

Hydrodynamic Modes in Dense Trapped Ultracold Gases

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We consider the hydrodynamic modes for dense trapped ultracold gases, where the interparticle distance is comparable to the scattering length. We show that the experimental determination of the hydrodynamic mode frequencies allows one to obtain quite directly the equation of state of a dense gas. As an example, we investigate the case of two equal fermionic populations in different hyperfine states with attractive interaction.

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Most of the fascinating recent work on ultracold gases [1] has been dealing with dilute situations. Naturally, even in this regime interactions play an important role, as in the case of Bose-Einstein condensation (BEC) where they strongly increase the size of the condensate compared to the free boson case. In this dilute regime the scattering length is small compared to the interparticle distance and the mean field approximation is valid. However, it is of great interest to explore the dense gas regime where scattering length and interparticle distance are comparable and mean field is no longer valid. This would lead to physical systems which are very simple examples of strongly interacting systems. These have much more complicated counterparts in condensed matter physics, such as liquid ^4He or ^3He , or the electron gas in metals. This regime is also of major experimental interest in the search for a BCS superfluid in fermion gases [2], since this is the range where the higher critical temperatures [3,4] will be found, which should make the transition more accessible. This dense regime corresponds to large scattering lengths, which can be reached in the vicinity of Feshbach resonances, as it has already been seen in optical traps [5]. Naturally, we assume that inelastic collisions, such as three-body recombination, will be small enough to be negligible.

In this paper we show that the experimental determination of the mode frequencies in the hydrodynamic regime allows one to obtain quite efficiently and directly the equation of state of a dense gas. Hydrodynamic equations are valid in the limit of low frequency (compared to elastic scattering time) and long wavelength, and have already been used to study the dilute Bose gas [1,6] (with very good experimental agreement) and the free Fermi gas [7]. In these cases the equation of state is known. We show that, rather surprisingly, the analysis of the equation giving the mode frequencies is not much more complicated when the equation of state is unknown and that one can conveniently invert the problem and get the equation of state from the mode spectrum. As an example, we apply our treatment to the case of two equal fermionic populations in different hyperfine states with attractive interaction; in particular, we investigate

the vicinity of the collapse, a very interesting physical situation analogous to ^7Li BEC collapse.

Although one might consider the extension to higher temperature, we work in the low temperature range where thermal effects are small and we neglect dissipation, so we deal with a perfect fluid. This should be valid for a strongly degenerate Fermi gas in its normal state (residual collisions would lead to a damping of the modes). Naturally, our results apply also to a superfluid when normal liquid effects (which would, in particular, produce damping) are negligible, such as low temperature Bose condensates or low temperature simple scalar BCS superfluids. We consider, for simplicity, an isotropic trapping potential $V(r)$ (mostly the harmonic case) but, somewhat surprisingly, most of our procedures can be generalized to the case of anisotropic harmonic traps. Also, we treat the 3D case, but lower dimensions can be handled in the same way.

With our hypotheses hydrodynamics reduces to the Euler equation $m n d\mathbf{v}/dt = -\nabla P - n\nabla V(r)$ supplemented by particle conservation $\partial n/\partial t + \nabla(n\mathbf{v}) = 0$ for density $n(\mathbf{r}, t)$ and thermodynamics. Since $\partial P/\partial n_0 = n_0 \partial \mu/\partial n_0$, the equilibrium particle density $n_0(r)$ satisfies $\mu(n_0(r)) + V(r) = \tilde{\mu}$, where $\tilde{\mu}$ is the overall chemical potential. Below, we refer for short to $\mu(n)$ as the equation of state. Linearizing these equations around equilibrium, one finds that the density fluctuation $n_1(\mathbf{r}, t) = n(\mathbf{r}, t) - n_0(r)$ oscillating at frequency ω satisfies $\nabla^2(n_1 \partial P/\partial n_0) + \nabla(n_1 \nabla V) + m\omega^2 n_1 = 0$. Of particular interest is the “neutral mode” solution $n_1^0(r)$, corresponding to the density fluctuation produced by a small shift $\delta\tilde{\mu}$ of the overall chemical potential, that is $n_1^0(r) = (\partial\mu/\partial n_0)^{-1} \delta\tilde{\mu}$. Since the result is still an equilibrium situation, this mode corresponds to $\omega = 0$ but is not physical since it does not conserve particle number.

