Perceptual Decision making
Part 2

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Perceptual decision making: Generic features

Stimulus of type A vs stimulus of type B,

whenever the stimuli can be ambiguous
Perceptual decision making: Generic

Behavioral observations - relationships between:

Identification (psychometric function)
Discrimination
Reaction times

→ « Categorical perception »

Liberman et al 1957

Pisoni & Tash, 1974

Random dots experiment
Shadlen et al 2006
Putting almost everything together:
Information theoretic viewpoint / Bayesian inference / Signal Detection Theory

Identification (psychometric function) / Discrimination / Reaction times

Results:

- Performance in identification $\Leftrightarrow$ ratio signal ambiguity / discriminability
  (each term being measured by a specific Fisher information)

- Optimal decoding:
  $\rightarrow$ drift = log likelihood (as already seen)
  $\rightarrow$ relationship: discriminability ($d'$, Fisher) / variance of the diffusion process

- Optimal coding $\rightarrow$ « Categorical perception »

Random dots experiment
Shadlen et al 2006
Encoding by a large population of stimulus specific cells feeding regions characterized by more categorical responses, as in, e.g.:

- **visual object categorization tasks**
  - Prefrontal cortex (PFC): neuronal activity not much affected by the physical properties of the stimulus itself, but reflects category membership (Freedman et al., 2001, 2002).

- **random dot experiments:**
  - Middle Temporal (MT) → Lateral Intra Parietal (LIP)
  - Decision (LIP): Experimental support to diffusion process (Kim & Shadlen, 1999; Shadlen & Newsome, 2001; Heekeren et al., 2004; Smith & Ratcliff, 2004; Huk & Shadlen, 2005)
Stimulus space

Neural coding layer: population code
Stimulus-specific cells

Category membership

Decoding: decision layer (identification)
Category-specific cells

Neural coding layer: population code
Stimulus-specific cells
Preliminary argument
Maximization of the mutual information $\rightarrow$ min of the decoding error

$$I(\mu, r) = \mathcal{H}(\mu) - \mathcal{H}(\mu|r)$$

$$\mathcal{H}(\mu) = -\sum_\mu q_\mu \ln q_\mu$$

$$\mathcal{H}(\mu|r) = \int \left( -\sum_\mu \mathcal{P}(\mu|r) \ln \mathcal{P}(\mu|r) \right) \mathcal{P}(r) \, d^N r$$
Mutual Information

\[ I(\mu, r) = H(\mu) - H(\mu | r) \]

Fano Bound

\[ H_b(P_e) + P_e \ln(M - 1) \geq H(\mu) - I(\mu, r) \]

where

\[ H_b(P_e) \equiv -P_e \ln P_e - (1 - P_e) \ln(1 - P_e) \]

is the binary entropy

Fano Bound (weaker form)

\[ P_e \geq \frac{H(\mu | r) - \ln 2}{\ln(M)} \]

\[ \Rightarrow \text{maximizing } I(\mu, r) \text{ optimizes the Fano bound} \]
Wanted at the output: \( P(.|x) = \{ P(\mu|x), \mu = 1, \ldots, M \} \)

Given:

\[ P(r|x) = \text{encoding of the stimulus} \]

\[ P(x|\mu) = \text{stimulus property} \]

Proper cost function: Bayes cost function (Bayes risk), averaged over all possible stimuli given the category, averaged over all possible categories.

Optimal inference: Min cost

\[ P(\mu|r), \mu = 1, \ldots, M \]

which can be shown to be an optimal estimator of \( P(\mu|x) \)

Optimal coding is obtained by maximizing the mutual information

\[ I(\mu, r) = \mathcal{H}(r) - \mathcal{H}(r|\mu) \]
Bayes cost function (Bayes risk) for a categorization task

If $x$ is the stimulus value that elicited the observed neural activity $r$, the **inference risk** (cost) $C$ for making use of a candidate estimator $g$, $g(.|r) = \{g(\mu|r), \mu = 1, \ldots, M\}$, instead of the true posterior probability $P(.|x) = \{P(\mu|x), \mu = 1, \ldots, M\}$, is the Kullback-Leibler divergence (or relative entropy):

$$C(x, r) \equiv \sum_{\mu=1}^{M} P(\mu|x) \ln \frac{P(\mu|x)}{g(\mu|r)}$$

Wanted:
min of the mean cost $\overline{C}$ (average of $C$ over $r$ given $x$, and then over $x$).

