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Self-organised critical hot spots of criminal activity

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In this paper[1] we introduce a family of models to describe the spatio-temporal dynamics of criminal activity. It is argued here that with a minimal set of mechanisms corresponding to elements that are basic in the study of crime, one can observe the formation of hot spots. By analysing the simplest versions of our model, we exhibit a self-organised critical state of illegal activities that we propose to call a *warm spot* or a *tepid milieu*[2] depending on the context. It is characterised by a positive level of illegal or uncivil activity that maintains itself without exploding, in contrast with genuine hot spots where localised high level or peaks are being formed. Within our framework, we further investigate optimal policy issues under the constraint of limited resources in law enforcement and deterrence. We also introduce extensions of our model that take into account repeated victimisation effects, local and long range interactions, and briefly discuss some of the resulting effects such as hysteresis phenomena.

1 Introduction

As in other fields, mathematical modelling of criminality has two main scopes. First, specific models are devised that can be directly confronted with detailed empirical data. In this spirit, computational criminology is a promising field mainly based on multi-agent approaches (see e.g. Bernasco 2009; Eck & Liu 2008; Berk 2008). But mathematical models are also useful in shedding light on mechanisms, and can then be used as quantitative approaches to help the analysis of more specific models as well as to guide policing decisions. In this second aspect, *stylized facts* concerning the level and distribution of criminal activity are thus both a benchmark and the object of explanations by the model.

Many different types of models have been developed in the criminology literature: economic and behavioural models (see e.g. Becker 1968; Bourguignon *et al.* 2003b), some

[1] An early and preliminary version of this work has been presented at the workshop *Crime Hot Spots: Behavioral, Computational and Mathematical Models* of the Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, January 29–February 2, 2007.

[2] 'Milieu': the people, physical and social conditions and events which provide the environment in which someone acts or lives (Cambridge Dictionary).

of them taking into account social interactions (see e.g. Glaeser *et al.* 1996; Gordon *et al.* 2009a), with main tools taken from game theory and models related to the general framework of discrete choice in socio-economic modelling; epidemic or predator–prey models making use of ordinary differential equations (ODEs) (see Crane 1991; Campbell & Ormerod 1998; Nuño *et al.* 2008); multi-agent simulations (see Eck & Liu 2008; Groff 2007); and reaction–diffusion models describing spatio-temporal patterns of criminal activity making use of partial diffusion equations (PDEs) (see Short *et al.* 2008, 2010; Pitcher 2010) – we refer to Gordon (2010) for a review on mathematical modelling of criminality. The type of model that we will introduce here makes use of PDEs within an economic/behavioural approach.

One of the well-documented and outstanding stylized facts in criminology is the formation, relative stability and persistence of *hot spots* where high levels of criminal activities are concentrated. In a relatively recent period, it has become apparent that there was an overall positive effect on increasing law enforcement in these areas, even if at the detriment of other neighbourhoods in a city. However, the mechanisms at play in the formation of hot spots, what determines their persistence, stability issues are still the object of much investigation with a variety of points of view. Very recent and interesting studies by Short *et al.* (2008, 2010) propose a model consisting of a system of reaction-diffusion equations to describe this phenomenon. We also refer to the bibliography in these works as well as in Bernasco (2009) for many references on spatial distribution of crime activities.

The present paper is a contribution to the study of formation, dynamics and other properties of hot spots of criminal activity. We propose a new model aimed at describing levels of criminal activity. Since we believe that such approaches are of interest in several different contexts, we choose to remain vague on the type of criminal activity under consideration. These can range from uncivil behaviour, traffic offences to petty crimes, theft and and burglaries. Therefore, in the following, we mostly speak of *illegal activity*.

Three main issues are addressed in the present paper. The first one is the formation and stability of hot spots. It is argued here that with a minimal set of mechanisms corresponding to elements that are basic in the study of crimes, one can observe the formation of hot spots. Actually, we distinguish here between what we propose to call *warm spot* or *tepid milieu* (depending on their intensity and spatial extension), where a self-organised critical state of illegal or uncivil activity maintains itself at a positive level but without exploding, and true hot spot where high level or peaks are being formed.

Then, we propose an approach to investigate optimal policing questions. With necessarily limited law enforcement means and tightening budgets, this question is of high practical importance. Lessons can be drawn from models, even very theoretical ones, like the one we discuss here.

Lastly, we set the stage for taking into account other effects: repeated victimisation, social influence factors or phenomena of diffusion of criminality and deterrence through social interactions. There are several ways the question of social influence can be addressed such as long range interaction, collective interaction or local diffusion and interaction. We briefly describe in the framework of our model a particular trait of crime and punishment which has been discussed in earlier works, in particular by Ormerod (2005), that of *hysteresis*. In few words, this phenomenon bears on the fact that harsher punishment can reduce drastically illegal activity that does not go back to its original high level if the punishment rate is gradually reduced to its former level.

The main purpose of this paper is to combine all of these complex features into a single system of a small number of equations. In the present paper we also analyse some of the simplest situations that arise in this context and argue for their relevance to crime study. We leave for further studies several mathematical aspects of the more complete system as well as applications to some more specific questions or further numerical investigations.

2 Dynamics of criminal activity

2.1 The basic model

We consider a spatial domain – a city – where the instantaneous criminal activity $u(x,t)$ at location x and time t is assumed to depend on an *instantaneous willingness* (or *propensity*) *to act*, $S(x,t)$. By 'instantaneous' we mean an average over some short time scale, but we do not specify this scale – it could be one hour, or one day, depending on the particular application considered. The quantity $S(x,t)$ is a spatio-temporal field that may combine different contributions to the realization of a criminal activity. Taking a coarse-grained approach, one may consider $S(x,t)$ to represent the typical propensity to commit a crime by agents at location x. Alternatively, one may consider $S(x,t)$ to represent the propensity of location x to be the subject of an illegal activity, and this depends on the availability (and instantaneous willingness to commit a crime) of potential offenders, on the presence of potential targets at this location, and on the strength of deterrent forces at this location. This point of view is in line with the *Routine Activity Theory* developed by Cohen & Felson (1979) and Clarke & Felson (1993). This general interpretation should be kept in mind, even though in the following we will mainly make use of the term 'willingness (or propensity) to act'. This propensity S is somewhat analogous to the 'attractiveness' in the model of burglary of Short *et al.* (2008), and to the 'setting' variable in Nadal *et al.* (2010) (although in the later it is a time independent quantity).

The 'city' is defined as an open set Ω in either $\mathbb{R}$ or $\mathbb{R}^2$. We assume the instantaneous illegal activity $u(x,t)$ at location $x \in \Omega$ and time t to be given by a non-linear function of the local propensity to act. We write

$$u(x,t) = \Lambda(S(x,t)). \tag{2.1}$$

We call Λ the *acting-out* function. It is assumed to satisfy $\Lambda(S) = 0$ for $S \leqslant 0$ and $\Lambda(S) > 0$ otherwise, with Λ an increasing function of $S > 0$ with limit 1 as S goes to infinity. We define $\beta = \Lambda'(0^+)$. As examples one may consider

$$\Lambda(S) = \begin{cases} 0 & \text{if } S \leqslant 0, \\ 1 - \exp(-\beta S) & \text{if } S > 0, \end{cases} \tag{2.2}$$

or

$$\Lambda(S) = \begin{cases} 0 & \text{if } S \leqslant 0, \\ \beta S & \text{if } 0 < S < 1/\beta, \\ 1 & \text{if } S > 1/\beta. \end{cases} \tag{2.3}$$

In the extreme case where one look at a single-agent behaviour, or at the level of a single illegal act committed in a given neighbourhood:

$$\Lambda(S) = \begin{cases} 0 & \text{if } S \leqslant 0, \\ 1 & \text{if } S > 0, \end{cases} \tag{2.4}$$

(which corresponds to $\beta \to \infty$). Actually, we will be mainly interested in the large β regime, having in mind that acting out is essentially a binary decision – to act or not to act. Therefore, it corresponds to a threshold type situation, where a sharp transition occurs at $S = 0$ which we believe is warranted in analysing criminal activity (a sharp transition occurs at the very onset of criminal activity).

One of the reasons of the introduction of both a *level* of criminal activity u and a propensity to act S is that the former cannot take negative values but the latter can. Indeed, a point where there is no criminal activity, that is, $u = 0$ may correspond to a variety of situations. It is natural to distinguish two cases with no criminal activity but with one very close to it ($S < 0$ but close to 0 and the other one where this possibility is remote (with S very negative). The dynamics of such two places need to be thought of differently and this is made possible by the introduction of the propensity to act S which is a characteristic of the location at a given time.

Here, for the sake of simplicity, we assume that there is one globally defined such acting-out function. In more complex models however, we might envision situations where Λ also varies with the space variable, for instance, where the territory is divided into patches and each one of them has its own function Λ. In particular, β could be taken to be a parameter that varies spatially.

The model we propose for the time evolution of S reads as follows:

$$\frac{\partial S}{\partial t} = -S(x,t) + W(x,t) - C(x,t)u_m(x,t). \tag{2.5}$$

Let us now explain the meaning of the quantities appearing in this differential equation (2.5). First, $u_m(x,t)$ is a moving average of past activity (crime rate). More precisely, u_m is defined through the equation:

$$\tau_u \frac{\partial u_m}{\partial t} = u(x,t) - u_m(x,t), \tag{2.6}$$

with $u(x,t)$ determined from S through (2.1). Note that (2.5) is written with a time scale normalised to 1. Thus τ_u gives the relative time scale for the moving average of the illegal activity: we assume $\tau_u \gg 1$.

Second, $C(x,t)$ is defined to be a local *cost* of the illegal activity. It represents the *perceived* cost of illegal activity resulting from police enforcement and harshness of punishment. The product $C(x,t)u_m(x,t)$ thus represents the *risk* associated with illegal activity as perceived by potential offenders. The argument, is that a higher past illegal activity at a given place increases the chances of a 'cracking down' by law enforcement authorities. In further versions of this model one may consider a non-linear cost, by replacing Cu_m by some non-linear function $C(u_m)$ involving thresholds, saturations etc.