We make the change $n_1(\mathbf{r}) = n_1^0(r)w(r)Y_{lm}$ (i.e., the local fluctuation of the chemical potential is a convenient variable) and obtain

$$rw'' + [2 + rL'(r)]w' - \left[\frac{l(l+1)}{r} + \frac{m\omega^2 r}{V'(r)} L'(r) \right] w = 0, \quad (1)$$

where we have set $L(r) = \ln(n_0(r))$ with $L'(r) = dL/dr$,

and $V'(r) = dV(r)/dr$. This equation for the mode frequencies has a quite simple form. In particular, as soon as $V(r)$ is known, the properties of the fluid appear only through $L(r)$, which is itself simply related to the equation of state. Therefore it appears much more convenient to model $L(r)$, rather than $\mu(n)$. Indeed Eq. (1) lends itself to a very large number of specific models with analytical solutions or quasianalytical solutions, as we will see below in the case of the harmonic trap. Before proceeding to this case, it is also interesting to note that Eq. (1) may be written with the form of a one-dimensional Schrödinger equation (with energy equal to zero) by making the further change $w(r) = \psi(r)/(r\sqrt{n_0(r)})$. The corresponding potential is found to be $(1/r + m\omega^2/V')L' + (1/4)L'^2 + (1/2)L'' + l(l+1)/r^2$ and is simply related to $L(r)$. This form is of particular interest when one has an explicit analytical solution for an approximate model, as we find below. One can then easily correct the results by a first order perturbation calculation.

Let us now specialize to the case of the harmonic trap $V(r) = \frac{1}{2}m\Omega^2 r^2$. It is then convenient to make the further change $w(r) = r^l v(r)$ which leads to

$$rv'' + [2(l+1) + rL'(r)]v' - (\nu^2 - l)L'(r)v = 0, \quad (2)$$

where $\nu^2 = \omega^2/\Omega^2$. We check on this equation that, whatever the equation of state $\mu(n)$ of the fluid, we have as expected the dipole mode ($l=1$) corresponding to the gas oscillating in the trap as a whole, at frequency $\omega = \Omega$. It corresponds [6] to $\nu = 1$ with $\nu^2 = 1$. Furthermore $\nu = 1$ gives also $\omega = \Omega\sqrt{l}$ whatever $L(r)$, i.e., independent of the equation of state of the fluid (and, in particular, whether it is a Bose or Fermi gas) [8]. This generalizes for an interacting fluid, at low temperature, results obtained by Griffin *et al.* and Stringari [6,9] for a Bose gas and by Bruun and Clark [7] for a free Fermi gas.

Note that Eq. (2) is invariant under the change of scale $r \rightarrow Kr$, provided we make the same change of scale for $L(r)$. So we take in the following the gas radius R as unity (consistently with hydrodynamics we use the Thomas-Fermi approximation). Next notice that Eq. (2) is only slightly modified by the change $y = r^\alpha$ provided again that the same change is made for $L(r)$. This gives

$$y \frac{d^2 v}{dy^2} + \left(1 + \frac{2l+1}{\alpha} + y \frac{dL}{dy}\right) \frac{dv}{dy} - \frac{\nu^2 - l}{\alpha} \frac{dL}{dy} v = 0. \quad (3)$$

A convenient feature of Eq. (3) is also that the absolute scale in density disappears in L' and only $\bar{n}(r) \equiv n(r)/n(0)$ enters. We introduce similarly a normalized local chemical potential $\bar{\mu}(r) \equiv \mu(n(r))/\mu(n(0))$, where $\mu(n(0))$ is simply obtained from the gas radius R by $\mu(n(0)) = \frac{1}{2}m\Omega^2 R^2$, leading to $\bar{\mu} = 1 - r^2$.

