Atom-dimer scattering amplitude for fermionic mixtures with different masses: \(s\)-wave and \(p\)-wave contributions

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We study near a Feshbach resonance, as a function of the mass ratio, the fermion-dimer scattering amplitude in fermionic mixtures of two fermion species. When masses are equal the physical situation is known to be quite simple. We show that, when the mass ratio is increased, the situation becomes much more complex. For the \(s\)-wave contribution we obtain an analytical solution in the asymptotic limit of very large mass ratio. In this regime the \(s\)-wave scattering amplitude displays a large number of zeros, essentially linked to the known large value of the fermion-dimer scattering length in this regime. We find by an exact numerical calculation that a zero is still present for a mass ratio of 15. For the \(p\)-wave contribution we make our study below the mass ratio of 8.17, where a fermion-dimer bound state appears. We find that a strong \(p\)-wave resonance is present at low energy, due to a virtual bound state, in the fermion-dimer system, which is a forerunner of the real bound state. This resonance becomes prominent in the mass ratio range around the one corresponding to the \(^{40}\text{K}-^{6}\text{Li}\) mixtures, much studied experimentally. This resonance should affect a number of physical properties. These include the equation of state of unbalanced mixtures at very low temperature but also the equation of state of balanced mixtures at moderate or high temperature. The frequency and the damping of collective modes should also provide a convenient way to evidence this resonance. Finally it should be possible to modify the effective mass of one of the fermionic species by making use of an optical lattice. This would allow one to study the strong dependence of the resonance as a function of the mass ratio of the two fermionic elements. In particular one could check if the virtual bound state is relevant for the instabilities of these mixtures.

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I. INTRODUCTION

One of the most fascinating aspects of the physics of ultracold gases [1] is the ability, in appropriate situations, to describe the interaction between nonidentical fermionic atoms in terms of a single parameter, namely the scattering length \(a\) for two different atoms. For example, in fermionic gases involving only atoms in the two lowest hyperfine states of a single element, such as \(^6\text{Li}\) or \(^{40}\text{K}\), the interaction is fully described by the scattering length between atoms belonging to these two different hyperfine states [1]. Moreover in the presence of a Feshbach resonance this scattering length can be experimentally modified almost at will by merely changing the applied magnetic field. The case of wide Feshbach resonances is particularly convenient since all the physics related to the origin of the resonance, namely the existence of a closed channel, is irrelevant for most purposes. This has allowed, in the case of two fermionic atomic species, the experimental realization of the BEC-BCS crossover where the scattering length goes from small negative values to small positive values through the resonance, where it diverges. For small negative values of the scattering length two different atoms have a weak effective attraction, which for high enough density and small enough temperature, leads to the formation of Cooper pairs analogous to the ones appearing in superconductors and corresponding superfluid properties. On the other hand for small positive values of the scattering length \(a > 0\), two fermionic atoms of different species form a bound state resulting in the appearance of molecules, or dimers, with binding energy \(E_b\). These dimers behave statistically as bosons, and again for high enough density and small enough temperature a Bose-Einstein condensate forms with superfluid properties. The BEC-BCS crossover occurs when one goes continuously from the Bose condensate to the BCS superfluid by merely varying the scattering length through the change of magnetic field.

The simplicity of the description of the interaction extends to the BEC limit, when one considers dimer-dimer interaction, while this is \textit{a priori} a more complex situation since dimers have an internal structure. Nevertheless the BEC limit is quite simple since the composite bosons one deals with are dilute. In this case again the interaction between these bosons is fully characterized by their scattering length \(a_4\). This is because in the low-temperature regime corresponding to the existence of the Bose-Einstein condensate, the kinetic energies of the dimers will be quite small compared to their binding energy, and the structure of these composite bosons will be irrelevant. This scattering length \(a_4\) has been obtained [2,3], in terms of the scattering length \(a\) for fermions, as \(a_4 \simeq 0.6a\) when the different fermions have equal masses.

On the other hand when one goes away from the BEC limit toward the unitarity limit, it is no longer true that the composite nature of the bosons is irrelevant. One might accordingly expect a more complex physics for this molecular gas. Remarkably the \(T = 0\) equation of state on this BEC side is nevertheless very well described [1] in a wide domain of density by the Lee-Huang-Yang [4] equation of state for these composite bosons [5], making again merely use of the dimer-dimer scattering length \(a_4\) between these bosons. This is as if these bosons were elementary. This is somewhat surprising.
More precisely, assuming, for example, two fermions species with equal densities $n = k_F^3/6\pi^2$, a typical fermion wave vector is of order $k_F$, and for $k_Fa \sim 1$ the typical kinetic energy of a fermion is now of the same order as the dimer binding energy which is $E_b = 1/(2\mu a^2)$ with $\mu$ being the reduced mass. In this case one has to consider the total energy dependent scattering amplitude and not only its zero energy limit given by the scattering length. This is valid for dimer-dimer scattering as well as for single fermion-dimer scattering. In this last case these single fermions may arise because of thermal breaking of dimers, or because we consider unbalanced fermionic mixtures. Actually for $k_Fa \sim 1$ these single fermions arise also because we may no longer consider isolated dimers since they are too close, and we have actually to deal with a complicated many-body problem. However, our point is that even the simplest minded description has to consider the energy dependence of scattering processes.

Actually it is possible to understand qualitatively the wide range of validity of the Lee-Huang-Yang equation of state in the following way. One assumes first that only the $s$ wave scattering properties are relevant, since higher partial waves give a quite small contribution at low energy. Then one supposes that the $s$-wave phase shift $\delta_0(k)$ stays small enough so that its low-energy expansion $\delta_0(k) \approx -k\tilde{a}$ remains valid, where $\tilde{a}$ is the scattering length relevant for the considered process, dimer-dimer or fermion-dimer scattering. In this case the standard low-energy expression for the scattering amplitude $f(k) = -1/(\tilde{a}^{-1} + ik)$ remains valid. This is what happens when the different fermions have equal masses. In this case the scattering amplitude has a fairly weak variation when $k$ is going from zero to the maximum value corresponding to dimer breaking, so the situation is not much different from evaluating the scattering amplitude at $k = 0$.

The purpose of the present paper is to show that this simplicity is linked to the fact that the two atomic species have equal masses $m_1 = m_\downarrow$ (we follow the standard use of referring to the two different atomic species as $\uparrow$ and $\downarrow$ atoms, even if there is no relation between this notation and the physical spin of these atoms). It disappears when the masses are quite different. In this case, even though the elementary scattering process between different fermions is very simple to describe, nevertheless the mere complexity due to the existence of a dimer leads to complexity in the involved scattering processes. In the present paper we will consider only the simplest situation which displays such a feature. We will deal with the fermion-dimer scattering, not the dimer-dimer scattering. The lonely fermion has a $\uparrow$ spin and scatters with a $\downarrow$ dimer. We will furthermore assume that the available kinetic energy is small enough so the dimer is not broken by the collision. In other words we will deal only with elastic scattering.

