Momentum Distribution of a Dilute Unitary Bose Gas with Three-Body Losses

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Using a combination of Boltzmann’s equation and virial expansion, we study the effect of three-body losses and interactions on the momentum distribution of a homogeneous unitary Bose gas in the dilute limit where quantum correlations are negligible. The comparison of our results to the recent measurement made at JILA on a unitary gas of 85Rb allows us to determine an experimental fugacity \( z = 0.5(1) \).

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In the past few years, ultracold gases have become a unique tool for the experimental study of strongly correlated systems. In atomic vapors, strong interactions can be achieved either by trapping the atoms in an optical lattice or by using Feshbach resonances. While the first route has been very successful and has led to groundbreaking discoveries such as the observation of the Mott transition in both Bose [1] and Fermi gases [2,3], Feshbach resonances could only be used to study strongly correlated molecular states [14,15]. Recent experimental results suggested new routes to overcome this challenge and that it might be possible to quantitatively study the unitary Bose gas. First, it was demonstrated that at finite temperature the increase of the three-body loss rate scaling as \( a^4 \) actually saturates when \( a \gg \lambda_B \), where \( \lambda_B = h/\sqrt{2\pi mk_BT} \) is the thermal wavelength [16,17]. Moreover, recent experimental results from JILA demonstrated universal local dynamics of the momentum distribution of a unitary Bose gas towards a quasi-equilibrium state [18] and have triggered several theoretical works on the dynamics of strongly correlated Bose gases near Feshbach resonances [19–21].

The stability of the unitary Bose gas hinges on the following argument [11]. First, the three-body losses are characterized by a coefficient \( L_3 \) such that \( \dot{N} = -L_3 n^2 N \), where \( N \) is the total atom number and \( n \) is the particle density. This phenomenological law defines a characteristic loss rate \( \gamma_3 = L_3 n^2 \). For a thermal gas, the cloud is brought back to equilibrium by elastic scattering at a characteristic rate \( \gamma_2 = n \sigma v \), where \( \sigma \) is the scattering cross section and \( v \) is the characteristic velocity of the atoms. At unitarity, the scattering cross section follows a universal scaling \( \sigma = 8\pi/k^2 \), where \( k \) is the relative wave vector of two scattering particles. In the presence of losses, the system can be kept in a quasiequilibrium state provided that the ratio \( \gamma_3/\gamma_2 \) stays small. It was shown both theoretically and experimentally [16,17] that at unitarity the three-body loss rate is given by

\[
L_3 = 36\sqrt{3}\pi^2 \frac{\hbar^3}{m^3(k_BT)^2}(1 - e^{-4\eta}),
\]

where \( \eta \) is a dimensionless parameter characterizing the probability of forming a deeply bound molecule at a short distance [22]. Plugging Eq. (1) into the expression for \( \gamma_3 \), we see that quasiequilibrium can be achieved as long as \( (1 - e^{-4\eta})\lambda_B^3 \) is small, i.e., when the system is not too deeply in the quantum degenerate regime.

In this Letter, we investigate the effect of three-body losses on the momentum distribution of a unitary Bose gas. Our analysis is based on a semianalytical resolution of Boltzmann’s equation. Since Boltzmann’s equation neglects all many-body correlations, our work is restricted to a low phase-space density regime where, as aforementioned, three-body losses can be treated perturbatively. We calculate the first correction to the momentum distribution and we compare it to the effect of two-body interactions. We show that in the dilute limit, both effects deplete the center of the momentum distribution proportionally to the phase-space density of the gas. Moreover, for realistic parameters, this depletion is dominated by three-body losses.

Consider a homogeneous Bose gas that we describe by a phase-space density \( f(p) \). In the presence of losses, \( f \) is the solution of Boltzmann’s equation that we write formally

\[
\partial_t f = I_{\text{coll}}[f] - L_3[f],
\]

where \( I_{\text{coll}} \) and \( L_3 \) are nonlinear operators describing, respectively, the elastic collisions and the three-body losses. At low phase-space density, we can neglect the bosonic stimulation and we have
\[ I_{\text{coll}}[f](p_1) = \int d^3p_2 d^3p_2' \frac{d\sigma}{d\omega'} \frac{|p_2 - p_1|}{m} (f_{f4} - f_{f1} f_{f2}). \]

(3)

Here, \( f_a \) stands for \( f(p_a) \), \( (p_1, p_2) \) [respectively, \( (p_3, p_4) \)] are the incoming (outgoing) momenta satisfying energy and momentum conservation and \( d\sigma/d\omega' = 8\pi^2 |p_1 - p_2|^2 \) is the differential scattering cross section towards the outgoing solid angle \( \omega' \).

