

London penetration depth of strongly coupled isotropic superconductors: Low-temperature behavior

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We proceed to a systematic exploration of the low-temperature dependence of the London penetration depth of isotropic superconductors within strong-coupling theory in the clean limit. For a sizable range of parameters, we find that strong-coupling effects can reasonably simulate a power law dependence, sometimes with excellent precision. In such cases it would be quite difficult to distinguish experimentally between a pure power law and the strong-coupling result. Physically we have been able to ascribe this temperature dependence to low-frequency phonons which produce a quasielastic scattering for electrons. The presence of these low-frequency phonons requires rather wide phonon spectra and their effectiveness in scattering implies a fairly strong coupling. [S0163-1829(96)00529-2]

I. INTRODUCTION

The temperature dependence of the London penetration depth $\lambda_L(T)$ plays an important role in the ongoing debate about the mechanism of high- T_c superconductivity. It is indeed an essential piece of information about this mechanism to know the symmetry of the order parameter.¹ While an s -wave-type order parameter (that is, having a fixed sign over the Fermi surface) finds a natural explanation in terms of a purely attractive interaction, an order parameter which changes sign over the Fermi surface, such as the d -wave one produced by the spin fluctuation mechanism, points toward some repulsive component in the pairing interaction. A change of sign for the order parameter implies the existence of nodes of the gap and hence of low-energy elementary excitations in the superconducting phase. These should manifest themselves experimentally in numerous low-temperature properties. However, the interpretation of many experiments is not so easy. In this respect the penetration depth is a very interesting quantity to measure since it is a thermodynamic quantity and its measurement is less likely to be perturbed by extrinsic defects than a dynamical quantity. Moreover, it is a bulk quantity since one explores the sample over typically 1000 Å, in contrast to photoemission or tunneling where only a few atomic layers at the surface are involved, which raises the fear that the surface might be perturbed or perturbing in some way.

The existence of low-energy excitations should produce for $\lambda_L(T)$ a low-temperature dependence stronger than the standard BCS behavior, which is essentially flat. Therefore many experimentalists have looked at this low- T behavior. In particular they have quite often tried to fit this behavior with a power law T^n since this would allow one to find if the nodes of the gap, if any, are located on points or lines on the Fermi surface. Indeed many experiments on Y-Ba-Cu-O and Bi-Sr-Ca-Cu-O (Ref. 2) have found near T^2 behaviors which have been interpreted as proving the existence of point nodes. More recently Y-Ba-Cu-O crystals³ and films⁴ have shown a T behavior, pointing toward lines of nodes at the Fermi surface. There is, however, an implicit assumption in this kind of conclusion. This is that no s -wave superconduc-

tor can give a low-temperature dependence compatible with experiment. This assumption has been mainly challenged by pointing out that strongly anisotropic s -wave superconductors will clearly give a low- T behavior stronger than standard BCS behavior. A related point is that normal layers will also produce low-energy excitations and a strong low- T dependence.⁵

However, even for a completely isotropic superconductor there is also another direction to look for possible physical effects which would spoil the standard interpretation. Indeed, when comparison is made with experiment, the standard reference theory for s -wave superconductivity is the weak-coupling BCS theory. On the other hand there are good indications that high- T_c superconductors might be in the strong-coupling regime. Indeed, in addition to the high value of T_c itself, resistivity as well as infrared experiments give for the inverse quasiparticle lifetime at T_c a value which is of order of T_c itself, in clear contradiction with the weak-coupling assumption which requires it to be much smaller than T_c . It is therefore worthwhile to wonder about strong-coupling effects on the penetration depth at low temperature. Although much work⁶ has been devoted to the strong-coupling theory of $\lambda_L(T)$ and qualitative suggestions have been made on strong-coupling effects,⁷ there has been to our knowledge no quantitative and systematic study of the low-temperature behavior. This is naturally easy to understand since, before the discovery of high- T_c superconductivity and the recent debate on the symmetry, there was no specific reason to look in detail at this specific point. It is our purpose in this paper to consider this theoretical question in detail, both as a basic exploration in strong-coupling theory and as a reference for the interpretation of experiments in high- T_c compounds. We will consider only isotropic superconductors. Our purpose is indeed to show that even in this case the low-temperature behavior can be qualitatively different from the BCS result. Naturally strong anisotropy will produce important additional changes at low temperature. We will come back on this point at the end of the paper. In the next section we give our procedure for this systematic exploration. In Sec. III we give our results and discuss their physical origin.

Finally we discuss in the last section the consequences of our findings.

II. POSITION OF THE PROBLEM

The fact that the penetration depth is a thermodynamic quantity is an important advantage for the study of strong-coupling effects. Indeed the input of any calculation in the strong-coupling Eliashberg theory is the so-called Eliashberg function $\alpha^2F(\Omega)$ which gives the strength of the electron-phonon interaction for a given phonon frequency Ω (naturally we might consider as well an attractive pairing interaction caused by the exchange of other kinds of bosons, but in the following we will consider phonons for simplicity). In principle knowledge of $\alpha^2F(\Omega)$ is equivalent to fixing an infinite number of parameters. In practice this forces us to consider a set of models for $\alpha^2F(\Omega)$ depending on a restricted number of parameters. These models are in general either inferred from experimental data or chosen for various theoretical reasons. Nevertheless, this procedure is not satisfactory since the chosen models contain necessarily some arbitrary ingredients: Either these are irrelevant and it would be better to know it, or they are important, in which case they are not under control.