Rem.: in other approaches, e.g., Beck et al 2008, the theoretical analysis is based on the optimal estimation of the continuous parameter, $x$.
But what we want here is the **optimal estimation of the discrete quantity**, the class $\mu$. 
Bayes cost function (Bayes risk) for a categorization task

If $x$ is the stimulus value that elicited the observed neural activity $r$, the inference risk (cost) $C$ for making use of a candidate estimator $g$, $g(.|r) = \{g(\mu|r), \mu = 1, \ldots, M\}$, instead of the true posterior probability $P(.|x) = \{P(\mu|x), \mu = 1, \ldots, M\}$, is the Kullback-Leibler divergence (or relative entropy):

$$C(x, r) \equiv \sum_{\mu=1}^{M} P(\mu|x) \ln \frac{P(\mu|x)}{g(\mu|r)}$$

Wanted:

min of the mean cost $\bar{C}$ (average of $C$ over $r$ given $x$, and then over $x$)

$$\bar{C} = -\mathcal{H}(\mu|x) - \int \left( \int \sum_{\mu} P(\mu|x) \ln g(\mu|r) P(r|x) \, d^N r \right) p(x) \, dx$$
For a given coding layer (not necessarily optimized) \( g(\mu|\mathbf{r}) = P(\mu|\mathbf{r}) \) minimizes the mean classification cost.

- **An unbiased estimator:** in the large signal to noise limit, one can show that \( P(\mu|\mathbf{r}) \) is an unbiased estimate of the posterior probability \( P(\mu|x) \), that is:

\[
\int P(\mu|\mathbf{r}) P(\mathbf{r}|x) \, d^{N} \mathbf{r} = P(\mu|x)
\]

- **Remark:** because of the processing chain \( \mu \rightarrow x \rightarrow \mathbf{r} \), the left-hand side of the above equation is not identically equal to \( P(\mu|x) \).

- One can also show that it is **efficient:** its variance is as small as possible, that is, it saturates the Cramér-Rao bound:

\[
\int \left( P(\mu|\mathbf{r}) - P(\mu|x) \right)^2 P(\mathbf{r}|x) d^{N} \mathbf{r} \geq \frac{\left( P'(\mu|x) \right)^2}{F_{\text{code}}(x)}
\]

where \( P'(\mu|x) = \frac{\partial P(\mu|x)}{\partial x} \) and \( F_{\text{code}}(x) = \) (neural) Fisher information.
Optimal estimator: the distribution minimizing $\overline{C}$ is

$$g(\mu|r) = P(\mu|r), \quad \mu = 1, \ldots, M$$

Link with mutual information. In the expression of the mean cost $\overline{C}$, replace $g(\mu|r)$ by the optimal estimator $P(\mu|r)$; $\overline{C}$ is then given by a difference between two information values:

$$\overline{C} = I(\mu, x) - I(\mu, r)$$

$\overline{C}$

$I(\mu, x) \equiv \mathcal{H}(x) - \mathcal{H}(x|\mu)$

- mutual information between category and stimulus
  - quantifies the category-stimulus dependency
  - is independent of the neural system

$I(\mu, r) \equiv \mathcal{H}(r) - \mathcal{H}(r|\mu)$

- mutual information between stimulus and neural code
  - quantifies the category-neural activity dependency
  - can be increased through learning – at best up to $I(\mu, x)$
From this inference cost, one gets that an optimal strategy for the neural system consists in:

1. Maximizing the mutual information
   \[ I(\mu, r) \equiv \mathcal{H}(r) - \mathcal{H}(r|\mu) \]
   (applying the “infomax principle” to the coding layer);

2. Building a decoding layer with \( M \) output cells such that the \( \mu \)th output cell has its activity \( g(\mu|r) \) equal to the optimal estimator, which is the conditional probability \( P(\mu|r) \).
information theoretic approach

Coding layer

Data-processing theorem:

\[ l(\mu, r) \leq l(\mu, x) \]

If \( f \) is a smooth function,

\[ \lim_{N \to \infty} l(\mu, r) = l(\mu, x) \]