Third, $W(x,t)$ is an *idiosyncratic willingness* (or *propensity*) *to act*, which might be understood as the local expected payoff of a theft, or as an *idiosyncratic propensity* for illegal activity at location x and at time t. Note that it is a dishonesty index, corresponding to the opposite of the honesty index considered in Bourguignon *et al.* (2003b), Gordon *et al.* (2009a) and Nadal *et al.* (2010). In the simplest setting where W is time independent, in the absence of illegal activity, (2.5) describes the relaxation of $S(x,t)$ towards $W(x)$.

Next, we specify possible dynamics for the cost term $C(x,t)$ and for the idiosyncratic propensity for illegal activity $W(x,t)$ (in particular in order to account for repeated victimisation). We further discuss the introduction of additional terms in order to take into account social influence and deterrent effects.

2.2 Global and local social influence

We start with the inclusion of a contribution from social interactions in the time evolution of the instantaneous willingness to act, S. This can be done in two ways: either the influence is through the observation of the criminal activity, in which case we add a term depending on a (possibly local) average of the criminal activity; or it comes through a more direct interaction, resulting in an *opinion dynamics*, in which case we add a term depending on a (possibly local) average of the willingness to act. We are thus led to the following system of equations:

$$\frac{\partial S}{\partial t} = -S(x,t) + W(x,t) - C(x,t)u_m(x,t) + \int_\Omega J(x,x')u_m(x',t)\,dx', \tag{2.7}$$

$$\tau_u \frac{\partial u_m}{\partial t} = \Lambda(S(x,t)) - u_m(x,t). \tag{2.8}$$

The choice of $u_m(x',t)$ in the integral term of the first equation above is related to the observations of *repeated offences*: Places at which or near which certain crimes have been committed incur a higher risk of this crime being repeated – as in repeated burglaries. This effect is discussed below and we include it here in the social influence term. But one may also consider other factors as relevant for social influence. So more generally, we consider the system

$$\frac{\partial S}{\partial t} = -S(x,t) + W(x,t) - C(x,t)u_m(x,t) + \int_\Omega J(x,x')A(x',t)\,dx', \tag{2.9}$$

$$\tau_u \frac{\partial u_m}{\partial t} = \Lambda(S(x,t)) - u_m(x,t). \tag{2.10}$$

Here, one may choose $A(x,t) = u_m(x,t)$ as above but other choices are possible: $A(x,t) = S(x,t)$ or $A(x,t) = S_m(x,t)$, ($S_m(x,t)$ being the time moving average of the local instantaneous willingness to act, the 'opinion' $S(x,t)$). One may also assume that agents are aware of their idiosyncratic willingness to act, W, so that it is this 'intrinsic opinion' of the agents which matters, hence $A(x,t) = W(x,t)$.

The kernel $J(x,x')$ giving the weight of the social influence may have a global part, which is a part independent of x and x', and a local part, with a non-zero value only for close locations. For the later, the simplest natural hypothesis is to assume a diffusive form. In such case one has the system of equations:

$$\frac{\partial S}{\partial t} = -S(x,t) + W(x,t) - C(x,t)u_m(x,t) + J\,\overline{A}(t) + D\nabla^2 A(x,t), \tag{2.11}$$

$$\tau_u \frac{\partial u_m}{\partial t} = \Lambda(S(x,t)) - u_m(x,t). \tag{2.12}$$

For definiteness, one may take $A(x,t) \equiv u_m(x,t)$ in the above system. To arrive at a properly posed system, we further require boundary conditions on $\partial\Omega$ imposed on S, as this also reflects on A, since it is a function of u_m itself a function of S. The most natural one is the no flux condition:

$$\frac{\partial S}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2.13}$$

where n is the outward unit normal vector field on $\partial\Omega$. But other conditions might be relevant and envisaged as well:

$$\frac{\partial u_m}{\partial n} = g(x,t) \quad \text{on } \partial\Omega, \tag{2.14}$$

where g is a given flux of criminal activity, or

$$\frac{\partial u}{\partial n} = -ku \quad \text{on } \partial\Omega \tag{2.15}$$

for a certain constant k (flux proportional to the instantaneous activity). One may also consider

$$u = g(x,t) \quad \text{on } \partial\Omega \tag{2.16}$$

that is one assumes an imposed level at the boundary. Such conditions may play an important role, especially if one isolates a city within its suburban environment.

In this paper, we will only touch on the case of a global social influence, Section 3.5.1.

2.3 Risk aversion, law enforcement and deterrence

2.3.1 *Adaptive cost*

The simplest way to deal with deterrence here is to assume an adaptive cost: where the criminal rate is higher than the average, more resources are allocated to this place at the detriment of elsewhere. We assume a constant total cost, in average over the city the cost is maintained to a given value C_0. We thus consider the simple adaptive rule:

$$\tau_C \frac{\partial C}{\partial t} = -C(x,t) + C_0 + \eta_C\, C_0 \left(\frac{u_m(x,t)}{\overline{u_m}(t)} - 1 \right). \tag{2.17}$$

Here and in the following, for any quantity $A(x,t)$ the global (spatial) average is denoted $\overline{A}(t)$:

$$\overline{A}(t) = \int_\Omega A(x,t)p(x)\,dx, \tag{2.18}$$

where $p(x)$ denotes the local population density which is assumed to be constant in time. In cases where one would have no criminal activity at all, hence $\overline{u_m}(t) = 0$, one would put 0 in place of the ratio $u_m(t)/\overline{u_m}(t)$. However, such unrealistic situation is clearly of no interest for the present study, and without loss of generality can be ignored. If the crime rate remains uniform, $u_m(x,t) = \overline{u_m}(t)$, the cost will relax to the same value C_0 everywhere. The time scale $\tau_C > 1$ depends on the reactivity of the population and on the law enforcement. With the above updating rule, and provided $\overline{C}(t=0) = C_0$, the mean cost over the population remains constant, $\overline{C}(t) = C_0$, and if we choose $\eta_C \leqslant 1$ the cost C remains non-negative at every location.

An alternative natural choice, equivalent as far as the fixed points of the dynamics are concerned is to impose at each instant of time a cost equal to a minimal value, plus a term proportional to the mean crime rate:

$$C(x,t) = C_0(1-\eta_C) + \eta_C\, C_0\, \frac{u_m(x,t)}{\overline{u_m}(t)}. \tag{2.19}$$

2.3.2 *Deterrence from social interactions*

By analogy with our modelling of criminal activity, we introduce a new variable $v(x,t)$, the instantaneous deterrent activity, with its associated field value $D(x,t)$, the propensity to contribute to deterrence, and the static reference value $D_0(x)$, with

$$\tau_v \frac{\partial v_m}{\partial t} = v(x,t) - v_m(x,t), \tag{2.20}$$

$$v(x,t) = \Lambda_D(D(x,t)), \tag{2.21}$$

where Λ_D is a function similar to Λ, and

$$\tau_D \frac{\partial D}{\partial t} = -D(x,t) + D_0(x) + \int_\Omega L(x,y)\, D(y,t)\, dy + R(D,u_m). \tag{2.22}$$

The last term characterises the reaction to a non-zero criminal activity. R should be an increasing function of D and u_m, if locally there is a large value of u and a large enough positive D. One may consider $R = r(D(x))u_m(x,t)$, with $r(D) = R_0 D$ ($R_0 > 0$) for $D \geqslant 0$ and 0 otherwise. One could also take $r(D) \geqslant 0$ to be a sigmoidal function (going to zero as D goes to $-\infty$, and to a maximal value R_0 for D going to $+\infty$).

To describe the effect of deterrence on the crime rate, we add to the S dynamics, equation (2.5), a new term proportional to both the (average) criminal and deterrent activity rates, so that

$$\frac{\partial S}{\partial t} = -S(x,t) + W(x,t) - C(x,t)u_m(x,t) - K\,u_m(x,t)v_m(x,t). \tag{2.23}$$

This is equivalent to have the dynamics (2.5) with an effective cost

$$C_{eff}(x,t) = C(x,t) + K v_m(x,t). \tag{2.24}$$

It will be interesting to compare (2.5) with the adaptive cost, rule (2.17), to the dynamics (2.23) with $C(x,t) = C_0(x)$ – hence comparing two different dynamical rules for the cost adaptation.

2.4 Repeated victimisation and learning

2.4.1 *Repeated victimisation*

Repeated victimisation is a well-known effect in the criminology literature (see e. g. Nagin & Paternoster 1993; Pease 1998; Johnson *et al.* 2009), and, for instance, in the case of burglaries is related to the so-called 'repeat burglaries'. To account for this effect, we make the idiosyncratic willingness to act depend on time in such a way that its value at a location x increases if the recent local activity increases. One way to do so is to introduce a new term representing the attractiveness to offenders of places with a history of criminal activity: we replace $W(x,t)$ by $W_0(x)+\rho(x,t)u_m(x,t)$ with $\rho(x,t) \geqslant 0$. It is also known that this effect dies out with time which is reflected in our model by taking the weighted average u_m. We are thus led to the following system of equations (leaving aside for simplicity the social influence and deterrent terms):

$$\frac{\partial S}{\partial t} = -S(x,t) + W_0(x) + \rho(x,t)u_m(x,t) - C(x,t)u_m(x,t),$$
$$\tau_u \frac{\partial u_m}{\partial t} = \Lambda(S(x,t)) - u_m(x,t). \tag{2.25}$$

Note that this term $\rho(x,t)u_m(x,t)$ goes in the opposite direction to the risk aversion one, $-C(x,t)u_m(x,t)$. One may read the above equation for S as having an effective cost $C_{eff} = C - \rho$, which is allowed to take negative values. Alternatively, if ρ is time independent, making use of the equation for $\frac{\partial u_m}{\partial t}$ one may rewrite the above system as

$$\frac{\partial S}{\partial t} = -S(x,t) + W(x,t) - C(x,t)u_m(x,t),$$
$$\tau_u \frac{\partial u_m}{\partial t} = \Lambda(S(x,t)) - u_m(x,t), \tag{2.26}$$
$$\tau_u \frac{\partial W}{\partial t} = -W(x,t) + W_0(x) + \rho(x)\Lambda(S(x,t)). \tag{2.27}$$

Note that the time constant for the dynamics of W is here the one of u_m. The last equation in this formulation allows for generalisations, by simply modifying the dynamical rule (2.27) for W.