In a first part of the paper we will consider $s$-wave scattering, since it is the dominant one at very low energy and is anyway expected to be an essential component of the scattering process. This is the natural extension of the case where the scattering is fully described by the scattering length. Actually, although we have performed numerical calculations for any value of the mass ratio $m_1/m_\downarrow$, we deal extensively with the limit where this mass ratio is large. Indeed in this case we will be able to obtain an analytical solution of the problem, which has the advantage of displaying quite explicitly the behavior resulting from the mass difference between the two fermion species. Naturally since we have already explored in a preceding work [6] the behavior of the scattering length in this limit, this paper can also be seen as a natural extension of our earlier work. But it has the major interest of displaying a strong structure in the $s$-wave scattering amplitude, which is actually essentially linked to the increase of the scattering length with the mass ratio [6–9].

Then in the second part of our paper we will turn to $p$-wave scattering. Although it does not contribute in the zero energy limit, this partial wave turns out to be quite important. This is physically linked to the appearance of new bound states which appear at the level of the three-body problem when the mass ratio increases. These bound states have been studied in great details by Kartavtsev and Malykh [10], for the present situation of two identical fermions with mass $m_\uparrow$ and a different fermion with mass $m_\downarrow$. They are not present in the $s$-wave channel and appear only for angular momenta $\ell \geq 1$. Among them are the well-known Efimov [11] states, which are quite remarkable for their spectrum structure with binding energy going up to infinity when the range of the interaction goes to zero. For $\ell = 1$ the Efimov states appear only [10,12] for a mass ratio $m_\uparrow/m_\downarrow = 13.6$. However, there are already bound states for smaller mass ratios, which are not Efimov states. A first bound state [10] appears for mass ratio $m_\uparrow/m_\downarrow = 8.17$.

Actually very important contributions in the fermion-dimer scattering properties arise even for quite lower mass ratios. The basic reason is that, even before the mass ratio threshold for bound states is reached, virtual bound states are already present. For nonzero angular momenta, in particular $\ell = 1$, the centrifugal barrier acts to inhibit their decay and provide them with fairly long lifetime. Hence they are not much different from real bound states. Moreover, since their energy is positive, they give rise to resonances, which are strong when they are located at low energy. In particular the $p$-wave contribution becomes very strong, due to quasiresonance, when the mass ratio approaches 8.17. However, we find that these effects are already important for the mass ratio $m_\uparrow/m_\downarrow = 6.64$ corresponding to the mixtures of $^6\text{Li}$ and $^{40}\text{K}$, which is probably presently the most investigated experimentally [13–15] and which is of very high current interest [16–24]. Actually this point has already been noted, for the specific case of $^6\text{Li} - ^{40}\text{K}$ mixtures by Levinsen et al. [24]. In this work, the authors study the more general problem of resonant interactions where the finite effective range is an important ingredient. This is by far the most frequent case in these mixtures. Our work therefore corresponds to focus on the limit of a vanishing effective range in their approach. Specifically, we are primarily interested in the mass dependence of the $p$-wave scattering amplitude, not in the finiteness of the effective range.

Here we will consider only the case of wide Feshbach resonance in three dimensions, because of its theoretical interest due to its simplicity. In the following we will assume that we are near such a Feshbach resonance and ignore any stability problem. Fortunately such a Feshbach resonance, which is fairly broad and reasonably stable, has been identified experimentally near 155 G [26] in $^6\text{Li} - ^{40}\text{K}$ mixtures. We will consider in detail the mass ratio dependence of the $p$-wave contribution. This is not only of theoretical interest since we
will show that, by making use of optical lattices, it is possible to vary experimentally the effective mass ratio of $^6\text{Li}$-$^\text{K}$ mixtures. This would allow one to study experimentally the effect of the $p$-wave contribution when it is quite strong, or conversely to eliminate it to a large extent which could be in particular quite useful if this contribution turned out to be experimentally detrimental. We note indeed that these virtual bound states could play a very important role in the three-body decay and in the stability of these mixtures, since they correspond to physical situations where three atoms stay close together for a fairly long time, increasing in this way the probability of three-body decay processes. In this way optical lattices could also be used to study and possibly remove instability problems.

In this $p$-wave study we will consider mass ratios below the value 8.17, corresponding to the appearance of the first $p$-wave bound state, since clearly beyond this threshold the physical situation will get more complicated. This justifies also that we consider only the s-wave and $p$-wave components of the full scattering amplitude, since the higher angular momentum components are expected to give very small contributions to the total scattering in the regime we investigate.

In practice the next Sec. II is devoted to the study of the s-wave contribution to the scattering amplitude. But in the first subsection we will present the basic general equations which will be used also for the $p$-wave contribution. Then in the rest of Sec. II we will study analytically in detail the case where the mass ratio $m \uparrow/m \downarrow$ is very large. We will conclude this section by considering numerically cases where this mass ratio corresponds to situations which can be reached experimentally. Then in Sec. III before our conclusion we study numerically the $p$-wave contribution.

II. S-WAVE SCATTERING

A. Basic equation

In the same way as in our earlier work \cite{6} we use the integral equation formulation of this dimer-fermion scattering problem first worked out by Skorniakov and Ter-Martirosian \cite{27}. It is convenient to derive it rapidly in the case where the fermion masses are different \cite{28} by writing directly \cite{3} the integral equation satisfied by the full dimer-fermion scattering vertex $T_3(p_1, p_2; P)$. Here $P$ is the (conserved) momentum-energy of the particles $P \equiv \{P,E\}$, with $P$ the total momentum and $E$ the total energy of incoming particles. $p_1$ is the momentum energy of the incoming fermion and $p_2$ its outgoing energy. Since we want the dimer-fermion scattering amplitude, we can take the center-of-mass referential for which $p_1 = k_0$. If we call $k_0$, the outgoing momentum of the fermion (i.e., $p_2 = k_0$), the total energy $E$ is the sum of the binding energy of the dimer and of the kinetic energies of the outgoing particles:

$$E = -\frac{1}{2\mu a^2} + \frac{k_0^2}{2\mu}.$$  \hspace{1cm} (1)

where $\mu = m_\uparrow m_\downarrow/(m_\uparrow + m_\downarrow)$ is the dimer reduced mass, $\mu_T$ is the atom-dimer reduced mass $\mu_T = m_\uparrow M/(m_\uparrow + M) = m_\downarrow (m_\uparrow + m_\downarrow)/(2m_\uparrow + m_\downarrow)$. Here $M = m_\uparrow + m_\downarrow$ is the mass of the dimer. The integral equation satisfied by $T_3(p_1, p_2; P)$ is [3]

$$T_3(p_1, p_2; P) = -G_\downarrow \langle P - p_1 - p_2 - \sum_q G_\downarrow \langle P - q \rangle \times G_\uparrow (q) T_2(P - q) T_3(q, p_2; P),$$  \hspace{1cm} (2)

where $\sum_q \equiv \int d\mathbf{q} d\mathbf{s}/(2\pi)^3$, with $q \equiv \{\mathbf{q}, \Omega\}$. Here $T_2(P)$ is the dimer propagator:

$$T_2(P) = \frac{2\pi}{\mu} \frac{1}{\sqrt{-\frac{2\mu (P^2 - 2M - E)}}},$$  \hspace{1cm} (3)