From [16], the loss rate operator for a unitary Bose gas can be written as

\[ \mathcal{L}_3[f](p_1) = \int d^3p_2 d^3p_3 \frac{A_3}{E_{123}} \phi(\Omega_3)^2 f(p_1) f(p_2) f(p_3), \]

(4)

where \( E_{123} = (p_1^2 + p_2^2 + p_3^2)/2m - (p_1 + p_2 + p_3)^2/6m \) is the energy in the center of mass frame of the three particles of momenta \( (p_1, p_2, p_3) \), \( A_3 = 2\pi^3 (k_B T)^2 L_3 \) and \( \phi(\Omega_3) \) is the hyperangular wave function describing the angular structure of the Efimov trimers that we normalize by the condition \( \int d^3\Omega_3 |\phi(\Omega_3)|^2 = 1 \).

In the absence of losses, the system thermalizes to a distribution \( G \) solution of \( I_{\text{coll}}[G] = 0 \). For a classical gas, the solution of this equation is a Gaussian distribution \( G(n, E; p) = n \lambda_0 e^{-np^2/2m}/\sqrt{2\pi} \), where \( \beta = 1/k_B T \) and \( E = \int (G(p)p^2/2m)d^3p = 3nk_B T/2 \) is the energy density.

In the quasistatic regime \( y \ll 1 \), three-body losses are small and we can use \( A_3 \) as an expansion parameter. Since for \( A_3 = 0 \) the system can reach a stationary thermal state, we expect the characteristic evolution time in the presence of losses to vary as \( A_3^{-1} \) and thus \( \partial_t \) must be considered to scale as \( A_3 \). We write then \( f = f_0 + f_1 + \cdots \) where \( f_j \propto A_3^j \). The expansion of Eq. (2) to first order in \( A_3 \) yields

\[ I_{\text{coll}}[f_0] = 0, \]

(5)

\[ \partial_t f_0 = I'_{\text{coll}}[f_0] - \mathcal{L}_3[f_0], \]

(6)

where \( I'_{\text{coll}} \) is the linearized collisional operator.

According to Eq. (5), \( f_0 \) is a Maxwell-Boltzmann distribution. However, since the system loses particles by three-body recombination, its atom number and its energy vary with time. We therefore have \( f_0(p, t) = G(n_t, E_t; p) \). We then have in Eq. (6)

\[ I'_{\text{coll}}[f_1] = \mathcal{L}_3[f_0] + \partial_t G + i\hbar \partial_t G. \]

(7)

Take \( f_1(p, t) = G(n_t, E_t; p)\alpha(p, t) \). Equation (7) then becomes

\[ C[\alpha] = \frac{1}{G} \mathcal{L}_3[G] + \partial_t G + i\hbar \partial_t \ln(G), \]

(8)

with

\[ C[\alpha] = \frac{1}{G} I'_{\text{coll}}[G\alpha] \]


\[ = \int d^3p_2 d^3p_2' \frac{d\sigma}{d\omega'} \frac{|p_2 - p_1|}{m} \times (\alpha_3 + \alpha_4 - \alpha_1 - \alpha_2), \]

(10)

and \( \alpha_k = \alpha(p_k) \) for \( k = 1, \ldots, 4 \). The operator \( C \) is symmetric for the scalar product [23]

\[ \langle \alpha|\alpha' \rangle = \int d^3p G(p)\alpha(p)\alpha'(p). \]

(11)