What is the first non-zero correction when \( N \) is large but finite?
Large signal to noise ratio limit

Mutual Information between category and neural activity

\[ I(\mu, r) = \text{information content of the signal} - \frac{1}{2} \left\langle \frac{\text{categorization uncertainty}}{\text{neural sensitivity}} \right\rangle \text{ input space} \]
Large signal to noise ratio limit

\[ I(\mu, r) = \text{information content of the signal} - \frac{1}{2} \left( \text{categorization uncertainty} \right) \]

- stimulus ambiguity
- categorization uncertainty
- neural sensitivity
- neural code property
Large signal to noise ratio limit

### Mutual Information between category and neural activity

\[
I(\mu, r) = \text{information content of the signal} - \frac{1}{2} \left\langle \frac{\text{categorization uncertainty}}{\text{neural sensitivity}} \right\rangle_{\text{input space}}
\]

\[
I(\mu, r) = I(\mu, x) - \frac{1}{2} \int \frac{F_{\text{cat}}(x)}{F_{\text{code}}(x)} \, p(x) \, dx + O\left(\frac{1}{N^2}\right)
\]

where

- \(F_{\text{cat}}(x) = - \sum_{\mu=1}^{M} \frac{\partial^2 \ln P(\mu|x)}{\partial x^2} \, P(\mu|x)\) (of order 1)
- \(F_{\text{code}}(x) = - \int \frac{\partial^2 \ln P(r|x)}{\partial x^2} \, P(r|x) \, d^N r\) (of order \(N\))

\(F_{\text{cat}}(x)\):

*category-related Fisher information* that characterizes categorization uncertainty. \(F_{\text{cat}}(x)\) is larger at the boundary between categories.

\(F_{\text{code}}(x)\):

*Fisher information* characterizing the sensitivity of \(r\) to small variations of \(x\).
\[ F_{\text{cat}}(x) \]

Category-related Fisher information

\[
F_{\text{cat}}(x) = -\sum_{\mu=1}^{M} P(\mu|x) \frac{\partial^2 \ln P(\mu|x)}{\partial x^2}
\]

Typically, \( P(\mu|x) \) has a smooth S-shape

\[ |P'(\mu|x)| \text{ is the greatest at the boundary} \]

\[ F_{\text{cat}}(x) \text{ is also the greatest at the boundary, as } F_{\text{cat}}(x) = \sum_{\mu} \frac{P'(\mu|x)^2}{P(\mu|x)} \]
More cells coding for boundaries

- Steepest slope of their tuning curve in these regions
- Sharper tuning curves between categories
Figure: Continuous morph from 'cat' to 'dog' (Freedman et al, 2001)

Figure: Average performance (Freedman et al, 2001)

Figure: Distribution of preferred stimuli for recorded IT neurons (Knoblich et al., 2002)
Reaction times
Link variance diffusion model / Fisher information (d’)

Optimization of the decoding layer

for a given coding layer
(not necessarily optimized)
Coding layer: population code, Poisson neurons, Gaussian like tuning curves.

Decoding layer:

- \( M \) cells, each one fully connected to the coding layer
- \( \mathbf{w} = \{ w_{\mu i} \} \) synaptic weights from coding to decoding cells for every \( \mu \in \{1, \ldots, M\} \), \( z_\mu = \sum_{i=1}^{N} w_{\mu i} r_i \)
- + nonlinearity
  \[ g(\mu|\mathbf{r}, \mathbf{w}) \equiv \text{activity of decoding cell } \mu: \]
  \[ g(\mu|\mathbf{r}, \mathbf{w}) = \frac{\exp z_\mu}{\sum_{\nu=1}^{M} \exp z_\nu} \]
  'soft winner-take-all'
  actual neural implementation via a local circuit (Yuille & Grzywacz 1989).

- \( g(\mu|\mathbf{r}, \mathbf{w}) \sim \text{probability of the category } \mu \text{ given } \mathbf{r} \)
  \( (\mu \in \{1, \ldots, M\}) \)
Coding layer: population code, Poisson neurons, Gaussian like tuning curves

**Learning**: adaptation of the synaptic weights $\mathbf{w}$ in order to minimize the mean Bayes cost $\overline{C}$.

Then we expect $g(\mu|\mathbf{r}, \mathbf{w})$ to have a distribution with mean $P(\mu|x)$,

$$\overline{g}_\mu = P(\mu|x)$$

and variance saturating the Cramér-Rao bound:

$$\nu(g_\mu) = \frac{(P'(\mu|x))^2}{F_{\text{code}}(x)}$$

Two categories.