However, it has been pointed out recently that it is possible to overcome this difficulty for thermodynamic quantities.⁸ Indeed the very structured Eliashberg function $\alpha^2F(\Omega)$ enters actually the imaginary-axis Eliashberg equations only through the very smooth spectral function $\lambda(\omega) = \int d\Omega \, 2\alpha^2F(\Omega) \, \Omega / (\Omega^2 + \omega^2)$ which gives physically the frequency dependence of the effective pairing interaction. It has been shown⁸ that it is possible in a systematic way to approximate very well (within a few percent) all the possible $\lambda(\omega)$ by a set of functions, depending on five parameters, which provide in this way a “representation” of all the possible spectra $\alpha^2F(\Omega)$. For the critical temperature and the gap, it was indeed found that the difference between the quantity calculated from the original spectrum and from its representation differed at most by a few percent. Although there are clearly many different ways to choose the functions which give the representation, we have found it convenient to use all the functions generated by a spectrum made of two Einstein peaks.⁸ In this case the five parameters are, in addition to the Coulomb pseudopotential μ^* , the frequencies Ω_1 and Ω_2 of the two peaks and their weights λ_1 and λ_2 , where $\lambda_1 + \lambda_2 = \lambda$, with λ being the total coupling strength. We set $r = \Omega_2 / \Omega_1$ and $\rho = \lambda_2 / \lambda_1$. These parameters are obtained from the original spectrum⁸ by $r = (1 + b\rho^{-1/2}) / (1 - b\rho^{1/2})$, where $1 + b^2 = \langle \Omega^2 \rangle / \langle \Omega \rangle^2$. And ρ is the solution of $r^{-\rho/(1+\rho)}(1+r\rho) / (1+\rho) = \langle \Omega \rangle / \Omega_{\log}$, which is easily solved numerically. Here $\langle \Omega^\beta \rangle = \int d\Omega \, \Omega^\beta \, 2\alpha^2F(\Omega) / \lambda \Omega$ and $\Omega_{\log} = \lim_{\beta \rightarrow 0} \langle \Omega^\beta \rangle^{1/\beta}$. We stress that the exploration of this representation will allow us to cover all the possible cases arising in strong-coupling theory.

In the present case the situation simplifies somewhat since we will consider the variation of the reduced penetration depth $\lambda_L(T) / \lambda_L(0)$ as a function of the reduced temperature T/T_c . Therefore the absolute values of the frequencies will naturally be irrelevant and, in addition to μ^* , we are left with only three parameters to vary, namely, the total cou-

pling strength λ , the ratio r of the peak frequencies which is a measure of the width of the spectrum, and the relative weight $p = \lambda_2 / \lambda$ of the high-frequency peak compared to the total coupling strength. Actually we will consider only the case $\mu^* = 0$ in this paper although we have also explored the case $\mu^* = 0.1$. The reason for this will be clear in the following. Indeed the Coulomb pseudopotential is a high-frequency contribution to the pairing interaction while we will see that all the interesting physics comes from low frequencies, as we might expect at low temperature. Therefore μ^* provides essentially an uninteresting renormalization of the coupling strength.

Our endeavor meets from the start with a basic problem, which is both conceptual and practical. Whereas it is straightforward to give the variation of a physical quantity as a function of various parameters, how can we study the low-temperature dependence of $\lambda_L(T)$ as a function of these parameters? Indeed we do not know of any general functional dependence of $\lambda_L(T)$ in this regime, which would then depend only on a few parameters. In fact this general problem is analogous to the one raised above for $\alpha^2F(\Omega)$ and the solution should be the same: We ought to expand $\lambda_L(T)$ on an appropriate functional basis and find the coefficients of this expansion. Actually the present experimental situation and the way most results are analyzed suggest a natural way to do something of this kind. We will try to find an approximate power law dependence for $\lambda_L(T)$ at low temperature, which provides us with an exponent and a prefactor to characterize this behavior. Naturally we will find in general that $\lambda_L(T)$ does not obey a power law. Therefore we will proceed specifically in the following way. We work with the superfluid density $\rho_s(T) = \lambda_L^{-2}(T) / \lambda_L^{-2}(0)$ and define apparent exponents between temperature T_i and T_f as $n = \ln \{ [1 - \rho_s(T_f)] / [1 - \rho_s(T_i)] \} / \ln(T_f / T_i)$. Both for experimental and theoretical reasons we do not take our sampling temperature too low. Indeed for the experiments we are interested in,⁹ the low-temperature behavior is pretty flat with some scatter and the resulting imprecision makes results obtained in this region rather meaningless. On the theoretical side we could naturally go in this region, but the variations of $\rho_s(T)$ would be minute. Moreover, we know theoretically that a power law will be a very bad description of $\lambda_L(T)$ in this regime. Therefore we have chosen our sampling temperatures to cover the “middle-low” temperature region $0.2 < T/T_c < 0.5$ where an experimentalist is most likely to look for a power law. Precisely we have taken $T_1/T_c = 0.2$, $T_2/T_c = 0.35$, and $T_3/T_c = 0.5$, and we have defined three exponents: n_1 between 0.35 and 0.5, n_2 between 0.2 and 0.5, and n_3 between 0.2 and 0.35 (see Fig. 1). Naturally $n_2 = 0.389n_1 + 0.611n_3$, but we find it convenient to use this redundant presentation because $n_3 - n_1$ appears as a kind of theoretical “error bar” for the middle exponent n_2 . A large value will tell us that a power law is a bad representation of the temperature dependence of $\lambda_L(T)$ in this region, while a small error bar will mean that a simple power law provides a good description. Most of the time we have found $n_1 < n_2 < n_3$, corresponding to a weaker T dependence at lower temperature, but it is important that the reverse order occurred also. Naturally our description could be improved, for example, by replacing the prefactor by a polynomial, but we have not explored this possibility.