while $G_\uparrow (q)$ and $G_\downarrow (q)$ are the single fermion $\uparrow$ or $\downarrow$ propagators. Just as in the equal mass case \cite{3} one can show that, in the right-hand side of Eq. (2), only the on-the-shell value of $T_2(q, p_2; P)$ with respect to the variable $q$ is needed. Hence we may restrict ourselves to consider only this on-the-shell value everywhere in Eq. (2). Finally we have also to take the on-the-shell value for $p_2 \equiv \{p_2, \omega_2\}$ to obtain the scattering amplitude, which means $\omega_2 = k_0^2/2m_\downarrow$. We are thus led to consider more specifically $T_3(\{k, k^2/2m_\uparrow\}, \{k_0, k_0^2/2m_\downarrow\}; \{0, E\})$ and the scattering amplitude will be obtained by taking $|k| = k_0$. Hence we introduce, instead of $T_3(p_1, p_2; P)$, a function $a_3(k, k_0)$ which is directly related to the scattering amplitude, as we will see below \cite{27}:

$$a_3(k, k_0) \equiv \frac{\mu_T}{2\mu^2 a} \left[ 1 + \frac{1 + \mu a^2}{\mu_T} (k^2 - k_0^2) \right] \times T_3(\{k, k^2/2m_\uparrow\}, \{k_0, k_0^2/2m_\downarrow\}; \{0, E\}).$$  \hspace{1cm} (4)

In this part we are only interested in the s-wave component of the scattering amplitude. This component is obtained from the above $T_3$ by averaging it over the angle between the incoming momentum $k$ and the outgoing one $k_0$, and we note $\bar{T}_3$ the resulting angular average. Noting merely $a_3(k)$ the corresponding average on $a_3(k, k_0)$ [naturally $a_3(k)$ still depends on $k_0 = |k_0|$], we see that it is simply given by Eq. (4) where $T_3$ has been replaced by $\bar{T}_3$. Obviously when we take the angular average of integral equation Eq. (2), we are left with an integral equation involving only $\bar{T}_3$, or equivalently $a_3$.

It is more convenient to write the resulting integral equation by making use of reduced units, obtained by setting $k = k/\alpha$, $k_0 = k_0/\alpha$, $q = \bar{q}/\alpha$ and $a_3(k)/\alpha = \bar{a}_3(\bar{k})$. We obtain

$$\bar{R} \equiv \frac{\bar{a}_3(\bar{k})}{1 + \sqrt{1 + R (\bar{k}^2 - \bar{k}_0^2)}} = \frac{1}{2R' k_0} \ln \frac{1 - R k_0^2 + k^2 + k_0^2 + R' \bar{k} \bar{k}_0}{1 - R k_0^2 + k^2 + k_0^2 - R' \bar{k} \bar{k}_0}$$

$$- \frac{1}{\pi R'} \int_0^\infty d\bar{q} \frac{\bar{q} \bar{a}_3(\bar{q})}{\bar{q}^2 - \bar{k}_0^2 - i\delta} \frac{1}{\bar{k} \bar{q}}$$

$$\times \ln \frac{1 - R k_0^2 + k^2 + q^2 + R' \bar{k} \bar{q}}{1 - R k_0^2 + k^2 + q^2 - R' \bar{k} \bar{q}}.$$  \hspace{1cm} (5)

where we have used for the involved mass ratios the simpler notations $R = \mu/\mu_T$, $R' = 2\mu/\mu_m$. In writing this equation we have used the fact that we consider only the case where the incoming dimer and fermion have kinetic energies low enough so that the dimer cannot be broken in the scattering process. This implies that $E < 0$, or $R k_0^2 < 1$ in our reduced
units. This makes all the arguments in the logarithms positive. Otherwise we should have included $-i\delta$ contributions in these arguments. In the case where $\bar{k}_0 \to 0$ one checks easily that this equation reduces to the one studied in [6], which gives the scattering length for the same process. It is clear that $\bar{\alpha}_3(\bar{k})$ is a complex quantity. From Eq. (5) one can write the coupled integral equations for its real and imaginary parts, which are convenient for the numerical solution of Eq. (5). In the case where the masses are equal $m_1 = m_1$, one can check that the resulting equations are identical to the ones obtained by Skorniakov and Ter-Martirosian [27] for this specific case.

Finally it is convenient to consider the relation between $\alpha_3(k, k_0)$ and the wave function $\psi_k(r)$ for the dimer-fermion scattering problem, as it is found in [27,29]. For large dimer-fermion distance $r$ it behaves as

$$\psi_k(r) \approx e^{ikr} + f_\theta(r) \frac{e^{ikr}}{r},$$

(6)

where $\theta$ is the angle between the incoming fermion momentum $k$ and its outgoing value $k_0$ (with $|k| = |k_0|$), and $f_\theta(r)$ is the scattering amplitude. Its Fourier transform reads

$$\tilde{\psi}_k(q) = \int dr e^{-iqr} \psi_k(r) = (2\pi)^3 \delta(k - q) - \frac{4\pi \alpha_3(k, q)}{q^2 - k^2 + i\delta}.$$  

(7)

Note that the second term has a sign opposite to the one found in [27], just because our definition for $\alpha_3$ has an opposite sign. The scattering amplitude is obtained from $f_{\theta}(\theta) = -\alpha_3(q, k_0)$ (with $|q| = |k_0|$).

When one takes the angular average of Eq. (6), the resulting $s$-wave component $\psi^s_k(r)$ of the wave function is given by

$$\psi^s_k(r) = \frac{e^{ik_0} - 1}{2ik} \sin(kr + \delta_0),$$

(8)

where $\delta_0(k)$ is the $s$-wave phase shift, linked to $s$-wave component $f_0(k)$ of the scattering amplitude by

$$f_0(k) = \frac{e^{2ik_0} - 1}{2ik}.$$  

(9)

**B. Very light mass**

Let us go now to the interesting regime where the mass of the lonely fermion $m_1$ is very light (or equivalently the mass $m_1$ of the two identical fermions is very heavy). In this case parameter $R$ is near zero and $R'$ is near 2. In our preceding paper [6], where we had $\bar{k}_0 = 0$, we have carefully expanded Eq. (5) near this limit. Then by a Fourier transform we have converted the resulting equation into a differential equation with respect to a variable which is called $\bar{\tau}$ below. We have studied this last equation, matching in particular the behavior for the small and the large values of the variable $\bar{\tau}$ by appropriate consideration of the boundary conditions. However, the important range, in order to obtain the scattering length, corresponds to the large $\bar{\tau}$ values and the proper behavior in this range can also be obtained from physical considerations, as we will see below. We can also proceed by continuity, requiring that in the present case the large variable $r$ behavior reduces to the one we have found in the limit $\bar{k}_0 \to 0$. More specifically we have seen in this preceding work that the careful expansion of the right-hand side of Eq. (5) was useful only to satisfy the perfect matching with the boundary conditions for the small values of the variable $\bar{r}$. On the other hand only the expansion of the left-hand side of Eq. (5) was necessary in order to obtain the large $\bar{r}$ information necessary to obtain the scattering length. Here we will take advantage of this fact to avoid the painful task of expanding the right-hand side of Eq. (5) and rather rely on the above arguments to obtain the required large variable $\bar{r}$ behavior. Hence we will set $R$ and $R'$ to their limiting value, namely $R = 0$ and $R' = 2$ in the right-hand side. In the left-hand side, following our preceding work, we will also set $1 + R(\bar{k}^2 - \bar{k}_0^2) \simeq 1$ because large values of $\bar{k}^2 - \bar{k}_0^2$ are irrelevant in our problem. However, we will naturally keep the overall factor $R$ in front of the left-hand side. This is the only difference with taking the full $m_1 = 0$ limit.