Simulations for a stimulus near the class boundary

*L. Bonnasse-Gahot & JPN, 2008*
Two categories. Neural activity at time $\tau \sim$ biased random walk

$$\alpha_\tau(r) \equiv \log \frac{g(\mu=1|r,w)}{g(\mu=2|r,w)} = z_{1,\tau}(r) - z_{2,\tau}(r) = \sum_i (w_{1i} - w_{2i}) r_i$$

Large $N$, for independent coding cells - that is for $P(r|x) = \prod_{i=1}^{N} P_i(r_i|x)$:

$$\alpha_\tau(r) \sim \text{Gaussian pdf, with mean:}$$

$$\bar{\alpha}(x) = \tau \sum_i (w_{1i} - w_{2i}) f_i(x) \equiv \tau \bar{\alpha}^0(x)$$

and variance:

$$v_\alpha(x) = \tau \sum_i (w_{1i} - w_{2i})^2 f_i(x) \equiv \tau v_\alpha^0(x)$$

Figure: Decision process

- Developing at first order in $1/N\tau$ shows that $g(\mu|r,w)$ is as well a Gaussian random variable.
From the **hypothesis of optimal decoding**, one finally gets:

### Mean and variance of the diffusion process

<table>
<thead>
<tr>
<th>Mean (drift)</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{1}{\tau_a} \ln \frac{P(1</td>
<td>x)}{P(2</td>
</tr>
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\( F_{\text{code}}^0(x) \) is the Fisher information rate specific to the neural code

\[ [ \text{for Poisson neurons } F_{\text{code}}^0(x) = \sum_i \frac{(f_i'(x))^2}{f_i(x)} ] \]

\( F_{\text{cat}}(x) \) is the category-related Fisher information

L Bonnasse-Gahot & JPN, 2012
Mean reaction time

\[ \bar{\tau}_{RT} = \frac{\text{threshold}}{\text{drift}} \tanh \left( \frac{\text{threshold} \times \text{drift}}{\text{variance}} \right) \]

Case of two Gaussian categories

\[ P(x|\mu) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x - x_\mu)^2}{2\sigma^2} \right) \]

\[ \rightarrow \text{drift proportional to the distance to the category boundary}, \]

\[ x_f \equiv \text{boundary between categories defined by } P(1|x_f) = P(2|x_f). \]

\[ \bar{\tau}_{RT} = \frac{\beta}{(x - x_f)} \tanh \left( \beta(x - x_f)F(x) \right) \]

The variance of the random walk/diffusion is here shown to be stimulus dependent (since it depends on the Fisher information), in contrast to most analysis where it is assumed to be a constant.
Ylinen et al 2005

Behavioral performances of two groups of subjects with respect to the perception of a phonological quantity.

Two categories:
/ᵻ/ (short vowel) and /ᵻː/ (long vowel) (contrast based on vocalic duration).

Two groups:
Native speakers of Finnish: this contrast is phonemic, i.e., these subjects have a distinct representation of the two categories.

Non native speakers (Russian): the vocalic quantity is not contrastive.

A Identification of categories
B reaction times
C discrimination (d')
Data modelling

Native speakers of Finnish

Non native (Russian) group

Reconstruction of the mean reaction time for the whole continuum.

- 7 perceptual distances $d'$ \rightarrow Fisher information rate for the whole continuum.
- The formula, giving the mean reaction time as a function of the stimulus, now only depends on two free parameters, $x_f$ (category boundary), and $\beta$, which we determine, for each group, from the minimization of the error variance in the regression between experimental values and predicted ones.
Decision making:
on (some of the many) topics not discussed in this course
Decision making: on (some of the many) topics not discussed in this course

- Decision making in socio-economic context

Behavioral economics, behavioral game theory, neuroeconomics
Herbert Simon  
*bounded rationality*
Daniel Kahneman, Amos Tversky
Colin Camerer
Ernest Fehr, Simon Gächter

Social psychology
Stanley Milgram
Solomon E. Asch *(Opinion and social pressure, 1955)*
Serge Moscovici

Roles of expected reward/punishment, emotion, trust, social pressure...

