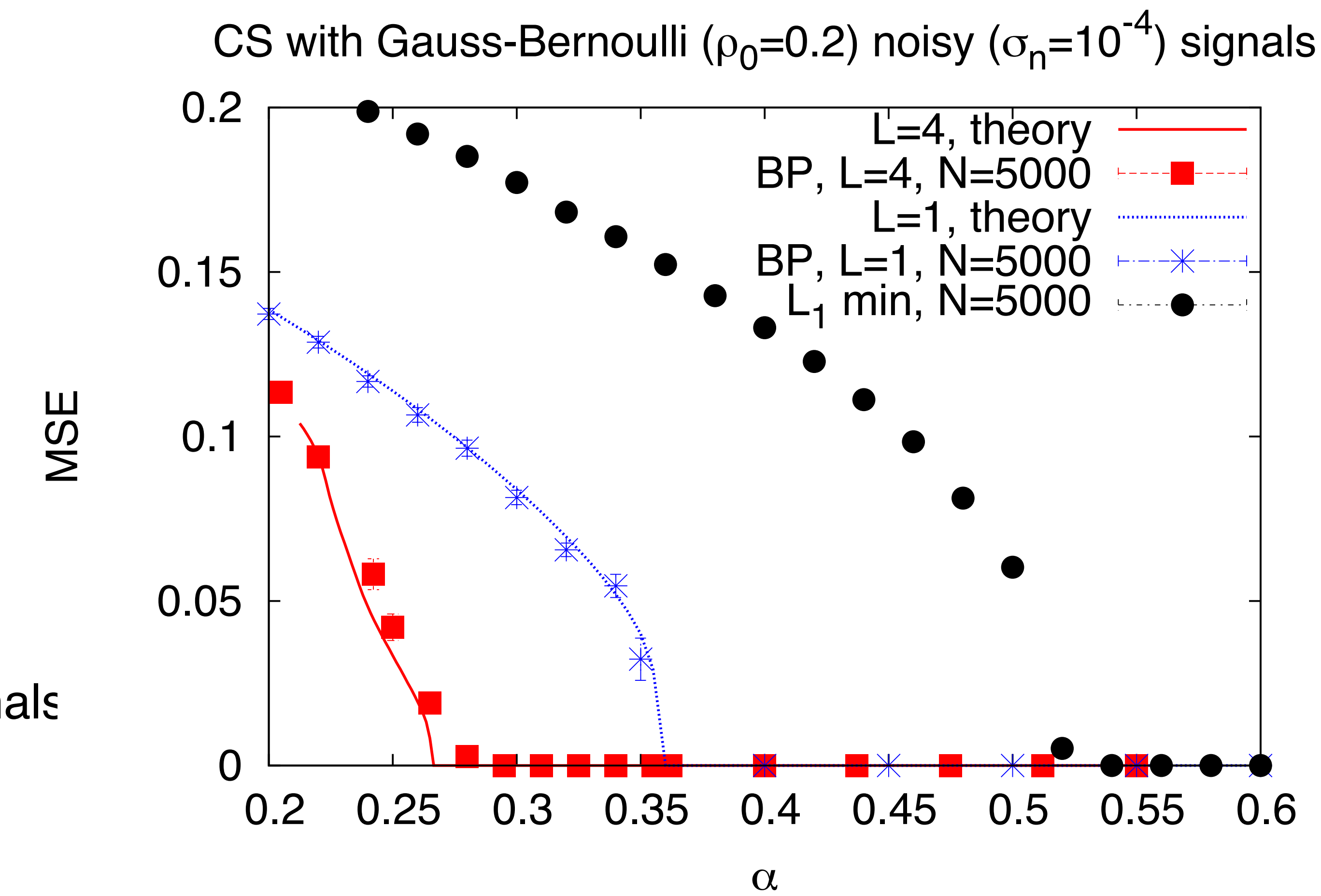