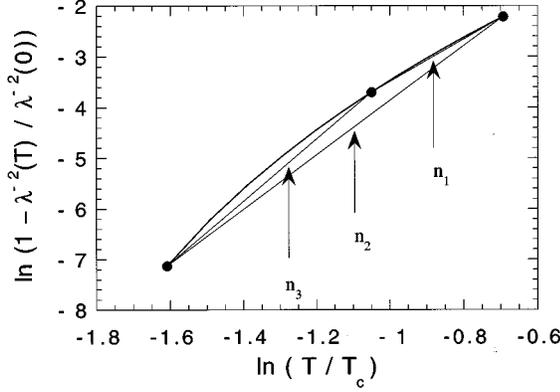


FIG. 1. Example of the calculation of our apparent power law exponents n_1 , n_2 , and n_3 from the log-log plot of $1 - \lambda_L^{-2}(T)/\lambda_L^{-2}(0)$ (this figure corresponds precisely to the case $r=4$, $\lambda=4$, and $p=0.3$ with $n_1=4.14$, $n_2=5.36$, and $n_3=6.13$). This figure suggests that it might be difficult to obtain experimentally a very good relative precision on the power law exponent.

III. RESULTS ON THE PENETRATION DEPTH

The London penetration depth is given by¹⁰

$$\lambda_L^{-2}(T) = \lambda_L^{-2} 2\pi T \sum_{n=0}^{\infty} \frac{\Delta_n^2}{\omega_n^2 + \Delta_n^2} \frac{1}{Z_n(\omega_n^2 + \Delta_n^2)^{1/2} + 1/2\tau_i}. \quad (1)$$

Here λ_L is the weak-coupling penetration depth at $T=0$ in the clean limit, given by $\lambda_L^{-2} \equiv (2/D)N_0e^2v_F^2$, where D is the dimensionality of the superconductor, N_0 is the density of states per spin at the Fermi level, τ_i is the impurity scattering time, and Δ_n and Z_n are, respectively, the gap function and the phonon renormalization function at the Matsubara frequencies $\omega_n = (2n+1)\pi T$. These last ones are given by the solution of the Eliashberg equations

$$\omega_n(Z_n - 1) = \pi T \sum_m \lambda_{n-m} \frac{\omega_m}{(\omega_m^2 + \Delta_m^2)^{1/2}}, \quad (2)$$

$$\Delta_n Z_n = \pi T \sum_m \lambda_{n-m} \frac{\Delta_m}{(\omega_m^2 + \Delta_m^2)^{1/2}},$$

where

$$\lambda(\omega) = \lambda \left\langle \frac{\Omega^2}{\Omega^2 + \omega^2} \right\rangle, \quad (3)$$

with $\lambda \equiv \lambda(0)$. As above the brackets $\langle \dots \rangle$ are for the average $\int d\Omega g(\Omega) \dots$ over phonon frequency with respect to the normalized Eliashberg function $g(\Omega) = 2\alpha^2 F(\Omega)/\lambda\Omega$.

Although this imaginary-axis expression, Eq. (1), is by far the most convenient for numerical calculations, it is useful for the physical interpretation of the results to write also the corresponding expression with integration on the real frequency axis. Indeed this allows us to express the penetration depth, which gives the superfluid response to the electromagnetic field, in terms of virtual pair-breaking excitations. This expression reads