In the absence of criminal activity, the term $-W(x,t)+W_0(x)$ in (2.27) enforces a relaxation towards a reference state $W_0(x)$, a time independent idiosyncratic value. The later may depend on the socio-economic characteristics of the agents in the neighbourhood, or, e.g. results from a gang maintaining a criminal activity in a neighbourhood. One may also consider cases without relaxation to such reference state, as done in Gordon *et al.* (2009a) and Nadal *et al.* (2010).

Repeated victimisation typically goes together with a diffusion effect – locations nearby a place with high criminal activity are likely to be subject of a higher crime activity. It is natural then to consider together with the repeated victimisation term a diffusive term as discussed in Section 2.2 with either $A(x,t) = u_m(x,t)$ or $A(x,t) = S_m(x,t)$, or even $A(x,t) = S(x,t)$.

It is useful at this point to compare our system of equations with the one in the model introduced and analysed by Short *et al.* (2010). In that model, and using its notations, the attractiveness for buglers at a location x is described by a variable $A(x,t)$ with reference level $B(x)$, which should be compared to our quantities $S(x,t)$ and $W_0(x)$ respectively. The attractiveness A is increased if a burglary has occurred at this location, relaxes to the reference level B in the absence of criminal activity, and diffuses to nearby locations. The burglars' activity is modelled through a density of burglars who arrive randomly at any location, diffuse in the neighbourhood until they make a burglary, and then leave the neighbourhood. In the present model the uncivil or criminal activity is assumed to be a non-linear function of the local variable S. We do not make explicit assumptions on the process leading to the criminal act, except through the choice of the acting out function which characterises the binary nature of the illegal activity, with a threshold phenomenon: u can be large as soon as $S > 0$. Note also that here we have an 'economics' approach, where the local attractiveness or willingness to act S is like a surplus: at a fixed point, it is given by W, a quantity that might be considered as the expected pay-off, plus possible additional pay-off from social interactions, and minus terms representing a cost.

2.4.2 *Learning*

In cases where the risk aversion has a positive long-term effect, agents may adapt their W values, or equivalently the attractivity of places may evolve according to local illegal activity, in a way which tends to make the local values similar to the 'norm', that of the population average. Then one adds a time dependence in W, with $W(x, t = 0) = W_0(x)$ and

$$\tau_w \frac{\partial W}{\partial t} = -W(x,t) + W_0(x) - \eta_w(G_m(x,t) - \overline{G_m}(t)) \tag{2.28}$$

with $G_m(x)$ is the moving average of $C(x,t)u(x,t)$, and $\overline{G_m}$ is the average of G_m over the city. This adaptation occurs on a time scale much larger than the one of S, hence $\tau_w \gg 1$. This time scale might be of the same order as τ_u, the one controlling the moving average of u, since both correspond to time scales affecting agents perception of time. The above learning rule corresponds to a long-term deterrent effect of punishment: for large mean suffered costs the willingness to act decreases.

The choice of Cu_m as the source of W modification can be discussed. Not that this term has the correct dimension (the one of W).

2.5 Links with neural models of decision making

The (family of) model(s) that we have introduced here is somewhat reminiscent of the 'Integrate and Fire' models in theoretical neuroscience (see e.g. Chow *et al.* 2004; Brunel & van Rossum 2007). In this context, (2.5) gives the dynamics of the post-synaptic potential S of neuron indexed by x, with firing rate u_m, and, if Λ is the binary function, u is the spiking activity (spike/no spike). Otherwise, Λ is the *transfer function* and u the spiking rate. The cost term corresponds to an *autapse*, a synaptic connection of the neuron onto itself. Here this connection is inhibitory. The term W corresponds to the external input minus the activation threshold. An apparent difference with a neural model is that here S can be negative, whereas the post-synaptic potential is a positive quantity. However, one can note that, if W and C are bounded from below, one can subtract a constant to

S and work with a positive variable. The social interaction term gives the inputs from other neurons, with excitatory synaptic weights $J(x, x')$ from cell x' to cell x. The model with social deterrence corresponds to a classical scheme of an assembly of cells composed of two sub-populations, one of excitatory neurons, described by S and u, and one of inhibitory neurons described by the variables D and v. In the present model, however, the interactions between these two populations are different from what one usually considers in neuroscience, although these interactions do have a biological correspondence: a term like $-K u_m v_m$ is analogous to having an heterosynaptic interaction – the synaptic input from a cell (here an inhibitory one) onto another one (here an excitatory cell) is gated by the activity of another cell (here of the target cell itself).

The dynamics considered here has still another important difference with the one of neural systems: after a spike, the membrane potential goes back to a resting state, and there is a 'refractory period', a short time during which no spike can be emitted. Here, in the case of a sharp activation function ($\Lambda = 0$ or 1), the reset would be equivalent to, say, reset S to W whenever $u = 1$.

In any case, the dynamics of S may be thought of as reflecting the integration dynamics at the neural level – that of a single neuron or of a network. Such neural dynamics is considered to be at the basis of any decision by the evidence accumulation in favour or against a particular alternative, an action being taken when this accumulation reaches some threshold.

Lastly, we have introduced the possibility of learning or adaptation, for the 'autapse' C and for the idiosyncratic term W. The chosen learning rules depend on the correlations between punishment and activity, which is reminiscent of (but not identical to) 'reinforcement learning rules' considered as the basis of behavioural learning at the neural level. In the neural context, what matters is the actual punishment (or reward). Here what we consider has ambiguously the meaning of either the expected punishment or the mean punishment incurred so far. It would indeed be interesting to explicitly take into account whether the agents have been punished or not. Particular dynamics based on the actual effect of punishment are considered in Gordon *et al.* (2009a) and Nadal *et al.* (2010).

3 Analysis

This section is devoted to a mathematical analysis of the large time limit of our dynamical system. The results are illustrated by numerical simulations.

3.1 Time independent W and C

3.1.1 *Single location dynamics*

Consider first W and C constant in time: $W(x, t) = W(x)$ and $C(x, t) = C(x)$. We have then a set of (spatially) uncoupled equations, so that, in this section, we can drop the index x for the moment, first studying the two coupled equations for given values of the scalars $C > 0$ and W:

$$\frac{dS(t)}{dt} = -S(t) + W - Cu_m(t), \tag{3.1}$$

$$\tau_u \frac{du_m(t)}{dt} = \Lambda(S(t)) - u_m(t). \tag{3.2}$$

We assume that the *acting out* function Λ satisfies $\Lambda(S) = 0$ for $S \leqslant 0$ and $\Lambda(S) > 0$ otherwise, with Λ a C^1 increasing function on $[0, +\infty]$, satisfying $\Lambda(0^+) = 0$ and having limit 1 as S goes to infinity. We define $\beta = \Lambda'(0^+)$. We will be particularly interested here on the large β limit.

The fixed points of the dynamics (3.2) are given by

$$S^* = W - C\Lambda(S^*), \tag{3.3}$$

$$u_m^* = \Lambda(S^*). \tag{3.4}$$

Given the properties of the function Λ one easily sees that for each given values of $\{W, C\}$ there is a unique solution. For $W < 0$,

$$S^* = W, \tag{3.5}$$

$$u_m^* = 0. \tag{3.6}$$

For $W > 0$, $S^* > 0$ is the unique intersection of the graphs $y = S$ and $y = W - C\Lambda(S)$ for $S \geqslant 0$. Clearly, the larger $W > 0$, the larger S^* and the larger the illegal activity u_m^*. The stability is determined by the linearisation of (3.2) near the fixed point. For $W < 0$, one gets the matrix $\begin{pmatrix} -1 & -C \\ 0 & -\frac{1}{\tau_u} \end{pmatrix}$ having eigenvalues -1 and $-1/\tau_u$.

For $W > 0$, one gets the 2×2 matrix

$$\begin{pmatrix} -1 & -C \\ \dfrac{\beta^*}{\tau_u} & -\dfrac{1}{\tau_u} \end{pmatrix} \tag{3.7}$$

with $\beta^* \equiv \Lambda'(S^*) > 0$. The eigenvalues are given by

$$\text{For } \beta^* C \leqslant \frac{(\tau_u - 1)^2}{4\tau_u}, \quad \lambda_\pm = -\frac{\tau_u + 1}{2\tau_u} \pm \frac{1}{2\tau_u}[(\tau_u - 1)^2 - 4\tau_u \beta^* C]^{1/2}. \tag{3.8}$$

$$\text{For } \beta^* C > \frac{(\tau_u - 1)^2}{4\tau_u}, \quad \lambda_\pm = -\frac{\tau_u + 1}{2\tau_u} \pm \frac{i}{2\tau_u}[4\tau_u \beta^* C - (\tau_u - 1)^2]^{1/2}. \tag{3.9}$$

The real part of the eigenvalues are always negative, so that the fixed point is always stable. In the case (3.9), convergence towards the fixed point occurs with damped oscillations. This is the case for β^* large enough, that is for W smaller than C if the function Λ has a steep slope at $S > 0$, as illustrated in Figure 1.