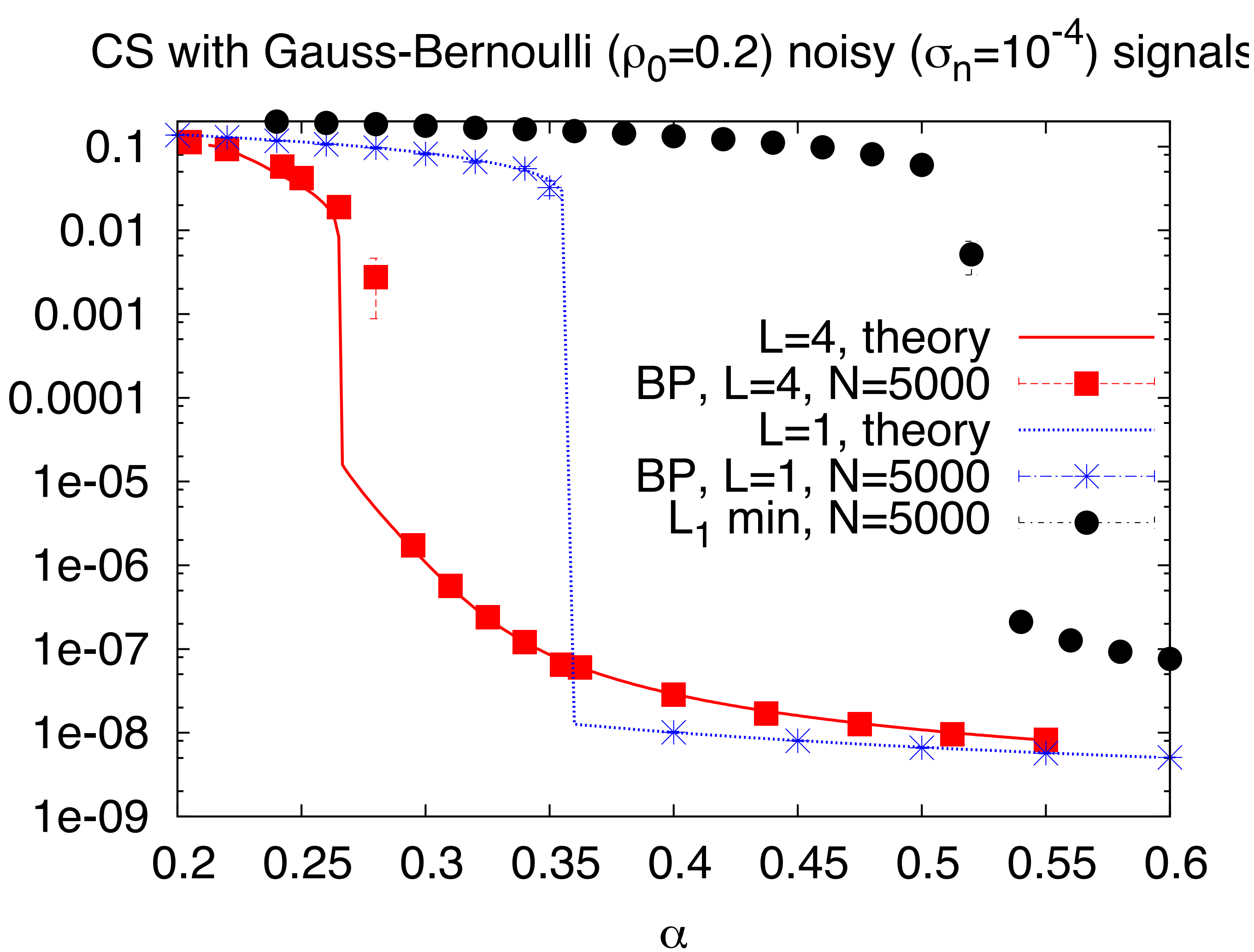


Thank you for your attention!