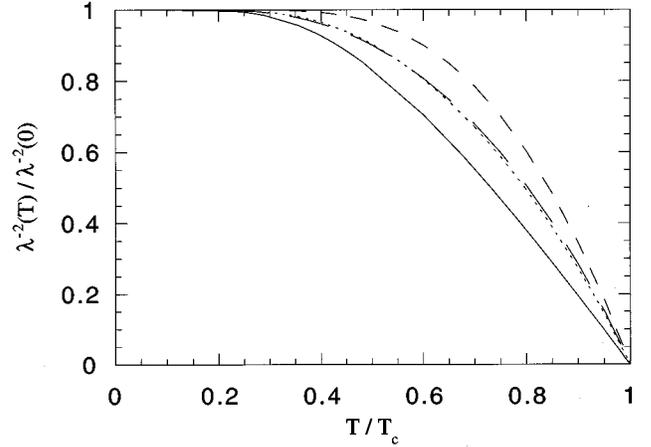


FIG. 2. Variation of the $\lambda_L^{-2}(T)/\lambda_L^{-2}(0)$ for an Einstein spectrum for a coupling strength $\lambda = 1$ (short dashed line), $\lambda = 4$ (dashed line), and $\lambda = 10$ (long dashed line). The solid curve is the BCS result.

$$\lambda_L^{-2}(T) = \lambda_L^{-2} \text{Im} \int_0^{\infty} d\omega \tanh\left(\frac{\omega}{2T}\right) \frac{\Delta^2(\omega)}{[\Delta^2(\omega) - \omega^2]^{3/2}} \times \frac{1}{Z(\omega) + 1/2\tau_i[\Delta^2(\omega) - \omega^2]^{1/2}}. \quad (4)$$

It reduces to Eq. (1) by deforming the integration contour toward the imaginary frequency axis. This result is easy to understand physically. If one omits the last term in the integral, this expression is the standard BCS result except for the frequency dependence of $\Delta(\omega)$. The last term corresponds to the mass renormalization of the excitations. Indeed, in addition to the constant factor $1/m^*$ due to the band structure effective mass coming in the prefactor λ_L^{-2} , one has to include the frequency-dependent mass renormalization function $Z(\omega)$. Naturally the imaginary part of $Z(\omega)$ will describe the effect of the finite lifetime of the excitations. The additional contribution $(1/2\tau_i)[\Delta^2(\omega) - \omega^2]^{-1/2}$ in the last term is just the corresponding effect of impurities on the excitation lifetime. In this paper we will actually be concerned only with the clean limit. Indeed we will see that the interesting low-temperature behavior of $\lambda(T)$ is controlled by lifetime effects due to inelastic scattering. When the superconductor gets very dirty, the elastic lifetime becomes much shorter than the inelastic one and the low-temperature behavior is similar to the standard weak-coupling BCS result and therefore uninteresting for our purpose. We note also that, if we have in mind a comparison with experiment, the dirty limit is not relevant for high- T_c compounds which are more or less on the clean side because T_c is so high. Therefore the clean limit is physically the interesting one. However, Eqs. (1) and (4) will be useful for comparison.

Let us start with the simplest case, namely, the Einstein spectrum, in order to have a reference for comparison. We display in Fig. 2 the results for an Einstein spectrum, which corresponds to take our parameter $p=0$ or 1 , the value of r being then irrelevant. We do not give the values of our exponents n_i for these Einstein spectra since they range typically from 5 to 10 and a power law is clearly a very poor way to describe the low-temperature behavior for them as it is

clear from Fig. 2. The $\lambda = 0$ curve is naturally the BCS result. At the beginning, when the coupling strength starts to increase, $\lambda_L(T)$ becomes flat at even higher temperature than in the BCS result. This is just the effect of the increase with λ of the ratio $2\Delta/T_c$. Indeed, as seen from Eq. (4), the flattening of $\lambda_L(T)$ starts when all excitations have essentially disappeared. Since the gap Δ gives the minimum energy for creating an excitation, the temperature at which $\lambda_L(T)$ becomes flat scales with Δ . However, when we have, say, $\lambda > 4$, this evolution is reversed and the low-temperature dependence gets stronger. When we reach $\lambda \approx 10$, we are already back to the $\lambda = 1$ behavior. Since we have specifically in mind here the low-temperature behavior, it is of interest to fully understand physically the origin of this somewhat surprising evolution.

Since this behavior appears on the strong-coupling side, we will understand it by considering the strong-coupling limit, where the coupling constant λ goes to infinity, while we let at the same time the phonons frequencies go to zero in order to keep, for example, $\lambda\langle\Omega^2\rangle = 1$ in order to keep the critical temperature fixed (here we do not restrict ourselves to an Einstein spectrum and consider rather a general spectrum). This limit has already been investigated in various papers.^{11-15,6} However, for our purpose it is more interesting not to let Ω go strictly to zero, but to take it very small in order to avoid singularities at low temperature. Because of the factor λ_{n-m} , the dominant contribution in the sums in Eq. (2) comes from the terms with $|\omega_m - \omega_n|$ at most of order a few times Ω . When Ω becomes very small, $\omega_m(\omega_m^2 + \Delta_m^2)^{-1/2}$ is almost constant over this range since Ω will be very small compared to Δ_m . Therefore the sum can be performed explicitly which leads, to leading order, to

$$Z_n = \frac{\pi}{2} \lambda \langle \Omega [2N(\Omega) + 1] \rangle \frac{1}{(\omega_n^2 + \Delta_n^2)^{1/2}}, \quad (5)$$