Looking now for simple situations where we can solve Eq. (3), we consider first the case of the noninteracting Fermi gas [7]. This gives $\bar{\mu} = \bar{n}^{1/p}$ with $p = 3/2$. Similarly, we can consider an interacting dilute Bose gas [6] where $\mu = gn$ (g being the coupling constant) leading again to $\bar{\mu} = \bar{n}^{1/p}$ with now $p = 1$. These two

cases imply $L(r) = p \ln(1 - r^\alpha)$ with $\alpha = 2$ in Eq. (3). Hence, we are led to consider for any α and p the model $dL/dy = -p/(1-y)$ for which Eq. (3) becomes

$$y(1-y) \frac{d^2 v}{dy^2} + [c - y(p+c)] \frac{dv}{dy} + p \frac{\nu^2 - l}{\alpha} v = 0, \quad (4)$$

with $c = 1 + \frac{2l+1}{\alpha}$. The general solution [10] of this equation, giving a nondivergent density fluctuation for $r = 0$, is the hypergeometric function $F(a, b; c; y)$, with $a + b = p + c - 1$ and $ab = -p[(\nu^2 - l)/\alpha]$. We have furthermore to require that the solutions satisfy the boundary condition that the outgoing particle current is zero everywhere on the sphere $r = 1$. This is not verified by the general solution, except if $a = -n$ where n is a nonnegative integer, in which case the solution is a polynomial [6]. This leads to the normal mode frequencies:

$$\frac{\omega^2}{\Omega^2} = l + \frac{\alpha}{p} n \left(n + p + \frac{2l+1}{\alpha} \right), \quad (5)$$

which agrees with Stringari [6] for $\alpha = 2$ and $p = 1$, and with Bruun and Clark [7] for $\alpha = 2$ and $p = 3/2$.

One may naturally wonder about the interest of these results for other values of our parameters α and p . These cases correspond to the density $\bar{n}(r) = (1 - r^\alpha)^p$ and the equation of state $\bar{\mu} = 1 - (1 - \bar{n}^{1/p})^{2/\alpha}$. Our point is that these corresponding models can be used to represent closely the equation of state $\mu(n)$ for a general fluid [with a given maximum density $n(0)$]. We show explicitly below that the flexibility offered by the two parameters α and p makes it a very convenient and efficient procedure. However, these general models do not seem very physical since, although their density properly vanishes at the gas radius, they give near this border $\bar{\mu} \approx \bar{n}^{1/p}$, whereas one should get the dilute gas behavior $p = 1$ (bosons) or $p = 3/2$ (fermions). However, just because the gas is dilute near $r = 1$, we do not expect this part of the gas to play a significant role. Similarly, these models give $\bar{n} \approx 1 - pr^\alpha$ for small r , whereas one expects only the case $\alpha = 2$ to occur for a regular equation of state. Nevertheless if $\bar{n}_0(r)$ is closely approximated over the whole range, one may expect this modeling to be already quite good, as we see explicitly below. Before going into this, let us consider the possibility of more refined models.

Indeed, it is clearly of interest to consider more complicated models which could display proper behavior near the center and the border of the cloud. Although we have not obtained such models with completely analytical solutions, we have found a large class of models with quasianalytical solutions which are in practice not different from fully analytical solutions. These are the models $dL/dy = -\sum_{k=0}^K p_k y^k/(1-y)$ (where we could take $\alpha = 2$ and $p \equiv \sum_{k=0}^K p_k = 1$ or $3/2$ in order to have the proper center and border behavior). To be simple and specific, let us take the case $K = 1$, giving $-dL/dy = (p_0 + p_1 y)/(1-y)$. This corresponds to the equation of state $\bar{n} = \bar{\mu}^p \exp[p_1(1 - \bar{\mu})]$ when $\alpha = 2$. In this case Eq. (3) becomes