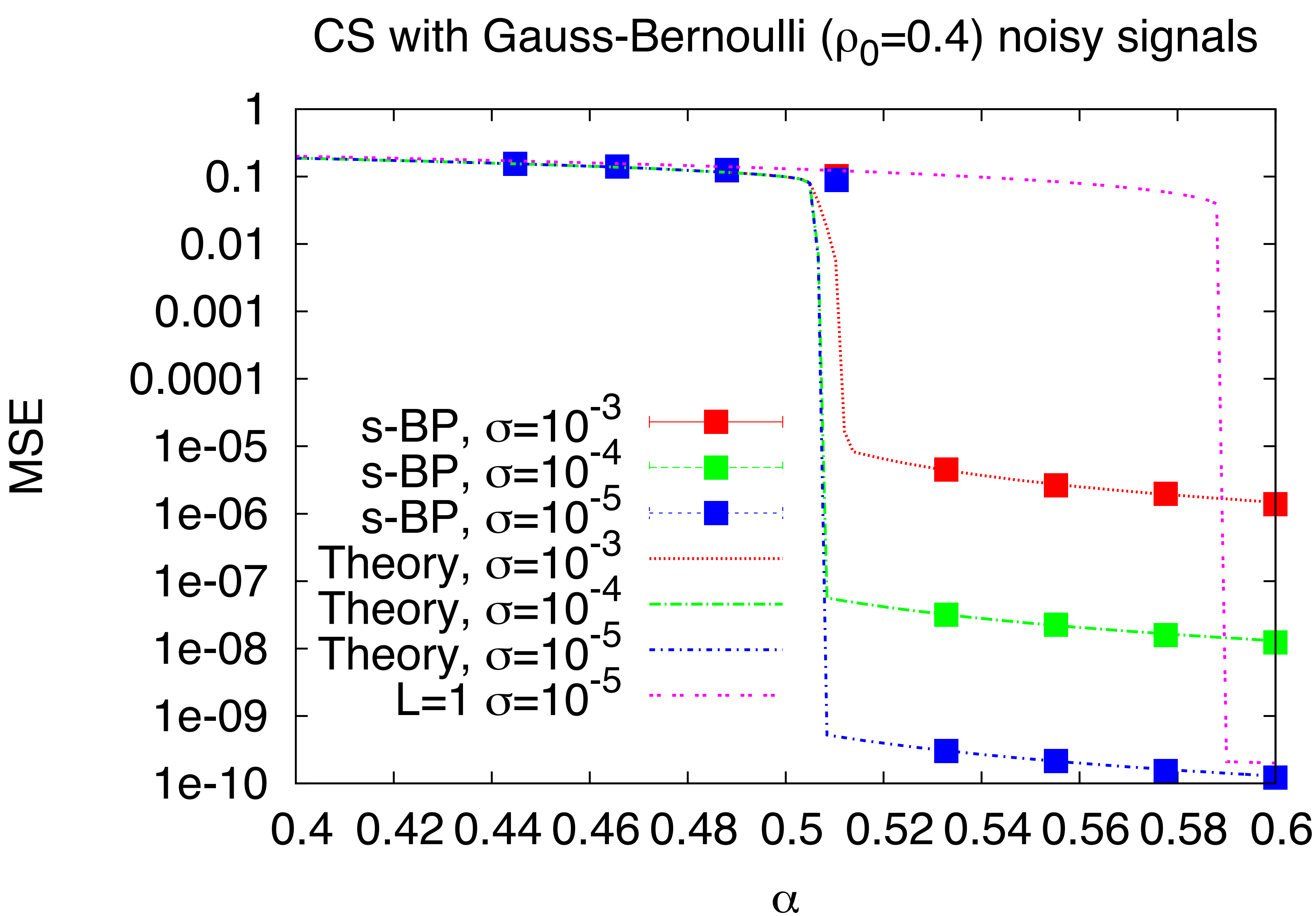
BONUS

Noise sensitivity

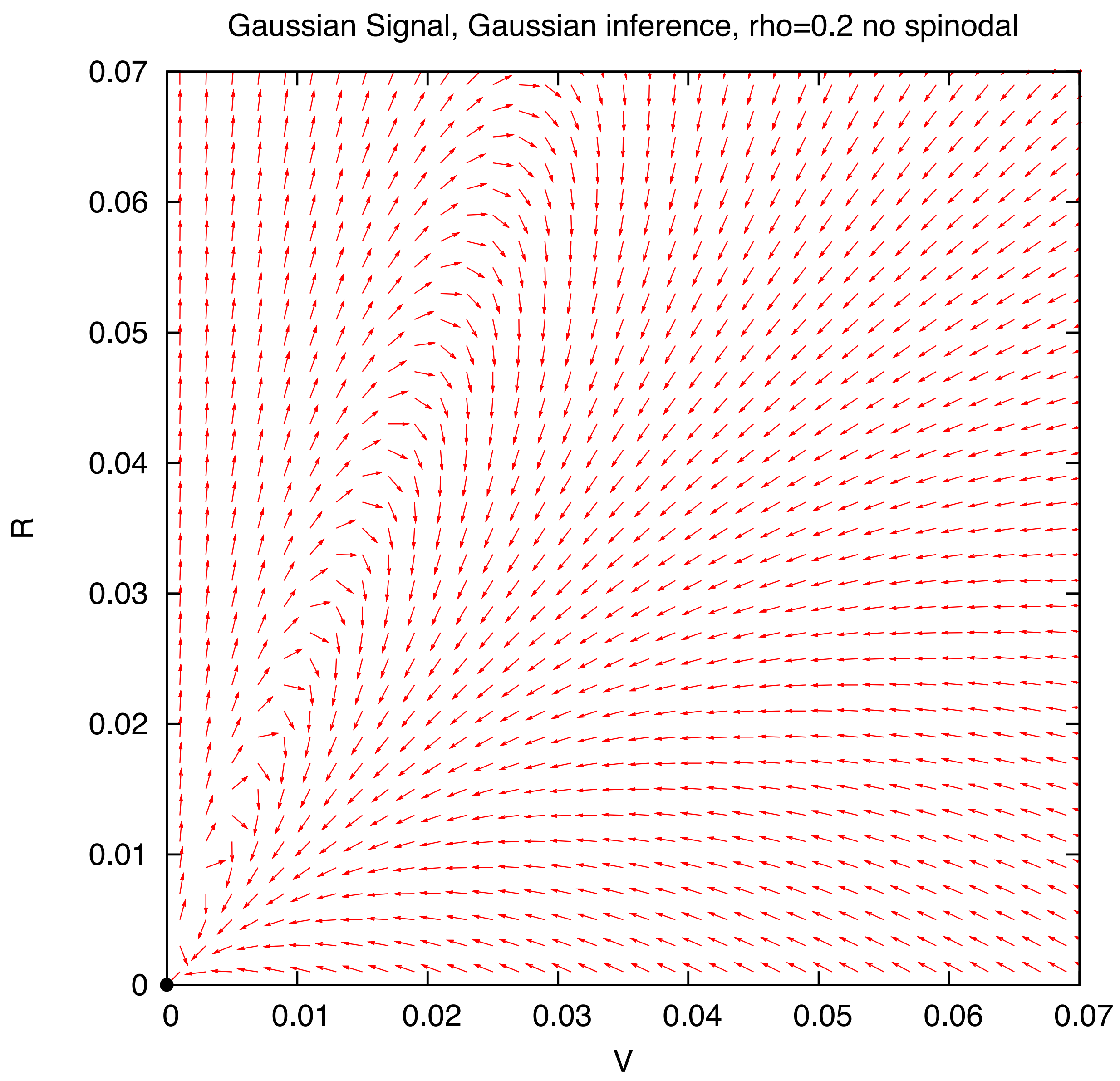