where $N(\Omega)$ is the Bose-Einstein distribution. The analytic continuation of this simple result toward the real axis,

$$Z(\omega) = i \frac{\pi}{2} \lambda \langle \Omega [2N(\Omega) + 1] \rangle \frac{1}{[\omega^2 - \Delta^2(\omega)]^{1/2}}, \quad (6)$$

can also be obtained directly from the real-axis Eliashberg equations. The physical meaning of Eqs. (5) and (6) is quite clear. The very-low-frequency phonons behave like quasi-impurities.¹³ When they are absorbed or emitted, they produce the same result as impurities with a scattering time τ_{ph} given by $1/\tau_{\text{ph}} = \pi\lambda\langle\Omega[2N(\Omega) + 1]\rangle$, as it can be seen by comparing Eqs. (4) and (6) (one would get the same result for $1/\tau_{\text{ph}}$ in the normal state). In other words in this limit $Z(\omega)$ is physically dominated by lifetime effects. When this result is carried into Eq. (1) one finds naturally the dirty limit with τ_{ph} as the scattering time:

$$\lambda_L^{-2}(T) = \lambda_L^{-2} \frac{2}{\pi\lambda\langle\Omega\coth(\Omega/2T)\rangle} 2\pi T \sum_{n=0}^{\infty} \frac{\Delta_n^2}{\omega_n^2 + \Delta_n^2}. \quad (7)$$

The last factor, which contains the sum over Matsubara frequencies, gives a very regular function of temperature. This function is given in Ref. 6. It goes to a constant¹⁶ when $T \rightarrow 0$. Therefore the low-temperature behaviour of $\lambda_L^{-2}(T)$ is

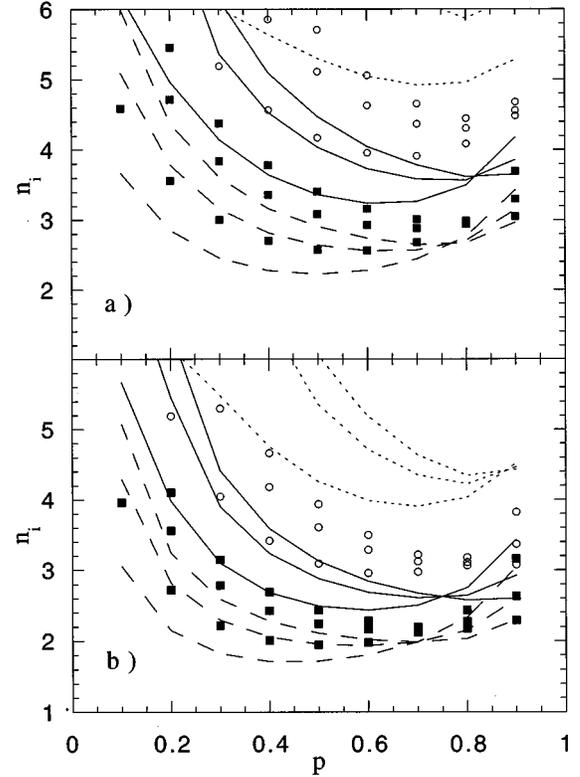


FIG. 3. Exponents n_1 , n_2 , and n_3 for $r=4$ (a) and $r=6$ (b), and $\lambda=2$ (dashed line), $\lambda=3$ (open circles), $\lambda=4$ (solid lines), $\lambda=6$ (solid squares), and $\lambda=8$ (dashed lines). The lines are just guides for the eye.

given by τ_{ph} . As long as $T \gg \Omega$, $\lambda_L^{-2}(T)$ diverges as T^{-1} , because the number of thermally excited phonons is proportional to T : When the temperature is lowered the number of scattering processes decreases which produces the growth of $\lambda_L^{-2}(T)$. Naturally this divergence saturates for $T \ll \Omega$, since at zero temperature only the possibility of spontaneous phonon emission is left which gives $1/\tau_{\text{ph}} = \pi\lambda\langle\Omega\rangle$. We come therefore to the conclusion that the increased low-temperature dependence found when the coupling gets very strong is due to the low-frequency phonons, which act as scatterers and whose number depends naturally on temperature.

We turn now to our investigation of the apparent exponent for the low- T behavior of $1 - \lambda_L^{-2}(T)/\lambda_L^{-2}(0)$. Since we have already explored the strong-coupling limit, we can restrict ourselves to “reasonable” values for the coupling constant λ . Specifically we consider λ going from 2 to 8. Indeed for $\lambda = 1$ we have found results rather similar to the BCS result with fairly large exponents (of order 4 or more) which have little interest. Similarly we do not show the results for a spectral width parameter $r=2$, since they do not depart very much from what is found for Einstein spectra. We consider first the results for $r=4$, which are displayed in Fig. 3(a). Since $p=0$ or 1 corresponds to Einstein spectra, we give only the results for $0.1 \leq p \leq 0.9$. While for $0.1 \leq p \leq 0.5$ it is plain that a power law does not agree with the theoretical results, remarkably it becomes a reasonable representation for $0.6 \leq p \leq 0.9$, mainly for high values of λ . Indeed, if we accept an uncertainty of $\pm 10\%$ on the expo-