We rewrite Eq. (5) in this limit by making use, instead of $\bar{\alpha}_3(\bar{q})$, of the more convenient function $F(\bar{q})$ defined by

$$F(\bar{q}) = \frac{\bar{q} \bar{\alpha}_3(\bar{q})}{\bar{q}^2 - \bar{k}_0^2 - i\delta}.$$  

(10)

We note that, since from Eq. (5) $\bar{\alpha}_3(\bar{q})$ is an even function of $\bar{q}$, $F(\bar{q})$ is an odd function of $\bar{q}$. Then Eq. (5) becomes

$$\epsilon^2(\bar{k}^2 - \bar{k}_0^2) F(\bar{k}) = \frac{1}{4k_0^2} \ln \left( \frac{1 + (\bar{k} + \bar{k}_0)^2}{1 + (\bar{k} - \bar{k}_0)^2} \right)$$

$$= -\frac{1}{2\pi} \int_0^\infty d\bar{q} \; F(\bar{q}) \ln \left( \frac{1 + (\bar{k} + \bar{q})^2}{1 + (\bar{k} - \bar{q})^2} \right)$$

$$= \frac{1}{2\pi} \int_{\infty}^{-\infty} d\bar{q} \; F(\bar{q}) \ln[1 + (\bar{k} - \bar{q})^2],$$

(11)

where in the last equality we have made use of the odd parity of $F(\bar{q})$ and, as in [6], we have set

$$\epsilon^2 = \frac{R}{2} \simeq \frac{m_1}{m_1}.$$  

(12)

We take now the Fourier transform of this equation, introducing

$$\tilde{F}(\bar{\tau}) = \int_{\infty}^{-\infty} d\bar{q} \; F(\bar{q}) \exp(-i\bar{q}\bar{\tau}),$$

(13)

and making use of

$$\int_{-\infty}^{\infty} d\bar{q} \; \ln(1 + \bar{q}^2) \exp(-i\bar{q}\bar{\tau}) = -2\pi \frac{e^{-\bar{\tau}}}{\bar{\tau}},$$

(14)

where we have restricted ourselves to the case $\bar{\tau} > 0$. We obtain

$$\epsilon^2 \left( \frac{d^2}{d\bar{\tau}^2} + \bar{k}_0^2 \right) \tilde{F}(\bar{\tau}) = \frac{e^{-\bar{\tau}}}{\bar{\tau}} \left( \tilde{F}(\bar{\tau}) + i\pi \frac{\sin(\bar{\tau}k_0)}{k_0} \right).$$

(15)

Since the $\epsilon = 0$ solution,

$$\tilde{F}_0(\bar{\tau}) = -i\pi \frac{\sin(\bar{\tau}k_0)}{k_0},$$

(16)

satisfies $(d^2/d\bar{\tau}^2 + \bar{k}_0^2)\tilde{F}_0(\bar{\tau}) = 0$, it is convenient to set

$$\tilde{F}(\bar{\tau}) = \tilde{F}_0(\bar{\tau}) + i\pi g(\bar{\tau})$$

(17)

to obtain the simpler equation,

$$\epsilon^2 \left( \frac{d^2}{d\bar{\tau}^2} + \bar{k}_0^2 \right) g(\bar{\tau}) = \frac{e^{-\bar{\tau}}}{\bar{\tau}} \tilde{g}(\bar{\tau}).$$

(18)
The fact that we find a second-order differential equation is directly related to our lowest order expansion of Eq. (5) in $m_f/m_\psi$. Going to higher order in this expansion would result in a higher order differential equation.

At this stage it is useful to notice the relation between $g(\vec{r})$ and the scattering wave function Eq. (8). Taking in Eq. (7) the angular average over the direction of $\vec{k}$, performing the angular integration over $\vec{r}$, setting $|\vec{k}| = |\vec{k}_0|$ and making use of the definition Eq. (10), we obtain

$$
4\pi \int_0^{\infty} dr \ r \sin(qr) \psi^2_{\vec{k}_0}(r) = \frac{2\pi^2}{k_0} [\delta(q - k_0) - \delta(q + k_0)] - 4\pi F(q),
$$

(19)

where we have added in the right-hand side the term $\delta(q + k_0)$ which is usually zero since $q = |q| > 0$ and $k_0 = |\vec{k}_0| > 0$. However, we have extended above the range of variation of $q$ to negative values and made use of the fact that the corresponding extension of $F(q)$ is odd. The added term is necessary to satisfy in Eq. (19) this parity property of $F(q)$. Going to reduced variables and taking the Fourier transform of Eq. (19) for $\vec{r} > 0$ we obtain with Eq. (16),

$$
\tilde{F}(\vec{r}) = \tilde{F}_0(\vec{r}) + i \pi \frac{\epsilon_{\vec{k}_0}}{k_0} \sin(\vec{k}_0 \vec{r} + \delta_0).
$$

(20)

Hence from Eq. (17) we have for large $\vec{r}$ the very simple relation,

$$
g(\vec{r}) = \frac{\epsilon_{\vec{k}_0}}{k_0} \sin(\vec{k}_0 \vec{r} + \delta_0).
$$

(21)

This provides us with the boundary condition necessary to solve the second-order differential equation [Eq. (18)], which is accordingly an effective Schrödinger equation for our scattering problem.

Except for the $\tilde{\bar{K}}_0^2$ term this equation [Eq. (18)] is the same as the one we have found in our preceding work [6] and we solve it by following the same procedure. Performing the change of variable $\vec{r} = 2 \ln(2/z)$, implying $z = 2e^{-r/2}$, we find that $\tilde{q}(z) = g(\vec{r}(z))$ satisfies

$$
z^2 \frac{d^2 \tilde{q}(z)}{dz^2} + z \frac{d\tilde{q}(z)}{dz} + 4k_0^2 \tilde{q}(z) - \frac{z^2}{L(z)} \tilde{q}(z) = 0,
$$

(22)

with $L(z) = 2e^{2 \ln(2/z)} = e^{2\vec{r}}$. We notice again that, in the range of the variable which is of interest, that is large $\vec{r}$, corresponding to small $z$, $L(z)$ is a quite slowly varying function of $z$. This allows one to treat it as a constant $L(z) \simeq L$. As discussed in detail in [6] this approximate treatment is increasingly accurate when the mass ratio $m_f/m_\psi$, that is, $e^2$, becomes increasingly small, which is precisely the limit we are interested in. We have found [6] that the important range for matching the solution of Eq. (18) to the asymptotic behavior, in the present case Eq. (21), is $\vec{r} \sim \vec{r}_0 = \ln(1/e^2)$. Hence we may take $L = e^2 \ln(1/e^2)$.