Because of energy and particle number conservation, the kernel of \( C \) is spanned by \( \alpha(p) = 1 \) and \( \alpha(p) = p^2 \). Finally, being a symmetric operator, its image is orthogonal to its kernel. To find the time evolution of the energy and the atom number, we project Eq. (8) on 1 and \( p^2 \). Using the structure of the kernel of \( C \), the collisional term vanishes and we obtain

\[ \dot{n}_t = -\frac{1}{2m} \mathcal{L}_3[G] \],

(12)

\[ \dot{E}_t = -\frac{1}{2m} \mathcal{L}_3[G]. \]

(13)

The explicit calculation of the rhs of these equations involves nine-dimensional integrals over the three momenta \( (p_1, p_2, p_3) \) in the three-body loss rate operator. This calculation can be performed analytically by introducing the momentum-space Jacobi coordinates [24] and we finally obtain

\[ \dot{n}_t = -L_n n^3, \]

(14)

\[ \dot{E}_t = -\frac{5}{9} \partial_t E L_n n^2. \]

(15)

where we recover the usual formula for three-body losses, as well as the recombination heating discussed in [16,17].

To find \( \alpha \), we project Eq. (8) on the range of \( C \) [i.e., orthogonally to \( \text{Span}(1, p^2) \)]. We then have

\[ C[\alpha] = P \left[ \frac{1}{G} \mathcal{L}_3[G] \right], \]

(16)

where \( P \) is the orthogonal projector on \( \text{Im}(C) \), and where we used the fact that \( \ln G \) is a linear combination of 1 and \( p^2 \) and thus lies in the kernel of \( C \) and \( P \).

Equation (16) is solved numerically by decomposing its solution over a basis of orthogonal polynomials [24]. The results are displayed in Fig. 1, where we observe a
flattening of the momentum distribution when the three-body losses strength is increased.

In the experiment described in [18], the cloud is not directly prepared in the quasistatic, strongly interacting state. Rather, the experimental sequence starts with a the weakly interacting Bose-Einstein condensate in a regime where losses can be neglected. The magnetic field is then ramped quickly to unitarity where the system can relax towards the quasiequilibrium described above. To get some insight on the relaxation of the system towards equilibrium, we consider the simpler case of a noncondensed gas for which the momentum distribution before the ramp is Gaussian. We write as before \( f = f_0 + f_1 \) with \( f_1 = f_{1,qs} + \delta f_1 \), where \( f_{1,qs} \) is the quasistatic solution and \( \delta f_1 \) satisfies the initial condition \( \delta f_1(p,t=0) = -f_{1,qs}(p,t=0) \), since at \( t = 0 \), \( f = f_0 \). Expanding Boltzmann’s equation to first order in \( f_1 \) and using the properties of \( f_{1,qs} \), we obtain for \( \delta f_1 \),

\[
\partial_t \delta f_1 = \Gamma_{\text{coh}}[\delta f_1].
\]

This equation shows that the relaxation towards the quasistatic regime is solely driven by two-body collisions and occurs at a rate \( \sim \gamma_2 \). This may seem paradoxical since one would rather expect the three-body characteristic rate \( \sim \gamma_3 \). However, as far as the phase-space density is concerned, the depletion of \( f \) at low momenta is quite small since the relative decrease of the peak momentum density is \( \propto n \lambda^3 \). Since \( 1/\gamma_3 \) is the time required to lose typically half the initial atom number, the dip should form on a time scale, \( \approx n \lambda^3 / \gamma_3 \approx 1/\gamma_2 \).

The three-body losses lead to a correction to the momentum distribution proportional to \( n \lambda^3 \). This scaling is similar to the first virial correction, and one may wonder if the three-body losses might not mask the effects of two-body interactions. To clarify this point, we calculated the leading order corrections to the occupation number \( \rho(p) = \hbar^2 f(p) \) using the scheme presented in [26]. In the virial expansion, the leading order term corresponds to the ideal Boltzmann gas. In the grand canonical ensemble, this term reads \( \rho^{(1)}(p) = ze^{-\beta \epsilon p} \), where \( z \) is the fugacity and \( \epsilon_p = p^2 / 2m \). The next order term is the sum of two contributions. The first one corresponds to Bose’s statistics and is simply \( \rho^{(2,a)}(p) = z^2 e^{-2\beta \epsilon p} \), while the second one is more involved and is due to interactions. Following [26], it is given by