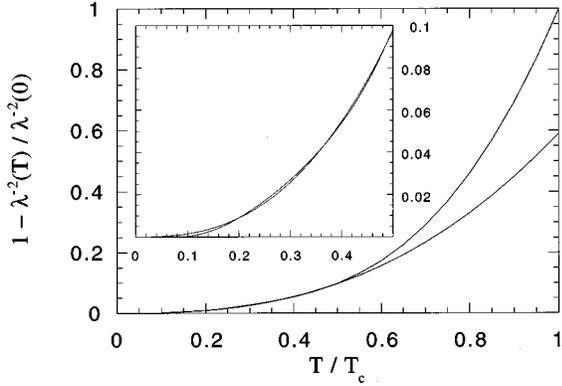


FIG. 4. $1 - \lambda_L^{-2}(T)/\lambda_L^{-2}(0)$ for $r=6$, $\lambda=4$, and $p=0.75$ together with its power law approximation $0.59 (T/T_c)^{2.6}$ over the full temperature range and (inset) for $T/T_c \leq 0.5$.

ment (it is not easy to obtain a better result experimentally), we find a power law for $0.7 \leq p \leq 0.9$ provided $\lambda \geq 3$. In particular for $p=0.8$ we find that for $\lambda=4$ ($n_i=3.55$), 6 ($n_i=2.95$), and 8 ($n_i=2.72$) the temperature dependence is remarkably well described by a power law. This is somewhat surprising since we know theoretically that this dependence is not a power law, and there is no obvious reasons why it comes so close to be one. We might indeed have anticipated that all our results would behave the way they do for, say, $p \leq 0.5$. Naturally the fact that $n_1 = n_3$ does not imply that we have exactly a straight line on our log-log plot, and we might worry that it has actually a sizable oscillation. One can check directly that it is not so: The difference with a straight line is quite small and could certainly not be seen experimentally. We give below a specific example of this.

When we look for larger values of the spectral width, we find that the range of parameters where a power law gives a good description gets larger and that the exponents decrease. This is seen on Fig. 3(b) where we present our results for $r=6$. For $0.6 \leq p \leq 0.8$ we obtain good power laws within $\pm 10\%$ down to $\lambda = 3$. For $p=0.75$ we find $n_i = 2.6$ for $\lambda = 4$. This exponent is already rather near experimental results. We show in Fig. 4 the result for $1 - \lambda_L^{-2}(T)/\lambda_L^{-2}(0)$ compared with an exact power law. It is clear that it would not be possible experimentally to make the difference between the power law and the strong-coupling result for temperature below $0.5T_c$. For $p=0.7$ and $\lambda = 8$ one finds a $T^{1.98}$ law. The results for $r = 8$ displayed in Fig. 5(a) show the same trends. The domain where we have a power law within $\pm 10\%$ extends now from $p=0.5$ to $p=0.8$ and starts almost from $\lambda = 2$. For $p=0.7$ and $\lambda = 4$ we find an exponent $n_i = 2.15$. Finally one sees in Fig. 5(b), where we show the results for $r = 16$, that exponents markedly below 2 and actually not so far from 1 can be obtained. For example for $p=0.6$ and $\lambda = 4$ we have $n_i = 1.51$. The domain with a power law within $\pm 10\%$ goes from $p=0.4$ to $p=0.7$, starting from $\lambda = 2$. We complete our results by displaying in Fig. 6 the prefactor of the approximate power laws that we have obtained for $r=4, 8$, and 16 (this prefactor is calculated between $T/T_c=0.2$ and 0.5 using our exponent n_2). We give only the results for the values of p where a power law is a good approximation. The prefactor that we

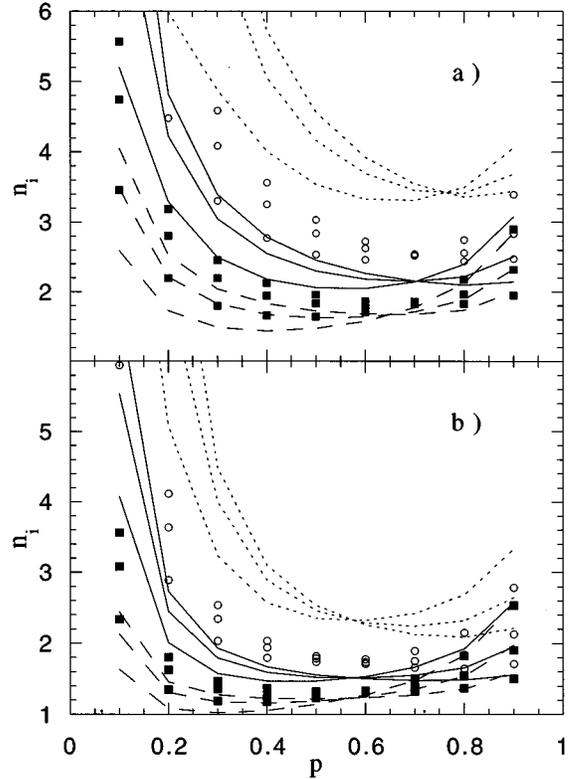


FIG. 5. Exponents n_1 , n_2 , and n_3 for $r=8$ (a) and $r=16$ (b), and $\lambda=2$ (dashed line), $\lambda=3$ (open circles), $\lambda=4$ (solid lines), $\lambda=6$ (solid squares), and $\lambda=8$ (dashed lines). The lines are just guides for the eye.

find is always of order unity. This means that the power law that we find at low temperature is not a small effect since its extrapolation for $T=T_c$ gives for $1 - \lambda_L^{-2}(T)/\lambda_L^{-2}(0)$ a result of order unity.