$$y(1-y)\frac{d^2v}{dy^2} + (q_2y^2 + q_1y + q_0)\frac{dv}{dy} + (r_1y + r_0)v = 0, \quad (6)$$

with $q_2 = -p_1$, $q_1 = -(c + p_0)$, $q_0 = c$, $r_1 = p_1 \frac{\nu^2 - l}{\alpha}$, and $r_0 = p_0[(\nu^2 - l)/\alpha]$. When we look for a series expansion of the solution $v = \sum_{n=0}^{\infty} a_n y^n$, we find the following recursion relation (with $a_{-1} = 0$): $[(n+1)(n+q_0)]a_{n+1} + [-n(n-1) + nq_1 + r_0]a_n + [(n-1)q_2 + r_1]a_{n-1} = 0$ which does not allow in general for a polynomial solution. For large n , this relation becomes asymptotically $a_{n+1} - a_n = -(q_2/n)a_{n-1}$. This leads to the standard behavior $a_{n+1} \approx a_n$ giving a convergence radius equal to 1. This is the same situation as for the hypergeometric function in Eq. (4) and leads in the same way to a singular behavior for $y = 1$ which disagrees with boundary conditions. But the above asymptotic relation may also have solutions $a_{n+1} \ll a_n$ implying $a_n \approx (q_2/n)a_{n-1}$ which gives $a_n \sim 1/n!$. This very rapidly convergent series has an infinite convergence radius and no singularity for $y = 1$. It corresponds to the physically acceptable solutions. Since we have only $y \in [0, 1]$, this solution is a quasipolynomial since the higher order terms in the series are very rapidly negligible. This is quite analogous to the polynomial solution of the hypergeometric differential equation. Naturally, these solutions arise only for special values of our parameters, which gives finally the mode frequencies. In practice, these parameters are found very easily in the following way. We solve iteratively the recursion relation for a_n with $0 \leq n \leq N$, and we require $a_{N+1} = 0$ (as if we had a polynomial solution). Since r_0 and r_1 are linear in $\nu^2 - l$, this is equivalent to find the roots of an equation of order N for $\nu^2 - l$. We then increase the value of $N = 1, 2, \dots$ until the roots have converged. For the lowest root, this is usually a very fast convergence, so one could obtain approximate analytical expressions. But the numerics is so easy that this seems unnecessary. All this analysis and procedure can be extended to the case of $K > 1$.

As an example, we turn now to the specific case of two equal populations of fermions in different hyperfine states. This may be the case of ^6Li or ^{40}K near a Feshbach resonance [11]. We assume an attractive interaction between unlike atoms with an interaction g , related to the (negative) diffusion length by $g = 4\pi\hbar^2 a/m$, and we take the Hartree approximation to describe this system. For total atomic density n , $\mu(n) = \hbar^2 k_F^2/2m - |g|n/2$ with $3\pi^2 n = k_F^3$. To solve directly this case, it is more convenient to rewrite Eq. (2) (taking $\alpha = 2$) with the variable $u \equiv k_F/k_F(0)$, where $k_F(0)$ is the equilibrium Fermi wave vector at $r = 0$. This leads to

$$PP'v'' + \left[\left(l + \frac{3}{2} \right) P'^2 + \frac{3}{u} PP' - PP'' \right] v' - \frac{3(\nu^2 - l)P'^2}{2u} v = 0, \quad (7)$$

with $P' = dP/du$ and $P'' = d^2P/du^2$. We have set $P(u) = 1 - u^2 - \frac{2}{3}\lambda(1 - u^3)$ with the coupling constant $\lambda = 2k_F(0)|a|/\pi$. This coupling constant goes from 0, for the very dilute regime corresponding to free fermions, to 1 when we reach at the center the instability where the gas is going to collapse.