- Intertemporal choices

Moreira et al 2016, Neurobiological bases of intertemporal choices: A comprehensive review
Ortega & Tishby, 2016 Memory shapes time perception and intertemporal choices
  (role of coding efficiency of sensorimotor representation, information theoretic approach)

- Executive control

*Cogmaster Course FCS1, ‘Action, décision, volition’ (E. Koechlin)*
Social conformity

- Conformity, Social norms – peer group pressure; conforming to the prevailing conventions; tradition, culture transmission

- Social psychology: Asch conformity experiment (1951, 1955)

When individual judgment conflicts with a group, the individual will often conform his judgment to that of the group
Social conformity

• **Conformity, Social norms** – peer group pressure; conforming to the prevailing conventions; tradition, culture transmission

Social psychology: **Asch** conformity experiment (1951, 1955)

When individual judgment conflicts with a group, the individual will often conform his judgment to that of the group

**Neurobiological correlates** of social conformity (**Berns et al 2005**)

[http://www.ccnl.emory.edu/greg/](http://www.ccnl.emory.edu/greg/)

- Brain regions classically associated with perception can be altered by social influences

- Independence (non-conformity) is found to be associated with subcortical activity changes indicative of emotional salience (amygdala activation ➔ emotional load associated with standing up for one’s belief)
Neurobiological correlates

Figure 1. Participants were presented with pairs of three-dimensional objects on a computer screen during a mental rotation period, and they had to decide whether the objects were the “same” (can be rotated to match) or “different” (no rotation can make them match). To induce social conformity, each trial began with the objects being shown first to a group of peers (Group; top panel). In actuality, the group was composed of actors, and their responses were predetermined. After a variable-duration decision phase, the collective response of the group was displayed to the participant. This ensured that the participant would see the group’s response. After 3 sec, the same pair of objects was displayed to the participant. In the example shown, the objects are different, but the group has unanimously said they are the same (the participant has not responded yet). The participant responded with a button press, indicating whether the objects were the same or different. Trial types were randomized across three conditions: group correct, group incorrect (as shown), and baseline (responses blinded to participant with an “X”; bottom panel). One run of 48 trials was performed with the group, and another run of the same 48 trials was performed with the group replaced by computers (bottom panel), in which the faces of the group were substituted with computer icons. The order of group and computer runs was counterbalanced across participants and gender.
Conformity was defined as agreeing with the exogenous source of information, either peers or computers, when the information was wrong. Conformity was measured behaviorally by the change in error rates of the participants between their baseline performance and the conditions in which exogenous information was presented.

The baseline error rate was computed for each participant from the trials in which no group (or computer) information was given (mean 13.8%, SEM 2%).

The error rate increased to 41% (SEM 5%) when the group gave wrong information, which was significantly greater than when the computers gave wrong information (mean 32%, SEM 4%) [paired t (32) 3.55, p < .001].
There were significant differences in RTs, and these differences depended on several factors (Figure 3). After adjusting for the effect of Same/Different stimuli, there was a significant lengthening of RT when external information was present \( F(2,579) = 20.27, p < .0001 \). But restricting the analysis to trials in which incorrect information was provided, there was no significant difference in RT between going with (i.e., Conformity) or against (i.e., Independence) the information \( F(1,190) = 2.65, p = .105 \), indicating that participants did not take longer for one behavior or the other. Notably for the subsequent image analysis, the source of the external information, either Group or Computers, also did not have a significant effect on RT \( F(1,576) = .554, p = .457 \).
Social conformity


- *Social Influence and Perceptual Decision Making: A Diffusion Model Analysis*
- *Social conformity is due to biased stimulus processing: electrophysiological and diffusion analyses*

Experiments: EEG

Analysis: **diffusion model analysis of reaction time data** to uncover the neurocognitive processes underlying social conformity in perceptual decision-making.
Social conformity

EEG experiments and Drift diffusion model analysis of reaction time data

Initial position

t₀: total non-decisional processing time

Drift rate

Decision A ("conform")
Decision B ("non-conform")

Task: dominant color? (blue or orange)

Half stimuli: half/half pixels in blue/orange
Half stimuli: dominant color well above chance level (performance 80%)
Social conformity

- Markus Germa et al, 2016

Social conformity is due to biased stimulus processing: electrophysiological and diffusion analyses

Experiment: EEG measuring stimulus evoked potentials (SEPs), lateralized readiness potentials (LRPs)

- Analysis of EEG signals: participants under social influence initially activated choices in line with the majority even when they finally chose against the majority.