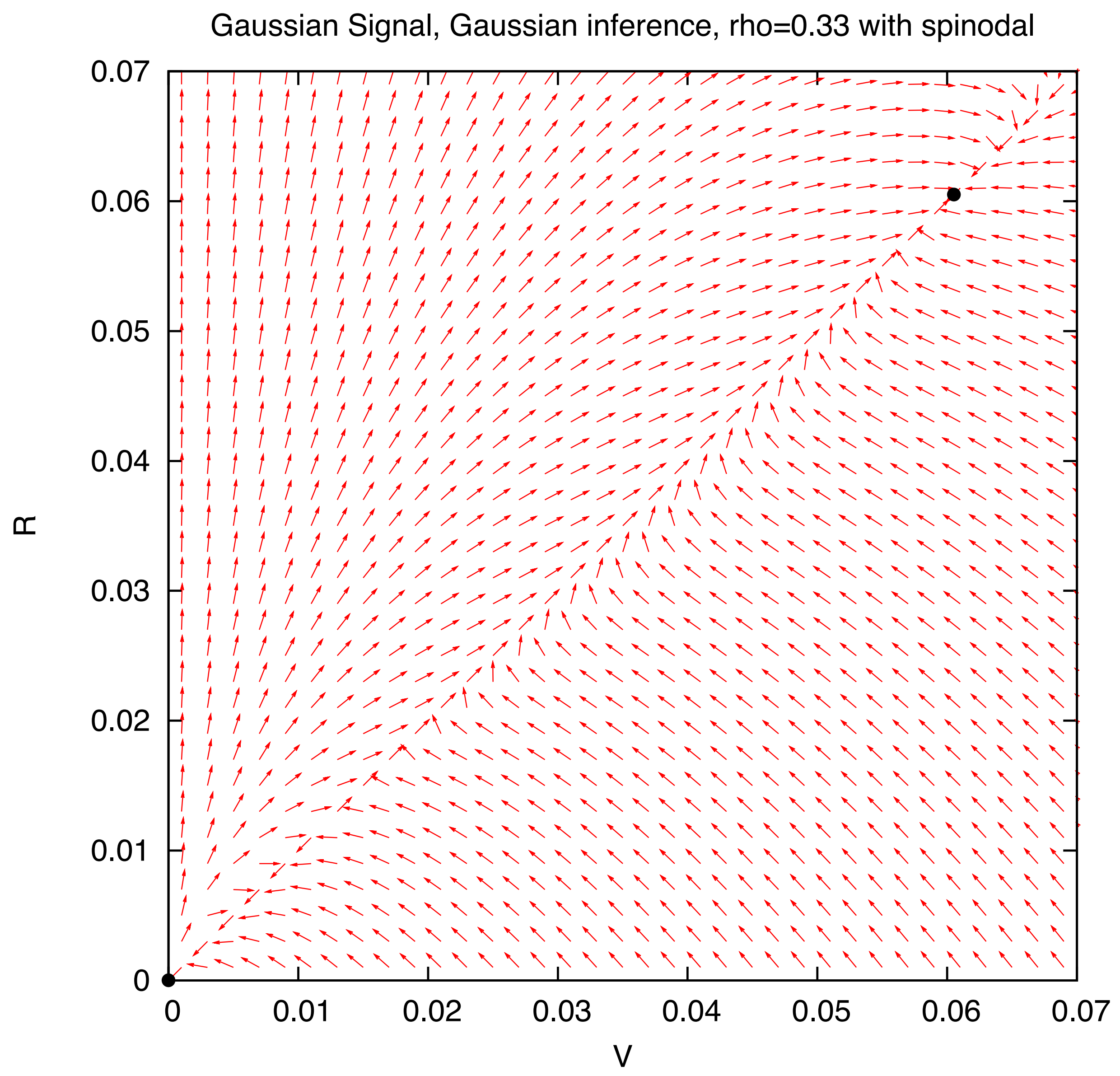


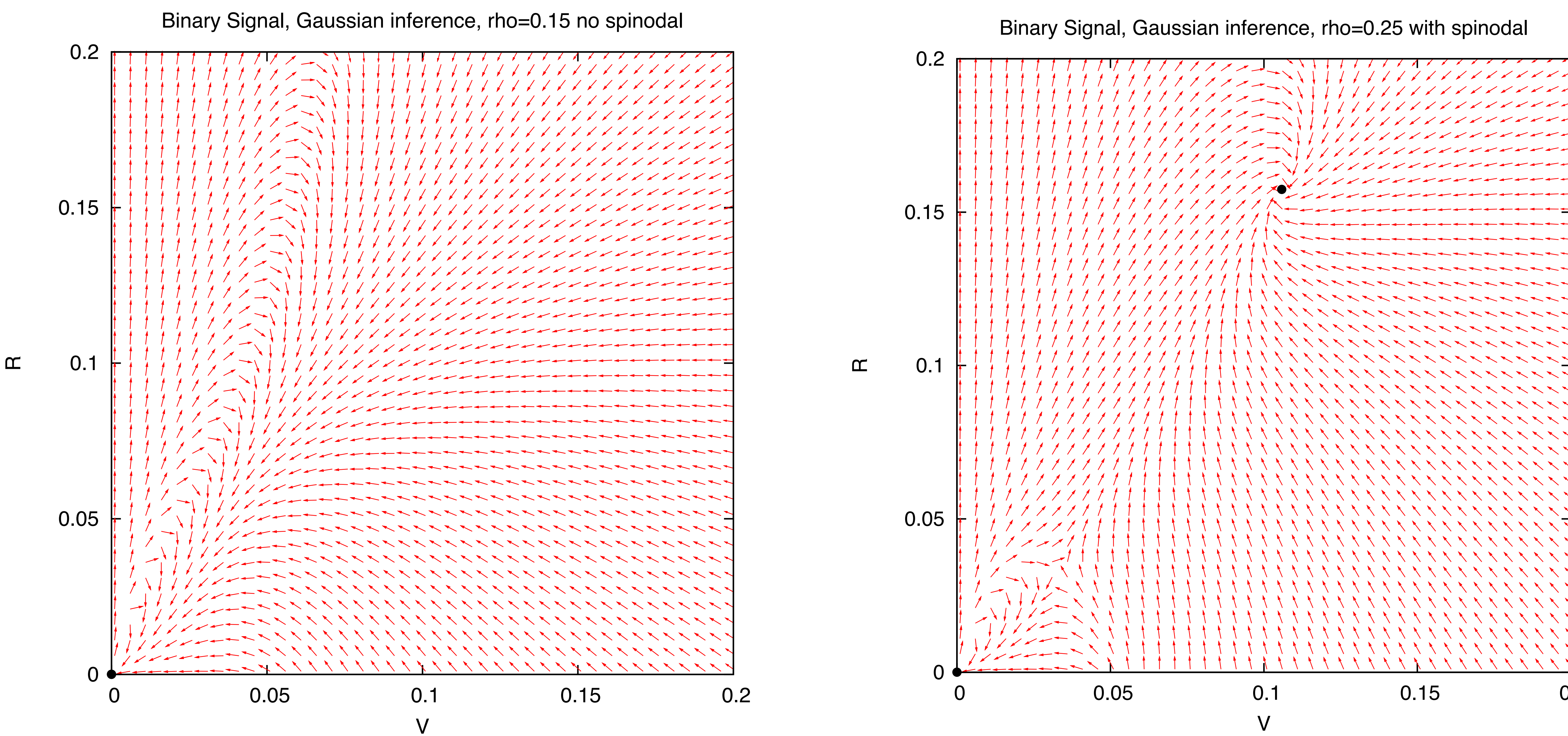
Noise sensitivity



Flow in the space $Q-q$, $E = q - 2m + \langle (x_i^0)^2 \rangle_0$






Binary Signal, Gaussian inference, rho=0.15 no spinodal
R
V
0.2
0.15
0.1
0.05
0
0
0.05
0.1
0.15
0.2
Binary Signal, Gaussian inference, rho=0.25 with spinodal
R
V
0.2
0.15
0.1
0.05
0
0
0.05
0.1
0.15