Then the further change of variable $z = \sqrt{\vec{L}} \ k x$ transforms Eq. (22) into

$$
x \frac{d^2 \tilde{q}(x)}{dx^2} + \frac{d\tilde{q}(x)}{dx} + 4k_0^2 \tilde{q}(x) - x \tilde{q}(x) = 0,
$$

(23)

for $\tilde{g}(x) = \tilde{q}(z)$. The solution of this equation which matches, as we will see shortly, the asymptotic behavior Eq. (21) is [30] the well-known Bessel functions $K_{2\vec{a}_0}(x)$ [or equivalently $K_{-2\vec{a}_0}(x)$], which goes continuously to the solution $K_0(x)$ that we have found in [6] for the case $\vec{k}_0 = 0$. A convenient integral representation for this function is [30]

$$
K_{2\vec{a}_0}(x) = \int_0^\infty dt \ e^{-x \cosh t} \cos(2\vec{a}_0 t).
$$

(24)

From this representation one obtains the small $x$ behavior,

$$
K_{2\vec{a}_0}(x) \simeq \Re \left[ \left( \frac{2}{x} \right)^{2i\vec{a}_0} \Gamma(2i\vec{k}_0) \right].
$$

(25)

where $\Gamma(x)$ is the Euler function. From this result one recovers in particular, for $\vec{k}_0 \rightarrow 0$, the known small $x$ asymptotic behavior $K_0(x) \simeq \ln(2/x) - C$ where $C$ is the Euler constant.

Since $2/x = \sqrt{\vec{L}} \ e^{\vec{r}}$ we see that, for large $\vec{r}$, our solution has indeed the required asymptotic behavior [Eq. (21)]. This is naturally within an unimportant constant prefactor since the homogeneous differential equation [Eq. (23)] does not allow one to find this prefactor. But, as soon as $\delta_0$ is found, this prefactor is given explicitly by Eq. (21). Comparing Eq. (25) with Eq. (21) in order to identify the phase shift, we find easily,

$$
\delta_0(\vec{k}_0) = \vec{k}_0 \ln L + \arg[i \Gamma(2i\vec{k}_0)] = \vec{k}_0 \ln[e^2 \ln(1/e^2)] + \arg[i \Gamma(2i\vec{k}_0)].
$$

(26)

In particular in the limit $\vec{k}_0 \rightarrow 0$ we can make use of $\Gamma(2i\vec{k}_0) \simeq 1/(2i\vec{k}_0) = C$, implying $\arg[i \Gamma(2i\vec{k}_0)] \simeq -2i\vec{k}_0 C$. Comparing to the limiting behavior $\delta_0(\vec{k}_0) \simeq -\vec{a}_0 \vec{k}_0$ we recover our result [6],

$$
\vec{a}_0 \equiv \frac{\vec{a}_0}{a} = -\ln L + 2C = \ln \frac{m_f}{m_\psi} - \ln \frac{m_f}{m_\psi} + 2C,
$$

(27)

for the fermion-dimer scattering length in the considered limit where the mass ratio $m_f/m_\psi$ is large.

Once the phase shift is known we obtain the scattering amplitude by merely making use of Eq. (9). This is obviously the simpler way. However, let us briefly show that we recover this result by calculating directly $\alpha_0(\vec{q}, \vec{k}_0)$ from our solution $K_{2\vec{a}_0}(x)$ of Eq. (23). From Sec. II A we have

$$
\tilde{f}_0(k_0) \equiv f_0(k_0) = -\lim_{\vec{q} \rightarrow \vec{k}_0} \tilde{a}_0(\vec{q}) = -2 \lim_{\vec{q} \rightarrow \vec{k}_0} (\vec{q} - \vec{k}_0) F(\vec{q}).
$$

(28)

the last equality resulting from Eq. (10). In the Fourier transform of the decomposition Eq. (17) we see that $F_0(\vec{q})$ does not contribute to the right-hand side of Eq. (28) since, as we have seen in Eq. (19),

$$
F_0(\vec{q}) = \frac{\pi}{2k_0} [\delta(\vec{q} - \vec{k}_0) - \delta(\vec{q} + \vec{k}_0)],
$$

(29)

so that, in this right-hand side, we are left only with the contribution of $\tilde{g}$:

$$
\tilde{f}_0(k_0) = -2i \lim_{\vec{q} \rightarrow \vec{k}_0} (\vec{q} - \vec{k}_0) i \pi \int_{-\infty}^{\infty} \frac{d\vec{r}}{2\pi} g(\vec{r}) \exp(i\vec{q}\vec{r}).
$$

(30)
We have seen that the solution of Eq. (23) is proportional to $K_{2i\tilde{k}_0}(x)$. Going back to the $\tilde{r}$ variable this translates into

$$g(\tilde{r}) = \alpha(\tilde{k}_0)K_{2i\tilde{k}_0}\left(\frac{2e^{-\tilde{r}^2/2}}{\sqrt{L}}\right), \quad (31)$$

where, as mentioned above, the prefactor $\alpha(\tilde{k}_0)$ cannot be obtained from the homogeneous equation [Eq. (23)] and has to be determined from the asymptotic behavior [Eq. (21)] for $g(\tilde{r})$. From the asymptotic behavior Eq. (25) we have

$$K_{2i\tilde{k}_0}\left(\frac{2e^{-\tilde{r}^2/2}}{\sqrt{L}}\right) \simeq \left|\Gamma(2i\tilde{k}_0)\right| \sin(\tilde{k}_0\tilde{r} + \delta_0), \quad (32)$$

where $\delta_0$ is given by Eq. (26). Comparing Eq. (31) with Eq. (21) we find

$$\alpha(\tilde{k}_0) = \frac{1}{\tilde{k}_0|\Gamma(2i\tilde{k}_0)|}. \quad (33)$$

The Fourier transform of $g(\tilde{r})$ is then

$$\int_{-\infty}^{\infty} \frac{d\tilde{r}}{2\pi} g(\tilde{r}) \exp(i\tilde{q}\tilde{r}) = \int_{0}^{\infty} \frac{d\tilde{r}}{2\pi} g(\tilde{r}) \exp(i\tilde{q}\tilde{r}) - (\tilde{q} \to -\tilde{q}), \quad (34)$$

where we have made use of the odd parity of $g(\tilde{r})$ directly linked to the odd parity of $F(\tilde{q})$. Going back to the variable $x = 2e^{-\tilde{r}^2/2}/\sqrt{L}$ we have

$$\int_{0}^{\infty} \frac{d\tilde{r}}{2\pi} g(\tilde{r}) \exp(i\tilde{q}\tilde{r}) = \frac{\alpha(\tilde{k}_0)}{\pi} \int_{0}^{\infty} \frac{d\tilde{r}}{2\pi} \exp(i\tilde{q}\tilde{r}) K_{2i\tilde{k}_0}\left(\frac{2e^{-\tilde{r}^2/2}}{\sqrt{L}}\right)$$

$$= \frac{\alpha(\tilde{k}_0)}{\pi} \left(\frac{\sqrt{L}}{2}\right)^{-2i\tilde{q}} \int_{0}^{\infty} dx x^{-2i\tilde{q} - 1} K_{2i\tilde{k}_0}(x), \quad (35)$$

where in the last step we have replaced the upper bound $2/\sqrt{L}$ by $\infty$, since in our case $L = \epsilon^2 \ln(1/\epsilon^2) \ll 1$ and, as it is clear from Eq. (24), $K_{2i\tilde{k}_0}(x)$ decreases exponentially rapidly for large $x$. The last integral can be performed analytically [30], leading to

$$\int_{-\infty}^{\infty} \frac{d\tilde{r}}{2\pi} g(\tilde{r}) \exp(i\tilde{q}\tilde{r}) = \frac{\alpha(\tilde{k}_0)}{4\pi} L^{-i\tilde{q}} \Gamma(i(\tilde{k}_0 - \tilde{q}))(i(\tilde{k}_0 + \tilde{q}))) - (\tilde{q} \to -\tilde{q}). \quad (36)$$