3.1.2 *Large β limit*

We consider a family of acting out functions indexed by the slope β at $S = 0^+$, as (2.2) or (2.3). More precisely, we assume that, when $\beta \to \infty$, the acting out function $\Lambda = \lambda_\beta$ converges to the step acting out function defined in (2.4). For βC large, when $0 < W < C$, at first order in $1/\beta C$ the fixed point is given by

$$S^* = \frac{W}{\beta C}, \tag{3.10}$$

$$u_m^* = \frac{W}{C}\left(1 - \frac{1}{\beta C}\right). \tag{3.11}$$

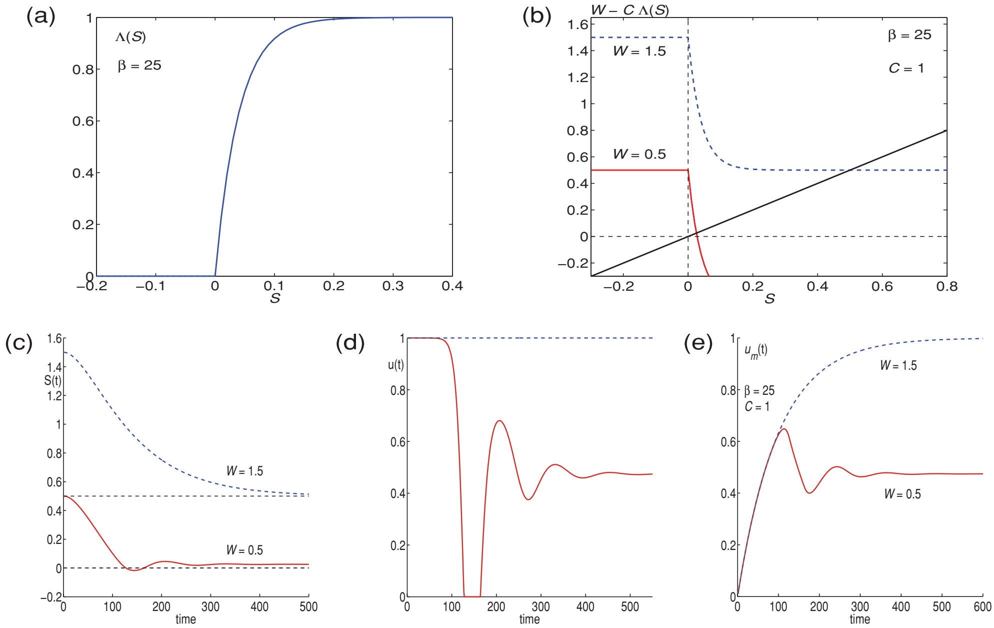

FIGURE 1. (Colour online) For the acting out function defined by (2.2) with $\beta = 25$ (a), illustration of the dynamics (3.2) and the resulting fixed points for a cost $C = 1$, and two particular values of W, one with $W < C$, $W = 0.5$ (solid curves), and one with $W > C$, $W = 1.5$ (dashed curves). (b) Fixed point obtained as the intersection of $y = W - C\Lambda(S)$ with $y = S$. For $W < C$ the fixed point is at a small value of S. (c)–(e) Dynamics of $S(t), u(t)$ and $u_m(t)$, showing damped oscillations for $W < C$.

As already seen, for $\beta C > \frac{(\tau_u - 1)^2}{4\tau_u}$ (that is essentially for βC larger than $\sim\tau_u/4$, since τ_u is large), convergence towards the fixed point occurs with oscillations, the frequency of which increases with β. For $W > C$, $S^* \sim W - C$ and $u_m^* \sim 1$.

Thus, in the limit $\beta \to \infty$, one has a well-defined limit for u_m^* and S^*, given by $S^* = W - Cu_m^* = 0$ for $0 < W < C$, and by $S^* = W - C$, $u_m^* = 1$ for $W > C$. Clearly, for $0 < W < C$ one cannot have a fixed point for $u = \Lambda(S)$, that is in this infinite β limit the instantaneous activity u remains irregular, with bursts of illegal activities, with however a well-defined mean activity $u_m^* = W/C$. Note that the limiting equation can be interpreted as a multi-valued differential inclusion that can be studied with special methods. Four our purpose here, it will suffice to interpret this equation as the limits of the continuous equations.

The main conclusions of this analysis for large β are:

(1) Whenever the idiosyncratic willingness to act W is larger than the cost C, the instantaneous willingness to act S is given by the gap between C and W, and the illegal activity is maximal, $u_m^* \sim 1$: one has here a hot spot.

(2) If the idiosyncratic willingness to act W is positive but smaller than the cost C, the dynamics leads to a stable state where the instantaneous willingness to act S is positive but almost zero, with a mean illegal activity level given by a ‘quality/price

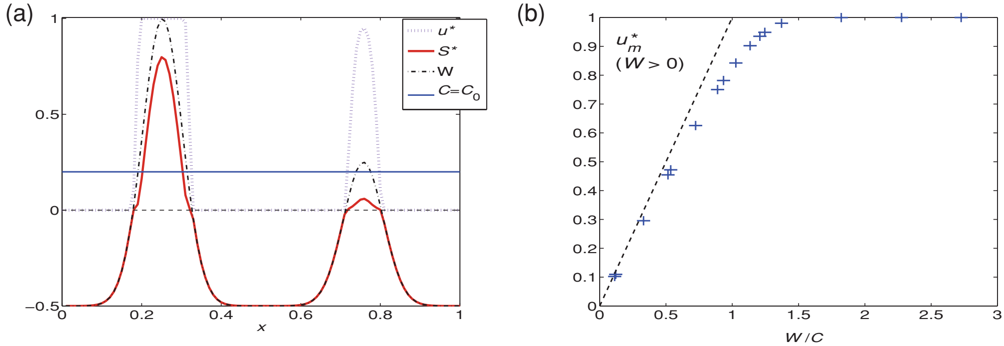

FIGURE 2. (Colour online) (a) Fixed points for the acting out function defined by (2.2) with $\beta = 50$, with a constant cost $C = 0.2$ (solid horizontal line), for a particular spatial pattern of idiosyncratic values $W(x)$ (dot-dashed curve), with a uniform population density on [0, 1]. Dot curve: u_m^*; solid curve: S^*. (b) For this simulation, plot of the activity u_m^* as function of W/C. For W/C of order 1 there is a crossover from the linear behaviour $u_m^* = W/C$ to the saturation at maximal activity, $u_m^* = 1$.

ratio', W/C. This state can be considered as a self-organised critical state (since $S = 0$ is the critical value at which there would be no illegal activity), where the illegal activity is positive but remains moderate (on the concept of self-organised criticality, see Bak 1997). We propose to call such state a *warm spot*. In some cases, such as the case of uncivil behaviour, one may assume that W is positive but small almost everywhere. Then essentially all the space is in this self-organised critical sate (with possibly some hot spots, as well as some quiet spots where $W < 0$): we propose to call this a *tepid milieu.*

Figure 2 illustrate the fixed points obtained for a particular spatial pattern of idiosyncratic values $W(x)$ having two underlying hot spots (positive values of W in two neighbourhoods) for a (moderately) large value of β.

3.2 Optimal control

Under the hypotheses considered in the previous section, it is clear that one would like to have the cost C larger than the idiosyncratic willingness to act W. Still ignoring possible social interaction effects, let us address the issue of optimal control under the constraint of limited resources.

Assume that on the spatial domain Ω under consideration, there is at each location $x \in \Omega$ a particular time independent value $W(x)$ of the idiosyncratic willingness to act, and continue to assume that the acting out function Λ is independent of the location and characterised by a large β value. Now we study the (realistic) scenario when there is a global limit imposed on available resources to enforce a cost on illegal activity. We want to compute the cost function $C(x)$ such that the mean illegal activity is minimised, under the constraint that the total cost – hence the mean cost averaged over all locations – is given.

We assume the criminal activities given by the fixed point equations, defined by (3.6) at locations x where $W(x) < 0$, and (3.11) where $W(x) \geqslant 0$. Since the crime rate u_m

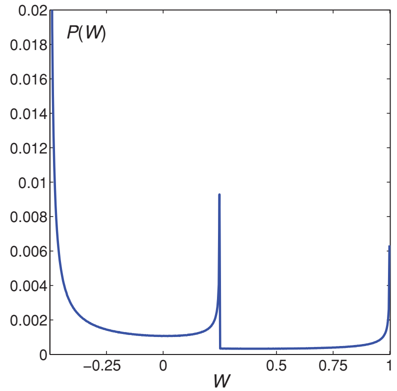

FIGURE 3. (Colour online) Probability density function of W associated to the spatial pattern $W(x)$ shown in Figure 4. In order to see the distribution at positive values of W, the plot is truncated at $P(W) = 0.02$ (the distribution has a large peak at $W = W_{min} = -.5$ with $P(W_{min})$ of order .3). This pdf $P(W)$ has a peak at W_{min}, $W_{max} = 1$, and at the value of W corresponding to the maximum of $W(x)$, $W = 0.25$.

depends only on W and C, we can consider C as a function of W, and search for the cost $C^{opt}(W)$ which minimises the mean activity. We thus consider the minimisation of $\overline{u_m} = \int u_m(W,C)P(W)dW$ with $P(W) = \int_\Omega \delta(W - W(x))p(x)dx$ (Figure 3 shows the pdf $P(W)$ for the spatial pattern $W(x)$ used in the numerical simulations, with a uniform population density on [0, 1], see Figures 2 and 4). Recall that $p(x)$ denotes the population probability density. Note also that the integral defining $\overline{u_m}$ simply represents the average of $u_m(W(x), C(W(x)))$ with respect to this distribution. That is, from now on, for any quantity $A(W(x))$ we define (differently from (2.18))

$$\overline{A} = \int_\Omega A(W(x))p(x)dx = \int A(W)P(W)\,dW. \tag{3.12}$$

The integration in the rightmost term is over the full support of W, which might be $]-\infty, \infty[$, or some finite interval, $[W_{min}, W_{max}]$ – in which case we can straightforwardly extend it to $]-\infty, \infty[$ by defining $P(W) = 0$ outside $[W_{min}, W_{max}]$.