We have solved Eq. (7) numerically, as a function of λ , for the first three monopole mode frequencies ($l = 0$). Results are given in Fig. 1. As expected, the frequencies decrease for increasing attractive interaction, since the gas gets more compressible when near the instability. However, we do not find the lowest mode frequency going to zero at the instability. This can be understood because the instability density is reached only at the center, and the rest of the gas still provides a restoring force accounting for the nonzero frequency (actually this instability limit cannot be reached experimentally since the modes correspond to infinitesimal density oscillations; finite oscillations will induce nonlinear effects and produce a collapse of the gas). This result for a Fermi gas is in sharp contrast with the one found for a Bose gas with attractive interaction [1]. In this last case, the gas cloud size is always of order of the extension l_{ho} of the harmonic potential ground state wave function. For hydrodynamics, this is a microscopic scale, so hydrodynamics is never valid and a full quantum calculation is required. On the other hand, the Fermi pressure makes the gas much larger than l_{ho} , which justifies the use of hydrodynamic. Indeed, at the collapse the typical Fermi wavelength λ_F is comparable to the scattering length $|a|$ while the radius of the cloud R_c is comparable to the size of a free fermion gas. This gives $R_c/l_{ho} \sim l_{ho}/|a| \gg 1$ in typical experiments.

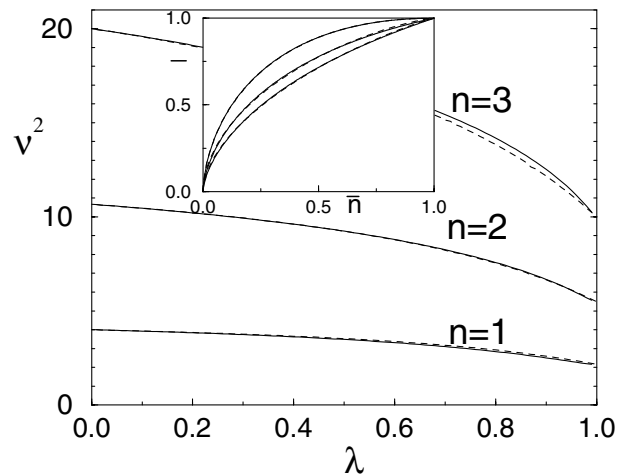


FIG. 1. Reduced mode frequency $\nu^2 = (\omega/\Omega)^2$ for a Fermi gas within the Hartree approximation as a function of the coupling constant λ . Solid line: exact result from the numerical solution of Eq. (7). Dashed line: approximate analytical solution. Inset: normalized Hartree chemical potential $\bar{\mu}$ as a function of the normalized density \bar{n} (solid lines) for $\lambda = 0.6, 0.8, 1$ ($\bar{\mu}$ increases with λ) and the corresponding modeling (dashed lines) by $\bar{\mu} = 1 - (1 - \bar{n}^{1/p})^2/\alpha$.

Actually, we believe that improving the hydrodynamic description by including quantum effects (hydrodynamic is not correct at the scale of the Fermi wavelength) will lead to modifications very near the instability and to a zero frequency mode at the instability $\lambda = 1$ (preliminary calculations support this view).

It is now of interest to consider an approximate solution of this same problem with the modeling we have discussed above. With the Hartree approximation, the relation between the reduced chemical potential $\bar{\mu}$ and the reduced density \bar{n} is $\bar{\mu} = (3\bar{n}^{2/3} - 2\lambda\bar{n})/(3 - 2\lambda)$. For each value of λ , we approximate $\bar{\mu}(\bar{n})$ by $\bar{\mu} = 1 - (1 - \bar{n}^{1/p})^{2/\alpha}$, where we obtain the parameters p and α by a least square fit. Then the mode frequencies are given by $\nu^2 = \frac{\alpha}{p} n(n + p + \frac{1}{\alpha})$ for $n = 1, 2, 3$. As seen in the inset of Fig. 1, the model is very close to the Hartree equation of state. The mean difference is maximum for $\lambda \approx 0.84$ where it reaches $6 \cdot 10^{-3}$. The results for the mode frequencies are given in Fig. 1 and they are surprisingly close to our exact results from numerical integration.