\[
\rho^{(2,b)}(p) = \frac{8\pi z^2}{m} \int_{C_{\gamma}} \frac{ds}{2\pi i} \int_0^{+\infty} \frac{dP P^2}{2\pi^2} \frac{e^{-\beta s}}{e^{-P^2/4m} - e^{-\beta s}} \times \frac{e^{-\beta(p^2/4m)}}{[s + p^2/4m - \frac{(p-p')^2}{2m}] [s + p^2/4m - \frac{(p+p')^2}{2m}]},
\]

where \( C_{\gamma} \) is a Bromwich contour [27]. We note that this expression is simply twice that obtained for spin 1/2 fermions [26]. To convert this momentum distribution to the canonical ensemble, we use the virial expansion of the equation of state of the unitary Bose gas, \( n \lambda^3 _{th} = z + 2b_2 z^2 + \cdots \), with \( b_2 = 9/4 \sqrt{2} \). We thus obtain

\[
\rho(p) = n \lambda^3_{th} e^{-\beta \epsilon p} + (n \lambda^3_{th})^2 [\xi(\lambda_{th} p / \hbar) - 2b_2 e^{-\beta \epsilon p}],
\]

where we took \( \rho^{(2)}(p) = \rho^{(2,a)}(p) + \rho^{(2,b)}(p) = z^2 \xi(\lambda_{th} p / \hbar) \).

In Fig. 2, we compare the effect of three-body losses with the virial corrections to the momentum distribution. We observe that for \(^7\text{Li}\), for which \( \eta = 0.2 \), the dip in the momentum distribution is dominated by three-body losses.

We now turn to the quantitative comparison of our results with the experimental data presented in [18]. In this experiment an ultracold, weakly interacting Bose-Einstein condensate is ramped abruptly to the Feshbach Resonance and after a 100-\(\mu\)s-long waiting time, the system reaches a quasiequilibrium characterized by the momentum

![FIG. 1 (color online). Deformation of the momentum distribution of a unitary Bose gas due to three-body losses. From top to bottom: \( n \lambda^3 (1 - e^{-4\eta}) = 0 \) (blue, Boltzmann gas), \( n \lambda^3 (1 - e^{-4\eta}) = 0.05 \) (orange), and \( n \lambda^3 (1 - e^{-4\eta}) = 0.1 \) (red).](image)

![FIG. 2 (color online). Correction to the Boltzmann gas: Three-body losses vs interactions. The correction to Boltzmann’s distribution is plotted for maximal three-body losses (\( \eta = 0\), red dashed line), \( \eta = 0.2 \), corresponding to \(^7\text{Li}\) (orange dotted line). The blue solid line corresponds to the correction, Eq. (19), due to Bose statistics and two-body interactions.](image)
In principle, the virial expansion is valid only in the limit of vanishingly small fugacities, and its accuracy is therefore questionable in the present case. Even though there is no reliable way to assess the accuracy of the virial expansion for unitary Bose gases, we note that for the equation of state of the unitary Fermi gas, the first-order virial expansion gives the correct result at a ≈10% level at z = 0.6 [28,29]. If we assume that the same level of accuracy is achieved in the case of bosons, our calculation should provide a quantitative description of JILA’s experiment. To further support our analysis we note that the temperature deduced from the virial expansion yields a three-body loss rate comparable to the one observed in [18].

The approach presented above provides a quantitative way to study unitary Bose gases in the dilute limit. In the case of the results presented in [18], we find that three-body losses are negligible and that the tail of the momentum distribution is well described by a first-order virial expansion at a fugacity z = 0.6(1). This value raises a series of open questions. First, is it possible to derive this value from a purely microscopic model describing the dynamics of a Bose gas projected from a weakly interacting regime to unitarity. Second, is it really universal? In our work, we assumed that, after the ramp, the thermalization was only driven by the two-body scattering length. However, for strongly interacting bosons, we know that three-body Efimov physics cannot be neglected and requires the introduction of the three-body parameter Rₜ. In this case, the fugacity would be a log-periodic function of the dimensionless parameter κₜRₜ, as suggested in [20]. This assumption could be tested by reproducing JILA’s experiment on different atoms to vary the value of Rₜ.

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