For $W < 0$, whatever the cost is, the equilibrium imposes $u_m^* = 0$, hence in this model the cost can be set to zero wherever the idiosyncratic willingness is negative:

$$\text{Wherever} \quad W < 0, \quad C^{opt}(W) = 0. \tag{3.13}$$

We can thus restrict the computation of the optimal cost function $C^{opt}(W)$ to $W \in [0, +\infty[$. Introducing a Lagrange multiplier λ to enforce the constraint on the mean cost, we consider the minimisation of

$$\mathscr{L} \equiv \int_0^\infty u_m(W,C)P(W)\,dW + \lambda\left\{\int_0^\infty C(W)P(W)dW - C_0\right\}, \tag{3.14}$$

where

$$u_m(W,C) = \begin{cases} W/C & \text{if } 0 < W < C, \\ 1 & \text{if } W \geqslant C. \end{cases} \tag{3.15}$$

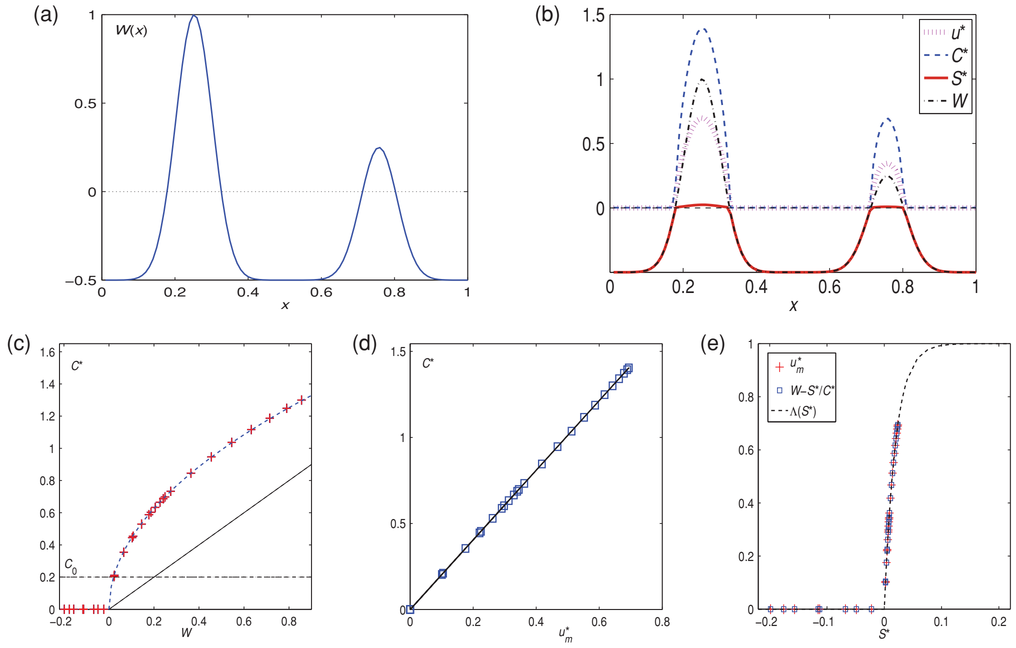

FIGURE 4. (Colour online) Case of an adaptive cost (as defined Section 2.3.1 and analysed Section 3.3). For the acting out function defined by (2.2) with $\beta = 50$, illustration of the fixed points of the dynamics (3.26) for a pattern of idiosyncratic values $W(x)$ shown on (a)–(b). Same W pattern and parameters as for Figure 2, with here a mean cost value C_0 equal to the value of the constant cost in Figure 2, $C_0 = 0.2$.

Assume that one can achieve a solution with $C > W$ everywhere. For $W > 0$, setting to zero the functional derivative of $\mathscr{L}$ with respect to C gives

$$-\frac{W}{C^2} + \lambda = 0 \tag{3.16}$$

which leads to C proportional to the squareroot of W, and with the constraint on the average of C one gets

$$C^{opt}(W) = C_0\, W^{1/2}/\overline{W^{1/2}}, \tag{3.17}$$

where we recall that the bar denotes the average over the space, or equivalently over the W distribution, which is here $\overline{W^{1/2}} = \int_0^\infty W^{1/2} P(W)\, dW$. For this optimal cost function, the mean crime rate is

$$\overline{u_m}^{opt} = \frac{(\,\overline{W^{1/2}}\,)^2}{C_0} \tag{3.18}$$

and at a location with idiosyncratic willingness to act $W \geqslant 0$, the mean crime rate is

$$u_m^{opt} = \frac{\overline{W^{1/2}}}{C_0} W^{1/2}. \tag{3.19}$$

It is worth noticing that, at this optimal solution, the crime rate is not reduced to the same value everywhere. Compared to the solution that would be obtained with a same value C_0 everywhere (that is $u_m = u_m(W, C_0)$), one has a strong reduction of the criminal activity where $W > C_0$, and where $W < C_0$ one has $u_m^{opt}/u_m(W, C_0) = \overline{W^{1/2}}/W^{1/2}$. Hence the price to pay to reach the minimal mean crime rate is to accept a larger criminal activity at locations where $W^{1/2}$ is smaller than the average value of $W^{1/2}$.

The above solution (3.17) is valid if one has indeed $W < C^{opt}(W)$ for all values of W with $P(W) > 0$, that is, if W_{max} is the largest value taken by $W(x)$ for $x \in \Omega$, if

$$W_{max}^{1/2}\, \overline{W^{1/2}} < C_0. \tag{3.20}$$

The equality $C_0 = W_{max}^{1/2}\, \overline{W^{1/2}}$ gives the minimal global amount of resource for which no location in Ω will have the maximal crime rate. In the unfortunate case where $W_{max}^{1/2}\, \overline{W^{1/2}} > C_0$, the optimal cost is proportional to the square root of W up to a threshold value W_c at which $C^{opt}(W_c) = W_c$, and then the cost is kept at this value maximal value $C_{\max} = W_c$ for all $W > W_c$:

$$C^{opt}(W) = \begin{cases} 0 & \text{if } W \leqslant 0, \\ W_c^{1/2}\, W^{1/2} & \text{if } 0 < W \leqslant W_c, \\ C_{max} = W_c & \text{if } W > W_c. \end{cases} \tag{3.21}$$

The threshold value $W_c < W_{max}$ is given by the constraint on the mean cost, which can be written here

$$C_0 = W_c^{1/2} \left(\overline{W^{1/2}} - \int_{W_c}^{W_{max}} (W^{1/2} - W_c^{1/2}) P(W)\, dW \right). \tag{3.22}$$

For the optimal cost function, one gets that the mean crime rate takes a very simple expression,

$$\overline{u_m}^{opt} = \frac{C_0}{C_{max}}, \tag{3.23}$$

where we recall that $C_{max} = W_c$ is solution of (3.22). One can check that this expression gives back (3.18) at $C_0 = W_{max}^{1/2}\, \overline{W^{1/2}}$. For illustrative purpose, consider the case of a uniform distribution of W on $[0, W_{max}]$ (and no negative values at all). One gets $\overline{W^{1/2}} = 2W_{max}/3$. The places with $u_m = 1$, can be avoided with a mean cost at least equal to two third of the maximal value of W. Otherwise, for $W_{max} > 3C_0/2$, $W_c = \frac{3}{2} W_{max}(1 - [1 - \frac{4C_0}{3W_{max}}]^{1/2})$.

Beside the constraint on the mean cost, there may exist a maximal value C_1 of the cost resulting from the maximal resources that can be provided locally at a given location. In that case, the solution (3.21) cannot be realized if $C_1 < W_c$, and the least worst choice is to have the cost given by the expression (3.21) with W_c replaced by C_1.

The optimal policy for the choice of the cost depends on the knowledge of the idiosyncratic willingness to act W, which is not an observable quantity. In the next section, we will see that the simple cost adaptation rules (2.17) and (2.19) proposed in Section 2.3.1 allow one to enforce the optimal policy from the observation of the illegal activity alone.

3.3 Cost adaptation

Let us now add to the dynamics the cost adaptation rule (2.17), still keeping $W(x)$ time independent. Since the cost $C(x,t)$ has a dynamics which depends on the mean crime rate averaged over all locations, $\overline{u_m}(t)$, the dynamics at the different locations are coupled, but coupled only through this collective variable:

$$\frac{\partial S(x,t)}{\partial t} = -S(x,t) + W(x) - C(x,t)\,u_m(x,t), \tag{3.24}$$

$$\tau_u \frac{\partial u_m(x,t)}{\partial t} = \Lambda(S(x,t)) - u_m(x,t), \tag{3.25}$$

$$\tau_C \frac{\partial C(x,t)}{\partial t} = -C(x,t) + C_0 + \eta_C\,C_0\left(\frac{u_m(x,t)}{\overline{u_m}(t)} - 1\right), \tag{3.26}$$

with $\overline{u_m}(t) = \int_\Omega u_m(x,t)p(x)\,dx$.

A a fixed point, the quantities of interest depend on the location x only through $W(x)$. Anticipating the uniqueness of the solution, we can consider all quantities as functions of W (instead of space x), and write the fixed point equations as

$$S^*(W) = W - C^*(W)\,u_m^*(W), \tag{3.27}$$

$$u_m^*(W) = \Lambda(S^*(W)), \tag{3.28}$$

$$C^*(W) = (1-\eta_C)\,C_0 + \eta_C\,C_0\,\frac{u_m^*(W)}{\overline{u_m^*}}, \tag{3.29}$$

with $\overline{u_m^*}(t) = \int u_m^*(W,t)P(W)\,dW$. For locations where $W < 0$, it is obvious that the solution, unique and stable, is given by $S^*(W) = W$, $u_m^* = 0$ and $C^* = (1-\eta_C)\,C_0$, the smallest possible value that can be taken by the cost C^*. We can now restrict the analysis to the set of locations with $W > 0$ and, again, we consider the limit of large β. In this limit, for $W > 0$ we can rewrite the fixed point equations (3.27) and (3.28) as

$$S^*(W) = \begin{cases} 0+ & \text{if } W \leqslant C^*(W), \\ W - C^*(W) & \text{if } W > C^*(W), \end{cases} \tag{3.30}$$

$$u_m^* = \begin{cases} W/C^*(W) & \text{if } W \leqslant C^*(W), \\ 1 & \text{if } W > C^*(W). \end{cases} \tag{3.31}$$

Let us consider the particular case $\eta_C = 1$. First, assume that the condition $W \leqslant C^*(W)$ is always true. Then one readily gets that C^* is proportional to the square root of W, which gives exactly the expression (3.17) of the cost. As already seen in the previous section, this optimal situation will be reached provided the condition (3.20) is met. Otherwise, there is a maximal cost $C^*_{max} = C_0/\overline{u_m^*}$ reached for every value of W larger than some critical value: again one gets exactly the optimal cost function, the one given by the expression (3.21) for cases where (3.20) is not valid.