A major interest of this approximate treatment is that it is easily inverted and allows to analyze readily experimental data. Let us consider, for example, the case where we have from experiment only the lowest monopole frequency as a function of particle number N in a trap of frequency Ω . It is clear that the information on the equation of state $\mu(n)$ is contained in such data. We show now how to obtain it explicitly. We will obtain $\mu(n)$ recursively: Knowing $\mu(n)$ for n between 0 and n_m , we find the increase $d\mu_m$ corresponding to an increase of n from n_m to $n_m + dn_m$. Hence, we will find a kind of first order differential equation which can be easily integrated numerically. The boundary condition for low n corresponds to recover the dilute gas results which are exactly known.

When $\mu(n)$ is known, the density n_m at the trap center is related to the particle number by $N = 4\pi \int_0^R dr r^2 n(r)$, where the cloud radius R is linked to n_m and to chemical potential μ_m at the trap center by $\frac{1}{2}m\Omega^2 R^2 = \mu_m = \mu(n_m)$: $n(r)$ is obtained by inverting $\frac{1}{2}m\Omega^2 r^2 = \mu(n_m) - \mu(n(r))$. So, for simplicity, in the argument we assume that n_m is known experimentally (but the procedure is basically unchanged if we work with N).

Also, for simplicity of the presentation, let us assume that our model depends only on a single parameter p instead of two (p and α) or more. Then our least square fit to the model $\bar{\mu} = \bar{n}^{1/p} \equiv m(\bar{n}, p)$ corresponds to make $\int_0^1 d\bar{n} [M(\bar{n}, n_m) - m(\bar{n}, p)]^2$ minimal, where $M(\bar{n}, n_m) \equiv \mu(\bar{n}n_m)/\mu(n_m)$ is the normalized chemical potential. This is equivalent to solve the equation $F(p, n_m) \equiv \int_0^1 d\bar{n} [M(\bar{n}, n_m) - m(\bar{n}, p)] (\partial m / \partial p) = 0$ which gives the dependence of p on n_m . Now, if we have an increase dn_m , this produces a change dp of our parameter p . They are linked by $(\partial F / \partial p) dp + (\partial F / \partial n_m) dn_m = 0$. On the other hand, dp and dn_m are also linked because our analytical result Eq. (5) for the mode frequency (which depends

on p) must be equal to the experimental result (which depends on n_m). This experimental data provides us with an explicit relation between $\partial F / \partial p$ and $\partial F / \partial n_m$. Then $\partial F / \partial p$ is just an integral which contains $M(\bar{n}, n_m)$ and can be calculated numerically. On the other hand, $\partial F / \partial n_m$ contains $\partial M(\bar{n}, n_m) / \partial n_m = \bar{n} \mu'(\bar{n}n_m) / \mu(n_m) - \mu(\bar{n}n_m) \mu'(n_m) / \mu^2(n_m)$, where $\mu'(n) \equiv d\mu/dn$. The second term introduces precisely the quantity $\mu'(n_m) = d\mu_m/dn_m$ we are looking for while all the other ingredients in $\partial F / \partial n_m$ can be calculated numerically from $M(\bar{n}, n_m)$. In this way, we obtain for $d\mu_m/dn_m$ an expression which can be explicitly calculated numerically when we know $\mu(n)$ for n varying between 0 and n_m . This leads by integration to a determination of $\mu(n)$ from the experimental mode frequency $\omega(n_m)$ as a function of the density n_m at the center of the trap or, equivalently, as a function of the particle number N in the trap. The generalization to more than a single parameter does not make any problem.

This inversion method is quite convenient since it gives a univocal answer for $\mu(n)$ from a given set of experimental data. Moreover, if more than a single mode is measured, the comparison between the results from the various modes will provide checks on the resulting $\mu(n)$. Finally, we can improve our modeling by making use of the richer models with quasipolynomial solutions we have already discussed.

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