Since $\Gamma(x) \simeq 1/x$ for $x \to 0$, we find from Eqs. (30) and (36),

$$\tilde{f}_0(\tilde{k}_0) = \frac{\alpha(\tilde{k}_0)}{2} [L^{i\tilde{k}_0} \Gamma(2i\tilde{k}_0) + L^{-i\tilde{k}_0} \Gamma(-2i\tilde{k}_0)]. \quad (37)$$

However, we have from Eq. (26),

$$\sin(\delta_0(\tilde{k}_0)) = L^{i\tilde{k}_0} \left(\frac{\Gamma(2i\tilde{k}_0)}{2\Gamma(2i\tilde{k}_0)}\right) + L^{-i\tilde{k}_0} \left(\frac{\Gamma(-2i\tilde{k}_0)}{2\Gamma(-2i\tilde{k}_0)}\right). \quad (38)$$

Taking Eq. (33) into account this leads to

$$\tilde{f}_0(\tilde{k}_0) = e^{i\tilde{k}_0} \sin(\delta_0(\tilde{k}_0)) \frac{1}{\tilde{k}_0}, \quad (39)$$

which, when we go back to unreduced units, is identical to Eq. (9) as it should be. This shows that our calculations and approximations in the very light mass limit are fully consistent.

C. Discussion

Let us now consider the implications of our result Eq. (26) for the phase shift and the corresponding scattering amplitude. Although the Euler $\Gamma$ function is easily obtained numerically, we have more insight if we make use of an excellent approximation. Since for real $x$ we have

$$\arg[i\Gamma(i\tilde{x})] = \arg[i\tilde{x}\Gamma(i\tilde{x})] = \arg[\Gamma(1 + i\tilde{x})] = \text{Im}[\ln(1 + i\tilde{x})], \quad (40)$$

we may use the excellent Stirling formula $\ln \Gamma(z) \simeq (z - 1/2) \ln z - z + (1/2) \ln(2\pi)$ which leads to

$$\delta_0(\tilde{k}_0) = -\tilde{k}_0\tilde{a}_3 + \frac{1}{2} \arctan(2\tilde{k}_0)$$

$$+ 2(C - 1)\tilde{k}_0 + \tilde{k}_0 \ln (1 + 4\tilde{k}_0^2). \quad (41)$$

If we were to correct the small imperfection of the Stirling formula for low $z$ by replacing $2C \simeq 1.154$ by 1, this would allow Eq. (41) to give the exact result $\lim_{\tilde{k}_0 \to 0} \delta_0(\tilde{k}_0) = -\tilde{k}_0\tilde{a}_3$ with $\tilde{a}_3$ given by Eq. (27), but the result would not be accurate for higher values of $\tilde{k}_0$. Our original result Eq. (26) and its simplified form Eq. (41) are displayed in Fig. 1 for $m_\ell/m$ = 10 and 100. In this last case the two results are essentially indistinguishable in the figure.

We can see that $\delta_0(\tilde{k}_0)$ displays first a strong decrease from zero down to large negative values after which it increases back to small negative values, and even becomes positive in the case of large mass ratios. For large $\tilde{a}_3$ approximate analytical results may be obtained for the location of the minimum and its value. They both scale as $\sqrt{m_\ell/m}$. The essential qualitative feature

FIG. 1. $\ell = 0$ phase shift $\delta_0(\tilde{k}_0)$ as a function of $\tilde{k}_0 = k_0 \alpha$, given by Eq. (26) (dashed line) and its asymptotic approximation given by Eq. (41) (solid line) for (a) $m_\ell/m = 10$ and (b) $m_\ell/m = 100$. 

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FIG. 2. $\ell = 0$ phase shift $\delta_0(k)$ for a repulsive square potential with strength $V_0$ and range $R$, as a function of $kR$, in the case $(2mV_0)^1/2R/\hbar = 10$. The oscillations beyond the maximum are due to the discontinuous rise of the potential.

FIG. 3. Modulus of the $\ell = 0$ scattering amplitude $|f_0(k_0)|$ as a function of $k_0$, for mass ratio $m_1/m_4 = 100$. Dashed line, asymptotic result; solid line, exact numerical result.

of this result, namely the strong decrease of the phase shift, is directly linked to the large positive value of the scattering length. For example, it is qualitatively fairly analogous to the one found for the $\ell = 0$ phase shift of a square repulsive potential with strength $V_0$ and range $R$, which is given by

$$\delta_0(k) = -kR + \arctan \left( \frac{k}{k'} \tan(k' R) \right). \quad (42)$$

where $k$ is related to the energy $E$ by $E = \hbar^2k^2/2m$ and $E - V_0 = \hbar^2k^2/2m$. This is plotted in Fig. 2 for $(2mV_0)^1/2 R/\hbar = 10$.

We consider now the consequences of the above result for the $s$-wave scattering amplitude which is obtained from Eq. (9). We are mostly interested in its modulus $|f_0(k_0)| = |\sin(\delta_0(k_0))|/k_0$. The result is plotted in Fig. 3 for $m_1/m_4 = 100$. The very interesting feature is the existence of several zeros occurring when $\delta_0(k_0) = n\pi$. The first few zeros are merely located at $k_0 = n\pi/\alpha$. Physically these zeros imply that there is no contribution from $s$-wave scattering to the total cross section. We see also that the scattering amplitude is strongly peaked for zero energy, a feature which is even more important for the corresponding contribution $\sigma_0(k_0) = 4\pi a^2 f_0^2(k_0)$ to the cross section. This feature is clearly linked to the strong decrease of the phase shift when $k_0$ starts from zero. As mentioned above this is directly linked to the large value of the scattering length, and consequently this is clearly not an artifact of our approximate treatment. This behavior implies that a rough model for the $s$-wave component of the effective fermion-dimer interaction could be a constant up to the energy $(\hbar^2/2a^3)^2/2\mu_T$ followed by no interaction at all for higher relative kinetic energy. This is quite different from the simple picture of a constant interaction, whatever the energy, which is valid, for example, when the fermions have equal masses.