For $\eta_C < 1$, one gets a sub-optimal solution, with

$$C^*(W) = \frac{1-\eta_C}{2}C_0 + \left[\eta_C \frac{W}{\overline{u_m^*}} + \frac{(1-\eta_C)^2}{4}C_0^2\right]^{1/2}, \tag{3.32}$$

and the global parameter $\overline{u_m^*}$ is determined by the normalisation condition $\overline{u_m^*} = \overline{W/C^*(W)}$. In our model, there is no criminal activity at all for $W < 0$, which might be considered as a too strong hypothesis. A plausible variant would be to have an acting out function taking non-zero but very small values for $W < 0$, or, one may add some noise in the dynamics of S, which would have the same qualitative effect. In such cases, a small but non-zero cost would be necessary as well, hence one would have to take a value of η_C smaller than one. We leave to future work a more detailed analysis of such case.

The main conclusion of this section is thus that a simple adaptation rule, which makes the cost proportional to the local crime rate, is sufficient to enforce the optimal policy without the knowledge of the unobservable variables which characterise the propensity to commit crime. In order to get rid of the hottest spots, the places out of control where the crime rate is maximal, one may have to increase the global available resource C_0. Within the present scheme, the minimal necessary global resource can be reached from a progressive increase of C_0 until the point where there is no more any domain with $u_m = 1$.

3.4 Social deterrence

Let us go back to the case C constant and homogeneous, $C(x,t) = C_0(x) = C_0$, and study the specific effect of deterrence as described by (2.20)–(2.23). Without any non-local interaction, the equations for a location with particular values $D_0 > 0$ and W_0 are then

$$\frac{dS(t)}{dt} = -S(t) + W_0 - (C_0 + K v_m(t))\, u_m(t), \tag{3.33}$$

$$\tau_u \frac{du_m(t)}{dt} = \Lambda(S(t)) - u_m(t), \tag{3.34}$$

$$\tau_v \frac{dv_m(t)}{dt} = \Lambda_D(D(t)) - v_m(t), \tag{3.35}$$

$$\tau_D \frac{dD(t)}{dt} = -D(t) + D_0 + R_0 D(t)\, u_m(t), \tag{3.36}$$

where $R = R_0 D(t)\, u_m$ with $0 < R_0 < 1$. Equation (3.33) can be rewritten as

$$\frac{dS(t)}{dt} = -S(t) + W_0 - C_{\mathit{eff}}(t) u_m(t), \tag{3.37}$$

$$\tau_v \frac{dC_{\mathit{eff}}(t)}{dt} = -C_{\mathit{eff}}(t) + C_0 + K\, \Lambda_D(D(t)), \tag{3.38}$$

with $C_{\mathit{eff}}(t) \equiv C_0 + K v_m(t)$. These equations have to be compared to the ones with cost adaptation but no deterrence, (3.26). Here the dynamics of the effective cost is more complex, but is purely local. We will not give a full analysis of this system, but focuses on the fixed points equations, which for $R_0 < 1$ can be written as

$$S^* = W_0 - C^*_{\mathit{eff}} u^*_m, \tag{3.39}$$

$$u^*_m = \Lambda(S^*), \tag{3.40}$$

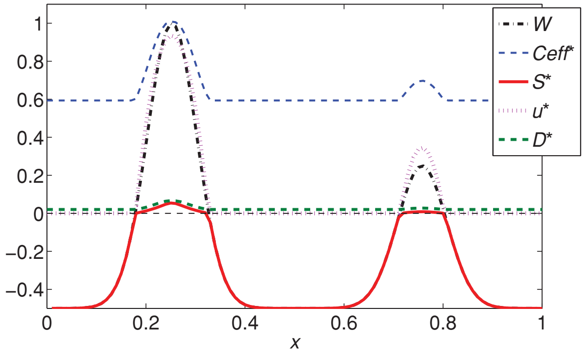

FIGURE 5. (Colour online) Case of social deterrence (as defined Section 2.3.2 and analysed Section 3.4), with a constant cost $C = C_0$. Plot of $W(x)$ (dash-dot curve), and of the fixed point values of the effective cost C_{eff} (upper dashed curve), D (lower dashed curve) and S (solid curve). Parameters: $C_0 = 0.2$, $D_0 = 0.02$, $R_0 = 0.8$, $K = 1.$; acting out and deterrence functions given by (2.2) with, respectively, $\beta = 50$ and $\beta_D = 25$; same values of C_0, β and same pattern $W(x)$ as in Figures 2 and 4.

$$C^*_{eff} = C_0 + K\,\Lambda_D(D^*), \tag{3.41}$$

$$D^* = \frac{D_0}{1 - R_0 u^*_m}. \tag{3.42}$$

For $W_0 < 0$, one still has the obvious solution $S^* = W_0$, $u^*_m = 0$, $v^*_m = \Lambda_D(D_0)$. For $W_0 > 0$, as before, we focus on the large β limit. The maximal possible value of C^*_{eff} is $C^{max}_{eff} = C_0 + K\Lambda_D(D_0/(1-R_0))$. From the analysis done before we know that if $W_0 > C^{max}_{eff}$ one has the solution $u^*_m \sim 1$, $S^* = W_0 - C^{max}_{eff}$, $D^* = D_0/(1 - R_0)$.

Otherwise, that is if $W_0 < C^{max}_{eff}$, S^* is not too large, $\Lambda(S^*) \sim \beta S^*$, and one gets $S^* = W_0/(1 + \beta C^*_{eff})$, with C^*_{eff} at least equal to C_0. Hence S^* goes to zero as β goes to ∞, and $u^*_m = W_0/C^*_{eff}$, with C^*_{eff} obtained as the solution of

$$C^*_{eff} = C_0 + K\,\Lambda_D\left(\frac{D_0}{1 - R_0 W_0/C^*_{eff}}\right). \tag{3.43}$$

This equation has a unique solution for $W_0 < C^{max}_{eff} = C_0 + K\Lambda_D(D_0/(1 - R_0))$ since Λ_D is a strictly increasing function of its (positive) argument.

The results are thus qualitatively similar to the case with an adaptive cost (see Figure 5), except that here the deterrent resources are purely local, hence less efficient. In the preceding case, available resources are concentrated on locations where a higher deterrent or law enforcement is needed, and the highest possible value of C that can be found in the stationary regime is $C_0 + \eta_C C_0(1 - \overline{u_m})/\overline{u_m}$, which is at most $C_0/\overline{u_m}$. Here the maximal deterrent effect is given by the maximal possible value of the effective cost $C + K u_m v_m$ which is the constant $C_0 + K$. In the cost adaptation case one has a multiplicative effect, whereas here one has an additive effect. There is thus a self-adaptation mechanism in the first case which is strongly limited in the present case. Limiting the criminal activity requires to tune the parameter K in order to have everywhere $C(x) > W(x)$. Such condition is much more likely to be met in the first case, thanks to the recruitment of resources from locations where there is no need for a strong cost, and to the multiplicative effect on the largest cost values.

3.5 Social influence

The effect of social influence on criminal behaviour has been discussed with different techniques and models (see Glaeser *et al.*, 1996; Campbell & Ormerod, 1998; Calvó-Armengol & Zenou, 2004; Ormerod, 2005).

Let us now briefly discuss the inclusion in our model of a social interaction in the time evolution of the instantaneous willingness to act, S, as proposed in Section 2.2. The main results are typical of those for models of discrete choice under social influence – or Ising type models (Random Markov Fields) with positive interactions–, as discussed in Gordon *et al.* (2009b): multiple equilibria generically exist. In particular, 'first-order'-type transitions will appear giving rise to hysteresis.

3.5.1 *Global social influence*

We consider here the simplest case, that of a global social influence, with a constant homogeneous cost, C, a time independent idiosyncratic willingness to act, $W(x)$, no social deterrent effect ($D = 0, v = 0$). We assume here that the influence is through the global mean criminal activity, $A(x,t) = u_m(x,t)$, with a positive weight of the social influence, $J > 0$. Equation (2.8) then reduces to

$$\frac{\partial S}{\partial t} = -S(x,t) + W(x) - C\, u_m(x,t) + J\, \overline{u_m}(t). \tag{3.44}$$

The possible fixed points are then given by the following system:

$$S^*(x) = W(x) - Cu_m^* + J\overline{u_m^*}, \tag{3.45}$$

$$u_m^* = u^* = \Lambda(S^*). \tag{3.46}$$

For x with $S^*(x) < 0$, this writes

$$S^*(x) = W(x) + J\overline{u_m^*}, \tag{3.47}$$

which can thus only happen for x for which $W(x) < -J\overline{u_m^*}$.
For $S^* > 0$, the fixed point equation (3.46) can be written in the form:

$$u_m^*(x) = \Lambda_C(W(x) + J\overline{u_m^*}), \tag{3.48}$$

where Λ_C is defined by $\Lambda_C(S) = 0$ for $S \leqslant 0$, and for $S > 0$ as the inverse of $\Lambda^{-1} + C\,Id$ (with Λ^{-1} the inverse of the function Λ restricted to $S > 0$ here). Therefore it suffices to determine $\overline{u_m^*}$ to completely solve the problem.

Given the pattern $W(x)$, one then gets the self-consistent equation for $\overline{u_m^*}$:

$$\overline{u_m^*} = \int_\Omega \Lambda_C(W(x) + J\overline{u_m^*})\, p(x)\, dx. \tag{3.49}$$

This equation is similar to fixed point equations encountered in Ising models in physics and in discrete choice model in the economics literature. With $J > 0$, this type of equation is

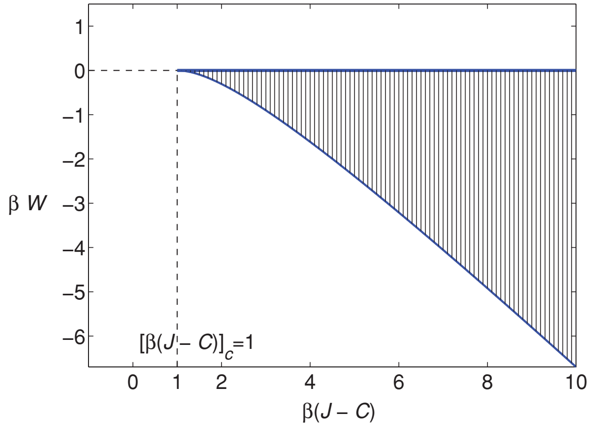

FIGURE 6. (Colour online) Global social influence: Domain (dashed area) of multiple solutions in the parameter space, in the case of a homogeneous W value, and with the acting out function (2.2).

known to have generically multiple solutions – that is, there is a wide range of parameters for which there exists several solutions. The analysis of the solutions of this equation is analogous to the one done in Nadal *et al.* (2006), Gordon *et al.* (2009b). It is sketched in the Appendix.