Naturally the parameters where we find very good power laws correspond to the region where one switches from $n_1 < n_2 < n_3$ to $n_3 < n_2 < n_1$ for the order of our exponents. We have found above that this crossing occurs in the region $p \approx 0.7 \pm 0.1$. However, since $p=0$ or 1 corresponds to an Einstein spectrum, the exponents must be in the same order for $p=0$ and 1 which implies that there must be another crossing region. This region is actually very near $p=1$ (for $r=4$ and $\lambda=3$ it occurs for $p=0.98$, and for higher values of r it goes closer to $p=1$). This means that the spectrum has a very small low-frequency component, which makes this kind of spectrum somewhat pathological and unlikely to be relevant experimentally. The exponents corresponding to this crossing are also fairly high which makes them quite uninteresting. Therefore we do not discuss this region further. In contrast the region which we have considered above corresponds to quite standard spectra and the exponents are similar to what is found experimentally.

It is worthwhile to try to understand the physical origin of this appearance of quasipower laws with small exponents. Since this feature develops when the spectral width gets large, it is useful to consider the large spectral width limit, which corresponds to letting go to infinity our parameter r in the two-peak representation. This is equivalent to letting the frequency Ω_1 of the lower peak go to zero, while we keep at

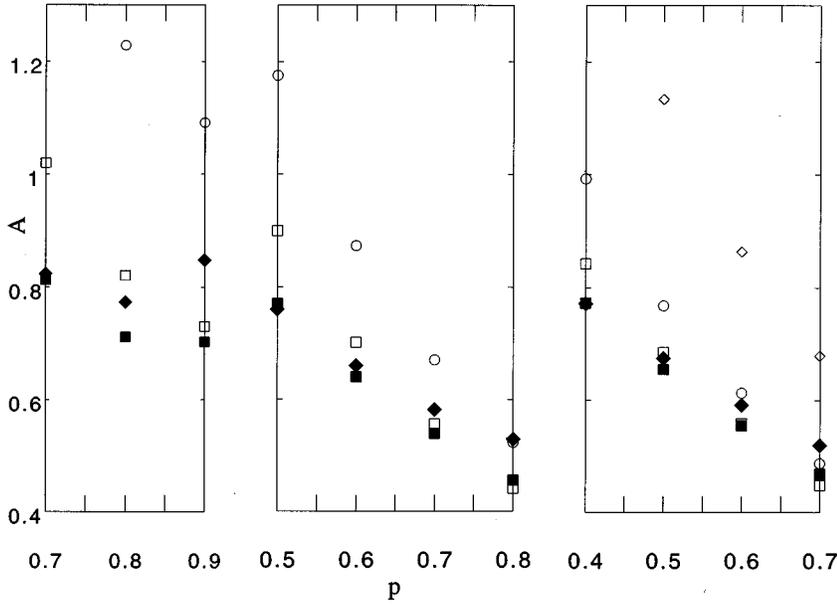


FIG. 6. The prefactor of the approximate power law for $1 - \lambda_L^{-2}(T)/\lambda_L^{-2}(0)$: $r=4$ (left), $r=8$ (middle), and $r=16$ (right); $\lambda = 2$ (open diamonds), $\lambda = 3$ (open circles), $\lambda=4$ (open squares), $\lambda=6$ (solid squares), and $\lambda=8$ (solid diamonds).

the same time the higher-frequency peak Ω_2 fixed (we can take, for example, Ω_2 as unity). Physically, in the same way as we have seen for the strong-coupling limit, the low-frequency phonons will behave as impurities. They disappear from the equation for the gap function. Therefore Δ_n as well as T_c is given by the Einstein spectrum result with frequency Ω_2 and coupling strength λ_2 . On the other hand the low-frequency phonons still contribute to Z_n by the $m=n$ term. This produces the same effect as introducing additional scattering with a lifetime $1/2\tau_{\text{ph}} = \pi \lambda_1 T$. This leads for the penetration depth to

$$\lambda_L^{-2}(T) = \lambda_L^{-2} 2\pi T \sum_{n=0}^{\infty} \frac{(\Delta_n^E)^2}{\omega_n^2 + (\Delta_n^E)^2} \times \frac{1}{Z_n^E [\omega_n^2 + (\Delta_n^E)^2]^{1/2} + \pi \lambda_1 T}, \quad (8)$$