- Analysis of reaction times: diffusion model analysis of reaction time data

  Conformity: correlated with a change in the drift rate, implying that participants predominantly accumulated stimulus information favoring the majority response
  → longer reaction times when decision against the majority.

→ the opinion of others can cause individuals to selectively process stimulus information supporting this opinion, thereby inducing social conformity.

→ this effect is present even when individuals do not blindly follow the majority but rather carefully process stimulus information
Females prefer males with the brightest yellow head

Why?

Hint:
this vulture gets its yellow colour from the consumption of excrements…
(it has been nicknamed in Spain « moniguero » = « dung-eater »)

Costly signaling / the handicap principle (Zahavi)

Evolutionary mechanism:
carotenoid pigments would diffuse passively to the skin,
the resulting yellow coloration could have become a useful signal in mating displays.

Signal reliability  ↔  cost
(exposition to gastro-intestinal parasites)

Modeling: Game theory

Question: link with this Course?
Neuroscience:

environment $\rightarrow$ stimulus $\rightarrow$ neural code $\rightarrow$ decoding, decision

Statistical (Bayesian) inference:

prior distribution $\rightarrow$ parameter $\rightarrow$ observations $\rightarrow$ estimation

Ethology: handicap principle (Zahavi, 70’s) / Game theory: costly signaling (90’s)

population distribution $\rightarrow$ hidden quality $\rightarrow$ signal $\rightarrow$ selection
Supplementary informations
unknown parameter $x \Rightarrow$ data, observations $r \Rightarrow$ estimator $\hat{x}(r)$

Case of an unbiased estimator: $\langle \hat{x} \rangle_x \equiv \int \hat{x}(r) P(r|\lambda) d^N r = x$

Quadratic error: $\sigma_x^2 \equiv \langle (\hat{x} - x)^2 \rangle_x = \int (\hat{x}(r) - x)^2 P(r|x) d^N r$

Cramer-Rao bound (1945):

$$\sigma_x^2 \geq \frac{1}{F(x)}$$

where $F(x)$ is the Fisher information

$$F(x) = -\int \frac{\partial^2 \ln P(r|x)}{\partial x^2} P(r|x) d^N r$$

Optimal bound: equality for specific cases ("efficient estimators")
Similar to the uncertainty principle in Quantum Mechanics.

Simple proof making use of Schwarz inequality $\langle u^2 \rangle \langle v^2 \rangle \geq \langle uv \rangle^2$
\[ F_{\text{cat}}(x) = - \sum_{\mu=1}^{M} P(\mu|x) \frac{\partial^2 \ln P(\mu|x)}{\partial x^2} \]

Typically, \( P(\mu|x) \) has a smooth S-shape

\[ |P'(\mu|x)| \text{ is the greatest at the boundary} \]

\[ F_{\text{cat}}(x) \text{ is also the greatest at the boundary, as } F_{\text{cat}}(x) = \sum_{\mu} \frac{P'(\mu|x)^2}{P(\mu|x)} \]
$F_{\text{cat}}(x)$ is the Fisher information that characterizes categorization uncertainty.

- Example: case of Gaussian categories

$$P(x|\mu) = \frac{1}{\sigma \sqrt{2\pi}} \exp -\frac{(x - x_\mu)^2}{2\sigma^2},$$

$F_{\text{cat}}(x)$ is the variance of the distance of $x$ to the centers of the categories:

$$F_{\text{cat}}(x) = \frac{1}{\sigma^2} \left[ < \left( \frac{x - x_\mu}{\sigma} \right)^2 >_x - < \frac{x - x_\mu}{\sigma} >^2_x \right]$$

with $< A_\mu >_x \equiv \sum_\mu P(\mu|x) A_\mu$.

- $F_{\text{cat}}(x)$ is larger at the boundary between categories.