For the simplest case of a homogeneous distribution of W values ($W = \overline{W}$ everywhere), in which case the fixed point equation simplifies to

$$\overline{u_m^*} = \Lambda(\overline{W} + (J - C)\overline{u_m^*}), \tag{3.50}$$

Figure 6 shows the *phase diagram*: in parameters space, the domain where the fixed point equation (3.49) has two solutions, one with $\overline{u_m^*} = 0$ (which would be the solution in the absence of social influence), and one with $\overline{u_m^*} > 0$ (see the Appendix for details).

We leave to future work a more detailed analysis of the fixed point solutions and their stability in more general cases.

3.5.2 *Hysteresis*

As noted by Gordon *et al.* (2009b), the domain of multiple solutions lies in the part of the parameters space corresponding to populations which, in average, are *not* willing to act (that is at negative values of $\overline{W}$ here). This means that hot spots of criminal activity may be caused purely by collective effects at locations where, with the same socio-economic characteristics, one could equally observe a ‘quiet’ spot. One can thus expect that such hot spots can be reduced, thanks to, or despite, the hysteresis effect associated to these multiple equilibria. This is the object of the discussion below.

There is a considerable literature on hysteresis in choice models and related models like Ising spin systems. We refer in particular to Sethna (2009) for various applications in physics, to Nadal *et al.* (2006) for choice models in a socio-economics context, and to Ormerod (2005) for a very interesting discussion of the implications of hysteresis in the context of crime modelling. We reconsider the later within our framework.

Let us consider the simple case of a homogeneous distribution of W (for which the phase diagram is shown on Figure 6). Figure 7 illustrate the solutions $\overline{u_m*}$ as function of

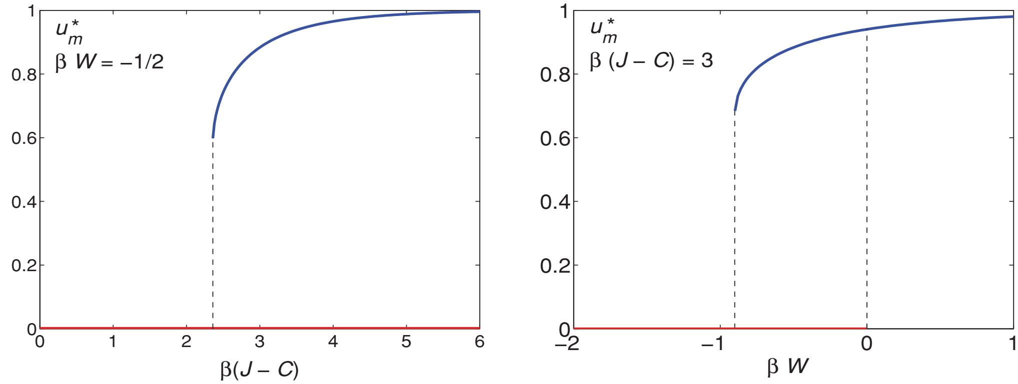

FIGURE 7. (Colour online) Fixed point solutions $\overline{u_m*}$ under a global social influence with a homogeneous distribution of W values, and the acting out function (2.2). (a) Solutions versus $\beta(J-C)$ keeping $\beta W = -0.5$ fixed. The solution $\overline{u_m*} = 0$ always exists. (b) Solutions versus βW keeping $\beta(J-C)$ fixed. The solution $\overline{u_m*} = 0$ only exists for $W < 0$.

the reduced parameter $\widetilde{J} \equiv \beta(J-C)$ at a given value of $\widetilde{W} \equiv \beta W < 0$, and as function of βW at a given value of $\widetilde{J} > \widetilde{J}_c$. In the first case, at small $\widetilde{J}$ there only exists the solution $\overline{u_m*} = 0$. At a critical value which depends on βW, $\widetilde{J}_c(\beta W)$, one enters the domain of multiple solutions: a solution with a high level of criminality appears abruptly. If the dynamics starts at a value $\widetilde{J} < \widetilde{J}_c(\beta W)$, the systems converge to the unique fixed point $\overline{u_m*} = 0$. If $\widetilde{J}$ is increased, the dynamical system will stay at this fixed point which is always stable. Only a strong external perturbation may make the system jump to the high criminal state in the range $\widetilde{J} > \widetilde{J}_c(\beta W)$. Once in this state, the system will remain on the upper branch of solutions if $\widetilde{J}$ is decreased – which will be the case if the cost C is increased –, until the critical point where this solution disappears. A that point the system jumps back to the quiet state with no criminal activity. Then the cost C can be decreased (hence $\widetilde{J}$ increases again), the system remaining in this quiet state.

For this particular simple model, the upper branch $\overline{u_m*} > 0$ exists for all $\widetilde{J} > \widetilde{J}_c(\widetilde{W})$. This needs not be the case in general (see e.g. Gordon *et al.* 2009b), that is one expects the upper boundary of the domain of multiple solutions in the plane $(J, \overline{W})$ to be generically given by a critical value of $\overline{W}$ which is a decreasing function of J. In the present simple case of a homogeneous W, the situation at a given value of $\overline{W}$ will look more similar to the behaviour of the solutions at a constant $\widetilde{J} > \widetilde{J}_c$ when one increases $\overline{W}$ from a low negative value to a positive value, as illustrated on the right panel of Figure 7. That is, there will be an upper critical value of J for which the quiet state, $\overline{u_m*} = 0$, disappears, leaving as a unique solution a fixed point with $\overline{u_m*} > 0$.

3.6 Repeated victimisation

Let us now briefly discuss some basic effects due to repeated victimisation as introduced Section 2.4.1. We restrict the analysis to the simplest case, that is, with time independent values of W_0, C and ρ, and without social interactions. We can thus consider a single location dynamics, as in Section 3.1.1. Then, the system of equations (2.25) yields here the system:

$$\frac{dS(t)}{dt} = -S(t) + W_0 + \rho u_m(t) - C u_m(t), \tag{3.51}$$

$$\tau_u \frac{du_m(t)}{dt} = \Lambda(S(t)) - u_m(t), \tag{3.52}$$

which we want to study in terms of the parameters W_0, C and ρ. The fixed points are given by

$$S^* = W_0 + (\rho - C)u_m^*,$$
$$u_m^* = \Lambda(S^*). \tag{3.53}$$

Obviously, the dynamics are the same as for the simplest case discussed in Section 3.1.1, with C replaced by $C - \rho$. If the later quantity is positive, all the results obtained as function of the cost (positive or null) apply. A location where $C - \rho < W_0 < C$ will be a hot spot (u_m^* close to 1) instead of being a warm spot in the absence of repeated victimisation.

If $C - \rho < 0$, one has a behaviour similar to the one discussed in the preceding Section 3.5, for a global social interaction and a homogeneous idiosyncratic willingness to act. Indeed, the system (3.53) gives the same equation as (3.50) for u_m^* with the parameter J replaced here by ρ. The analysis carried for (3.50) yields that, for $W_0 < 0$ and ρ larger than some critical value $\rho_c(W_0)$, there are two stable solutions, one with $u_m^* = 0$ and one with $u_m^* > 0$, and the associated hysteresis phenomenon (see Section 3.5 and the Appendix).

Let us describe it here. If, to start with, the system is in this high crime state, it may be driven back to the quiet state $u_m^* = 0$ by increasing C (or with deterrence measures leading to a decrease in ρ), until the value $\rho_c(W_0)$ is reached, at which point the solution with $u_m^* > 0$ does not exist any more. Once this is done, the cost can be decreased back to its original value with a crime rate remaining at the fixed point $u_m^* = 0$.

To the contrary of the case of social influence, the multiplicity of solutions here is the result of a purely local mechanism, which consists in the feed-back of the criminal activity onto itself. Indeed, for ρ large enough, it becomes a self-reinforcement process.

4 Conclusion

The main goal of this paper was to introduce a set of reaction–diffusion equations modelling the spatial and time distribution of uncivil or illegal activities under the assumption that a ‘mean field’ type, or ‘coarse grained’, approach is justified. The model is related to, but different from, both economics and behavioural modelling approaches to criminality – such as the ones of Becker (1968), Bourguignon *et al.* (2003*a*) or Gordon *et al.* (2009a) – and the reaction–diffusion modelling approach of Short *et al.* (2008, 2010). The basic assumptions are:

- Criminal activity is determined from an instantaneous *propensity to act* through an *acting out function* which exhibits a sharp transition or *threshold* mechanism between acting and non-acting;

- Locations are characterised by an *idiosyncratic willingness to act* which introduces inhomogeneity in the model;
- To represent *deterrence*, a cost is introduced which represents the law enforcement presence and efficiency, as well as severity of punishment;
- Social interaction is taken into account by diffusion (both local and non-local) as well as through a global term;
- A further aspect of deterrence is introduced through local social interaction going in the opposite direction to the local willingness to act;
- Deterrence and idiosyncratic willingness to act have their own dynamics. In particular, locations that have high levels of criminality in the past are described by an idiosyncratic willingness to act which becomes higher and therefore induces more criminal activity (repeat offences).

This model is discussed in this paper and some particular cases are precisely investigated. The main conclusions that we can derive from the analysis of these cases are as follows.

There are always stable equilibria, but according to the parameters, multiple stable equilibria can arise as well. This gives rise to hysteresis-type phenomena.

An important result coming from the analysis of the simplest version of the model is the existence of a self-organised critical state, where a positive but not large illegal activity is sustained with the instantaneous propensity to act maintained at a value close to zero (the critical value below which there is no illegal activity). This may correspond to two different situations. One is the case of a neighbourhood where the values of the idiosyncratic propensities to act are large, but where the cost imposed by law enforcement and deterrence is high enough in order to keep the level of criminal activity under control. What would be otherwise a hot spot is limited to what we call a *warm spot*. The other situation corresponds to a general mode, pervasive through most locations and gives rise to what we call a *tepid milieu*. We believe that this is, in particular, a relevant model for uncivil behaviour, in which case it is reasonable to assume that the idiosyncratic propensity to act is positive and small almost everywhere (with possibly quiet spots of negative values and hot spots of high values).