It is also interesting to note that this qualitative behavior should also be valid even if finite range corrections are taken into account, with respect to the zero-range interaction which is the framework we have taken from the start. These finite range corrections are of importance in narrow Feshbach resonances, and these happen to occur very frequently experimentally. We expect our qualitative picture to hold even in this case because, as we have just explained, it is basically linked to the increase of the scattering length with mass ratio. This increase is linked to the physics occurring when the atoms are far apart, as it is clear, for example, from the value of the matching distance $\bar{r}_0 = \ln(1/\epsilon^2)$ we have found above. Hence finite range corrections, which are of importance only when atoms are at short distance, should not modify our qualitative behavior of the scattering length and, for example, the zeros will still be present.Finite range corrections should only lead to small shifts in the location of these zeros.

These results we have obtained appear quite naturally in the small $m_4/m_1$ limit. Their qualitative features are interesting enough to check that they are not just a curiosity of this simple, but not so physical, limit and that they rather stay valid qualitatively in the physical range for atomic mass ratios. We have checked that our result is in semiquantitative agreement with the exact numerical result from Eq. (5). This is displayed first for $m_1/m_4 = 100$ which is somewhat beyond realistic values. We see from Fig. 3 that our approximation is in fair agreement with the exact one for reasonably small energies (the discrepancy at and near zero energy is just due to the difference between our approximate value for the scattering length and the exact one). There is, however, an increasing disagreement for higher values of the energy, which is not so surprising. Our approximate phase shift is varying with energy somewhat less rapidly than the exact one.

If we go now to the more realistic value corresponding to a hypothetical mixture of $^6$Li and $^{172}$Yb displaying a Feshbach resonance, we obtain for this mass ratio $m_1/m_4 = 173/6 \simeq 29$ the results reported in Fig. 4. It is quite interesting to see that the results are not deeply modified compared to the low-energy part of the preceding figure. Both the approximate and the exact results display a single zero. Just as in the preceding case, the exact zero is at a somewhat lower energy than what our approximate analytical expression gives. It is interesting that the relative location of this zero is in this order. Indeed our figure shows that the approximate zero is near the dissociation threshold of the dimer, and that it is no longer present for slightly lower values of the mass ratio $m_1/m_4$. On the other hand the exact zero is only halfway to the dissociation threshold.
Since the existence of this zero is a landmark of a strongly varying \( s \)-wave scattering amplitude, it is of high interest that it exists for even lower, and accordingly more realistic, mass ratio \( m_1/m_1 \). We have found specifically that the exact zero is disappearing for \( m_1/m_1 \approx 15.27 \). This result is in agreement with a recent result of Helfrich and Hammer [31]. They have calculated, in particular, as a function of the mass ratio, the fermion-dimer \( s \)-wave component of the breakup threshold, and found indeed in their Fig. 3 that it goes to zero for a mass ratio in the vicinity of 15.

If we take the lighter fermion to be \( ^6\text{Li} \), a mass ratio of 15 would imply for the heavier one a mass quite near the one of rubidium. Unfortunately the two stable isotopes of rubidium are bosonic, but there are two fermionic ones, \(^{86}\text{Rb} \) and \(^{84}\text{Rb} \), which have half-life of order of a month. Finally if we take the case of the \(^6\text{Li} - ^{40}\text{K} \) mixture, which is displayed in Fig. 5, we see that the scattering cross section (obtained by squaring the amplitude) near the dissociation threshold is only 5\% of its zero energy value.

Finally it is worth noting that the disappearance of this zero in the scattering amplitude when the mass ratio is lowered is strikingly similar to the same disappearance [6,9] in the function \( a_3(k,0) \), which comes in the calculation of the scattering length \( a_1 \). In this last case the disappearance occurs at a somewhat lower value of the mass ratio, and \( a_1(k,0) \) has no simple physical meaning, which makes the interpretation difficult. However, one could go continuously from this function to the scattering amplitude by varying the last parameter from 0 to \( k \). Hence the two functions are closely related, and the simple physical explanation for the zero in the scattering amplitude (it occurs when the phase shift reaches \( \pi \)) gives a corresponding understanding for \( a_3(k,0) \).

### III. P-Wave Scattering

We turn now to the \( p \)-wave contribution to the scattering amplitude. The analysis is done with the Skorniakov and Ter-Martirosian equations as detailed in Sec. II A. The only difference is that, instead of averaging Eq. (4) to obtain the \( s \)-wave component, in order to obtain the \( p \)-wave component \( a_{3p}(k,k_0) \),

\[
\alpha(k,q) = 1 - Rk_2^2 + k^2 + q^2, \quad \beta(k,q) = R'kq,
\]

and defining the kernels,

\[
K_s(k,q) = \frac{1}{2\beta(k,q)} \ln \left| \frac{\alpha(k,q) + \beta(k,q)}{\alpha(k,q) - \beta(k,q)} \right|,
\]

and

\[
K_p(k,q) = \frac{1}{2\beta(k,q)} \ln \left| \frac{\alpha(k,q) + \beta(k,q)}{\alpha(k,q) - \beta(k,q)} \right|.
\]

Then the equations for the \( s \)-wave and \( p \)-wave components can be written:

\[
K_{3p}(k_0) =\frac{2}{\pi} \int_0^\infty dq \frac{q^2 a_{3p}(q,k_0)}{q^2 - k_0^2 - i\delta} K_{3p}(k,q),
\]

where \( a_{3p}(k,k_0) \equiv \tilde{a}_3(k) \) used in Sec. II.

Since we are interested in this part in mass ratios smaller than in the preceding Sec. II, we provide first the results for the \( s \)-wave component for the specific mass ratios of interest. The results for \( |f_0(k_0)| \) in terms of \( k_0 \) are given in Fig. 6, for the mass ratios \( m_1/m_1 = 1,2,4,8 \). We have also inserted the result for \( m_1/m_1 = 6.64 \), corresponding to the \(^6\text{Li} - ^{40}\text{K} \) mixture, already displayed in Fig. 5. Except for the fairly slow increase of the atom-dimer scattering length with the mass ratio found at \( k_0 = 0 \), all the results are of order unity for \( k_0 = 1 \) and keep decreasing for increasing \( k_0 \).

We turn now to the \( p \)-wave component results. We first give in Fig. 7 the results for mass ratios \( m_1/m_1 = 1,2,4 \), and 6,64, stopping at the value corresponding to the \(^6\text{Li} - ^{40}\text{K} \) mixture. For equal masses the \( p \)-wave component is always small, as could be expected, and it can clearly be omitted when one deals with the atom-dimer scattering properties. Hence the atom-dimer vertex can, to a large extent, be taken as a constant proportional
the atom-dimer scattering length. No complication is thus expected to arise from this side when many-body properties will be investigated. The situation is qualitatively similar for $m_1/m_1 = 2$ and even 4; The $p$-wave component is fairly small and can be neglected compared to the $s$-wave contribution. However, we see that this $p$-wave component rises steeply when the mass ratio goes from 4 to 6.64. This last ratio is at the border of the resonant domain.