where, as explained above, Δ_n^E and Z_n^E are the solution of Eliashberg equations for an Einstein spectrum with frequency Ω_2 and coupling strength λ_2 . As one might have expected, this result is just the general formula, Eq. (1), with the impurity lifetime replaced by the contribution from low-frequency phonons. It is clear from Eq. (8) that, because of the term $\pi \lambda_1 T$, the low-frequency phonons depress $\lambda_L^{-2}(T)$ below what would be obtained from an Einstein spectrum alone. Hence they give for $\lambda_L(T)$ at low-temperature a dependence which is stronger than the standard BCS one. Therefore the low-frequency phonons are responsible for development of the low-temperature dependence when the width of the spectrum increases. However, there is no obvious reason why this dependence can get close to a power law. Nevertheless, we note that Eq. (8) gives a linear T dependence for $\lambda_L(T)$ at low temperature (naturally our limiting case corresponds to a situation where $\Omega_1 \ll T$). We show in Fig. 7 the result of the calculation when the coupling strength of the high-frequency peak is equal to $\lambda_2 = 3$. The coupling strength λ_1 of the low-frequency phonons takes the values $m\lambda_2/(10-m)$ and the

integer m goes from 0 to 9. The values of λ_1 for $m \geq 6$ are quite large and are just given to show the trend. It is interesting to note that the heavy line in this figure, which corresponds to $\lambda_1 = 1.28$, is quite similar to the experimental results of Ref. 3.

IV. DISCUSSION

In this paper we have explored the effect of strong coupling on the low-temperature behavior of the penetration depth in the simplest model, namely, for an isotropic attractive interaction leading to s -wave pairing. Quite surprisingly and unexpectedly we have found that strong-coupling effects can mimic a power law dependence, for a sizable range of parameters, sometimes with excellent precision. Physically we have been able to ascribe this strong T dependence to low-frequency phonons which produce a quasielastic scattering for electrons, but we have not found any deeper theoretical reasons for these quasi-power laws. The presence of these low-frequency phonons below T_c requires rather wide

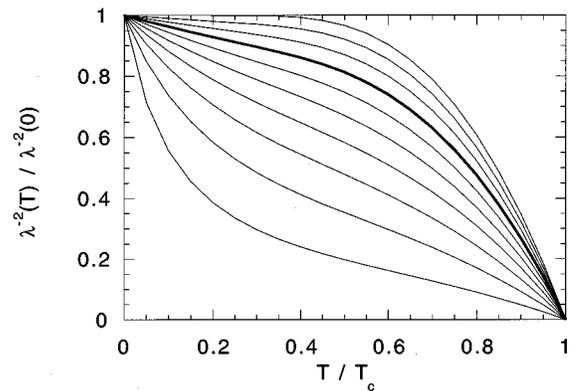


FIG. 7. $\lambda_L^{-2}(T)/\lambda_L^{-2}(0)$ in the large spectral width limit for $\lambda_2=3$. The coupling strength λ_1 takes the values $m\lambda_2/(10-m)$ and the integer m goes from 0 to 9. The heavy line corresponds to $\lambda_1 = 1.28$.

phonon spectra and their effectiveness in scattering implies fairly strong coupling.

Let us try to apply our results to the case of high- T_c superconductors. Although we have found that one can obtain in principle a linear low-temperature dependence from strong-coupling effects, it seems completely unlikely that such an interpretation applies to the linear dependence found recently in Y-Ba-Cu-O (Refs. 3,4) because our parameters are too extreme in terms of phonon frequency and coupling strength. On the other hand our results are of interest for other high- T_c compounds: If we take the experimental results showing a dependence not so far from a T^2 law, as is quite often found² in Bi-Sr-Ca-Cu-O, our results are rather close to providing an alternative explanation although our coupling strength is a bit too high and the spectrum we require somewhat too wide to agree with the experimental information presently available.

On the other hand we have only investigated here the isotropic case. It is clear that, if we consider some anisotropy, the effects that we have found will be increased. Indeed, as we see from Eq. (8), the inverse lifetime due to low-frequency phonons is in direct competition with the size of the gap. Anisotropy will make the gap smaller at some places on the Fermi surface, thereby increasing the effect of low-frequency phonons and leading to the possibility of hav-

ing a power law at low T for reasonable values of the parameters. In this way strong-coupling effects might be quite relevant to the understanding of experimental results. In this respect it is worth pointing out that most high- T_c microscopic theories claiming to perform a reliable critical temperature calculation are strong-coupling theories with coupling constants at least of order unity. This is in particular the case for spin fluctuations microscopic theories such as the Monthoux-Pines¹⁷ or Radtke-Ullah-Levin-Norman¹⁸ calculations. How strong the coupling should be in these compounds is still naturally a matter of debate. However, strong-coupling effects are generally completely overlooked in the interpretation of experiments on the low-temperature behavior of $\lambda_L(T)$. Our results show that this might be dangerous. Another way to put it is to say that, when something happens at low temperature, one should not only look at the fermionic degrees of freedom for an explanation, but one should also consider the possibility that bosonic degrees of freedom might be at least partly responsible for the effect.

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