This domain is now displayed in Fig. 8, where the results for mass ratios $m_1/m_1 = 6.64, 7, 7.5, 7.75, 7.875, and 8$ are plotted. We see that the hump, present for $m_1/m_1 = 6.64$ around $k_0 \approx 0.45$, develops into a strong resonance at lower and lower energy when $m_1/m_1$ is increased. Although we have sampled mass ratios which are very close to each other, we see that the resonance grows very rapidly with increasing mass ratio. The physical origin is clearly the development of a virtual bound state at positive energy. The lifetime of this state is directly related to the width of the resonance. Since the resonance peak gets quickly narrower, the lifetime grows very rapidly with increasing mass. The link with the bound state which appears \cite{10} at zero energy for $m_1/m_1 = 8.17$ is confirmed if we look at the position of the resonance peak as a function of the mass ratio. This is shown in Fig. 9. We see that the position $k^*_0$ of the resonance peak extrapolates to zero for a mass ratio which is in close vicinity of 8.17. More precisely when the resonance peak reaches $k^*_0 = 0$, it will be infinitely sharp, corresponding to an infinite value for $|f_1(k_0) = 0|$. Hence in this case $a_{p}(k, 0)$ will be infinitely large, which implies that the homogeneous Eq. (46) \cite{17} [i.e., without the term $K_p(k, 0)$ in the right-hand side] has a solution for zero energy $k_0 = 0$. This is just stating the well-known result that bound states are solutions of the homogeneous integral equation corresponding to Eq. (46). It is easy to find the lowest $m_1/m_1$ for which this homogeneous integral equation has a solution. We find that this occurs for $m_1/m_1 = 8.172$ in full agreement with Kartavtsev and Malykh \cite{10}.

Let us now come to the total scattering cross section $\sigma(k_0)$, which is likely to be the easiest direct physical quantity to measure experimentally. Since we neglect angular momenta higher than $\ell = 1$, we have for this cross section,

$$\frac{\sigma(k_0)}{4\pi} = \sum_\ell (2\ell + 1)|f_\ell(k_0)|^2 = |f_0(k_0)|^2 + 3|f_1(k_0)|^2.$$  

(47)

The result is plotted in Fig. 10 for $m_1/m_1 = 1, 2, 4,$ and 6.64. We see that, while up to $m_1/m_1 = 4$, the cross section is
as a function of the resonance, since for lower wave vectors the scattering is essentially negligible. One should see something similar to a threshold effect in density. We note that the resonance should also have an important effect even on a balanced mixture of $^{40}\text{K}$ and $^6\text{Li}$, on the BEC side, when the temperature is raised. Indeed the dimers will be partially broken by thermal excitation which provides a natural source of free $^{40}\text{K}$ atoms, and effects analogous to the ones arising in the unbalanced mixture. Similarly, going to high temperature, we expect this resonance to have a marked effect on the virial coefficients since these are systematically related to the three-body problem [33]. Naturally the resonance is also expected to affect strongly the transport properties. Both the effects on the equation of state and on the transport properties should appear in the frequency and the damping of the collective modes [1].

Another way to understand the effect of this resonance on the $^6\text{Li}^{40}\text{K}$ mixtures is to remark that the long-lived virtual bound states responsible for it should behave in many respects in a way analogous to real bound states, as it has already been noted by Levinsen et al. [25]. Hence on the BEC side the physical description should involve not only free fermions and dimers, but also the existence of trimers. Obviously all the physical properties should be affected by the presence of this additional fermion species. A related point is that the existence of these trimers should affect the dimer-dimer scattering properties. In other words we have only explored the simpler fermion-dimer scattering properties, but we expect that the dimer-dimer scattering amplitude will display related energy structure, and they should not be so complicated to explore with the methods of our preceding work [3].

Finally it is worth stressing that the use of optical lattices should provide a very convenient and powerful way to explore the resonance we have pointed out. Indeed the natural mass ratio of $6.64$ between $^{40}\text{K}$ and $^6\text{Li}$ atoms happens to be just at the border of the strongly resonating mass ratio domain. A rather weak optical lattice could be used to slightly tune the effective mass of $^{40}\text{K}$ or $^6\text{Li}$ (by making an appropriate choice of the light frequency, only one atomic species is affected), producing strong modification of the resonance and hence of the physical properties of the mixture. In this way one could go into the strongly resonating domain, or even reach the threshold for the appearance of real bound states; or one could go in the other direction and basically get rid of the resonance, which would allow one to prove that it is responsible for specific physical properties of the mixture. More specifically the three-dimensional potential $V_{\text{opt}}(r) = s E_R [\sin^2(K_0x) + \sin^2(K_0y) + \sin^2(K_0z)]$ with $E_R = K_0^2/2m$ provides, within second-order perturbation theory, an effective mass $m'$ given by $m/m' = 1 - s^2/32$. Increasing the $^{40}\text{K}$ effective mass to reach the bound state ratio $8.17$ requires $s = 2.45$ which corresponds to a fairly weak optical potential. This simple picture of effective mass modification works only if the size of the involved objects (dimer, trimer) is large compared to the optical wavelength $\lambda$. Taking $\lambda \sim 500$ nm this should be a valid approximation in the vicinity of the Feshbach resonance, where the scattering length becomes quite large. Even if corrections to this simple picture are necessary, the qualitative physical trends should remain valid.
IV. CONCLUSION

In this paper we have studied the $s$-wave and $p$-wave contributions to the fermion-dimer scattering amplitude as a function of the mass ratio. When masses are equal the physical situation is quite simple. The $p$-wave contribution is completely negligible, and the modulus of the $s$-wave contribution is essentially constant. Hence the energy dependence of the scattering properties is inessential, and the fermion-dimer scattering length $a_{\text{f-d}}$ is enough to fully characterize these properties.

When the mass ratio is increased the situation becomes much more complex, both for $s$ wave and for $p$ wave. For the $s$-wave contribution we have obtained an analytical solution in the asymptotic limit of very large mass ratio. This allows one to have explicitly the behavior of the $s$-wave scattering amplitude. We find that it displays a large number of zeros.

For the $p$-wave contribution the behavior gets rapidly quite complex as soon as the mass ratio has a sizable value. This is basically due to the appearance of a fermion-dimer bound state when the mass ratio reaches the value $m_1/m_\downarrow = 8.17$, and we have limited our study to mass ratios below this threshold. Even below this threshold the existence of the bound state appears through the existence of virtual bound states at positive energy. These give rise to a resonance in the scattering amplitude. This resonance gets stronger and goes to lower energy when the mass ratio is increased toward the limiting value of $m_1/m_\downarrow = 8.17$, at which the resonance corresponds to a divergence at zero energy. It is very interesting that the mass ratio $m_1/m_\downarrow = 6.64$, corresponding to the $^{40}\text{K}^{4}\text{Li}$ mixtures, much studied experimentally, is at the border where the $p$-wave resonance becomes important. Roughly, below this mass ratio we find a fairly small bump in the modulus of the $p$-wave scattering amplitude, which is fairly unimportant in the overall scattering properties and may be omitted in a first approach. On the other hand above this mass ratio, the $p$-wave resonance grows very rapidly and becomes the dominant feature of the scattering. We have stressed explicitly that the use of optical lattices could allow one to vary experimentally the mass ratio throughout this very interesting range of mass ratio. One could in this way disentangle the role of this $p$-wave resonance in the physical properties of these $^{40}\text{K}^{4}\text{Li}$ mixtures, which are likely to be quite complex. In the same way one could check if these virtual bound states play an important role in the instabilities of these mixtures.


[28] This has been done also recently by Iskin and Sá de Melo [8].