We have also discussed optimal policy issues under the constraint of limited resources in law enforcement and deterrence. We have shown that a simple adaptive scheme allows to enforce the optimal policy, limiting as much as possible the average criminal activity, with most if not all hot spots being reduced to warm spots.

Finally, the consequences of repeated victimisation have been only explored here in the simplest setting. Some warm spots may, unsurprisingly, become hot spots when repeated victimisation is taken into account. In such case, within our framework the only way to reduce such a hot spot to a warm spot is to increase the cost or take deterrent measures leading to a decrease of the strength of repeated victimisation. Another consequence is the appearance of multiple stable equilibria: locations where one would not expect to see a high criminal activity may become hot spots due to the self-reinforcement mechanism. These hot spots can be reduced through a temporary increase of the cost, taking advantage of the hysteresis phenomenon.

Many aspects have not been analysed in details in the present paper, and we propose our model for further mathematical investigation. As of matter of fact, it opens the way to many new problems. Of particular interest will be the study of the interplay between the repeated victimisation effect (increase of the idiosyncratic propensity to act at locations subject to recent illegal activity), and of social interaction (diffusion of the propensity to act). It remains to see if diffusion of hot spots, or oscillating behaviour may emerge. It would be interesting to see whether this model produces, as in Short *et al.* (2010), the possibility that owing to the action of police on one hot spot, it might be displaced to be reformed elsewhere. In fact, here we could be more precise and ask if two locations could swap their natures, oscillating between warm and hot states as triggered by crackdowns by the police. Indeed, as a result of adapting deterrence, oscillating behaviour might emerge. This would shed a new light on the nature of reformation of hot spots and could account for why this phenomena might be underestimated in the statistics. Actually, we conjecture that often, rather than being displaced and forming entirely new hot spots, the displacement might just enhance already existing warm spots.

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Appendix A Global social influence: Fixed point solutions

The goal of this appendix is to briefly discuss the existence and multiplicity of solutions of the fixed point equations in the case of a global social influence (see Section 3.5). This will be done in terms of the following parameters: the slope β, the cost C, the parameters characterising the distribution of the idiosyncratic willingness to act and the strength J of the global social interaction.

As in the rest of this paper, we are interested in the nature of the solutions for a family of acting out functions indexed by the slope β at $S = 0^+$, with $\Lambda(S) = 0$ for $S < 0$ and Λ a smooth (say C^1) strictly increasing function for $S > 0$. For what concerns $P(W)$, we consider a family of distributions parametrised by the mean value $\overline{W}$ and the variance σ_W^2. One can see from the equations that there is an arbitrary scale: if $\sigma_W \neq 0$, by a rescaling of β, C and J, without loss of generality one can assume $\sigma_W = 1$. If $\sigma_W = 0$, that is, if W takes the same value $\overline{W}$ everywhere, the set of free parameters reduces to $\{\beta(J-C), \beta\overline{W}\}$.

Let us first discuss the simplest case of a homogeneous value of W, hence $P(W) = \delta(W-\overline{W})$. Then at the fixed point of the dynamics all quantities are homogeneous, and the fixed point equation for the average crime activity (which is also the activity at any location) is simply given by

$$\overline{u} = \Lambda(\overline{W} + (J-C)\overline{u}). \tag{A 1}$$

If one assumes the acting out function to be concave on $\mathbb{R}^+$, with $\beta = \Lambda'(S=0^+) = \max_{S\geqslant 0} \Lambda'(S)$, one gets (i) a unique solution $\overline{u} > 0$ for $\overline{W} > 0$; (ii) for $\overline{W} = 0$, a unique solution $\overline{u} = 0$ for $J - C < 1/\beta$, and two solutions for $J > J_c = C + 1/\beta$; and (iii) for $\overline{W} < 0$, there exists a line $J_c(\overline{W}) \geqslant J_c = J_c(0)$ such that for $J < J_c(\overline{W})$ there is a unique solution $\overline{u} = 0$, and for $J > J_c(\overline{W})$ there exist three solutions (0 which is always stable, an unstable solution and another stable solution which is positive). In the parameter space $(J, \overline{W})$, the domain with multiple solutions, which lies in the half-plane $J > J_c$, is delimited above by the half-line $\{J \geqslant J_c, \overline{W} = 0\}$, and below by the curve $J = J_c(\overline{W})$. For the particular acting out function (2.2), the critical line $J = J_c(\overline{W})$ is given by

$$-\beta\overline{W} + 1 = \beta(J-C) - \ln\beta(J-C). \tag{A 2}$$

Figure 6 shows the domain of multiple solution in the parameter space, which here is $\{\beta(J-C), \beta\overline{W}\}$.

In the general case, that is, for a non-homogeneous distribution of W, as we have seen if we can compute $\overline{u_m^*} = \overline{u^*}$ at the fixed point, we have the solution for any x. The average illegal activity $\overline{u_m^*}$ is obtained as solution of the fixed point equation (3.49), which we rewrite here, in a slightly different form, as an integral over the distribution of W values, $P(W) = \int_\Omega \delta(W - W(x))p(x)dx$ (Figure 3 shows the pdf $P(W)$ for the spatial pattern $W(x)$ of Figure 4):

$$\overline{u_m^*} = \int \Lambda_C(W + J\overline{u_m^*})\, P(W)\, dW. \tag{A 3}$$

The analysis of this fixed point equation (A 3) can be done as in Gordon *et al.* (2009b). One then expects the nature of the solutions to depend in particular on the smoothness of the distribution $P(W)$ and on the number of its maxima. It is convenient to write $W = \overline{W} + w$ and $P(W)\,dW = P_0(w)\,dw$ where P_0 has zero mean and unit variance.

To start with, recall the definition of Λ_C: it is defined by $\Lambda_C(S) = 0$ for $S \leqslant 0$, and for $S > 0$ as the inverse of $(\Lambda^{-1} + C\,Id)$ (with here Λ^{-1} the inverse of the function Λ restricted to $\mathbb{R}^+$). Now let

$$\widetilde{\Lambda}_C(s) \equiv \int \Lambda_C(w+s) P_0(w)\, dw \tag{A 4}$$

and let $\widetilde{\Phi}_C$ be the inverse of $\widetilde{\Lambda}_C$:

$$u = \widetilde{\Lambda}_C(s) \Leftrightarrow s = \widetilde{\Phi}_C(u), \tag{A 5}$$

where $\widetilde{\Phi}_C(u)$ is a monotonous, increasing, function of u, with $\lim_{u\to 1} \widetilde{\Phi}_C(u) = +\infty$, and, if $P_0(w)$ has a non-bounded support, $\lim_{u\to 0} \widetilde{\Phi}_C(u) = -\infty$, or, if the support of $P_0(w)$ is $[w_{min}, w_{max}]$, $\lim_{u\to 0} \widetilde{\Phi}_C(u) = -w_{max}$.

The fixed point equation reads (from now on we drop the superscript $*$)

$$\overline{u} = \widetilde{\Lambda}_C(\overline{W} + J\overline{u}), \tag{A 6}$$

or equivalently,

$$\overline{W} = \widetilde{\Phi}_C(\overline{u}) - J\overline{u} \equiv F_C(J;\overline{u}). \tag{A 7}$$

Although we do not discuss here the dynamical stability, in view of the results in Gordon *et al.* (2009b) one can conjecture that the fixed point is stable if it satisfies

$$\frac{\partial}{\partial \overline{u}} F_C(J;\overline{u}) > 0 \tag{A 8}$$

with

$$\frac{\partial}{\partial \overline{u}} F_C(J;\overline{u}) = \widetilde{\Phi}'_C(\overline{u}) - J, \tag{A 9}$$

where $\Phi'_C(u)$ is the derivative of the function $\Phi_C(u)$ with respect to its argument u.

If we define the J_c by

$$J_c = \min_u \widetilde{\Phi}'_C(u), \tag{A 10}$$

it is clear that, if $J_c > 0$, for $0 \leqslant J < J_c$ any solution is stable. Actually, one can see that for $J < J_c$ there is always a unique (and stable) solution.

For $J > J_c$ there exists several solutions – in the simplest cases, two stable ones separated by one unstable solution. The value J_c of the social influence strength is thus a critical value in parameter space at which multiple equilibria appear. One can check that, in the case of a homogeneous distribution of W and a concave acting out function, (A 10) gives the correct critical value, $J_c = C + 1/\beta$.

For $J > J_c$, in the parameter space $\{J, \overline{W}\}$ at fixed C, the boundary of the domain with multiple solutions is given by the marginal stability condition:

$$\frac{\partial}{\partial \overline{u}} F_C(J;\overline{u}) = 0. \tag{A 11}$$

This implies that this boundary is given by the graph of the Legendre transform $\mathscr{L}\widetilde{\Phi}_C$ of the function $\widetilde{\Phi}_C$, that is by the equation

$$\mathscr{L}\widetilde{\Phi}_C(J) = \overline{W}. \tag{A 12}$$

Depending on the probability distribution of the idiosyncratic willingness to act, this graph in the $(J, \overline{W})$ plane has two or more branches, with two extreme branches (a concave one at low $\overline{W}$ values, and a convex one at large $\overline{W}$ values) which merge at $J = J_c$ with a common tangent as $J \to J_c$ from above (see Gordon *et al.* 2009b for details given for a similar model). This generalises the results obtained for the homogeneous distribution of W values, illustrated on Figure 6.

References

Bak, P. (1997) *How Nature Works: The Science of Self Organised Criticality*, Oxford University Press.

Becker, G. (1968) Crime and punishment: An economic approach. *J. Polit. Econ.* **76**, 169–217.

Berk, R. (2008) How you can tell if the simulations in computational criminology are any good. *J. Exp. Criminol.* **4**(3), 289–308.

Bernasco, W. (2009) Modeling micro-level crime location choice: Application of the discrete choice framework to crime at places. *J. Quant. Criminol.* **26**(1), 113–138.

Bourguignon, F., Nunez, J. & Sanchez, F. (2003*a*) A structural model of crime and inequality in Colombia. *J. Eur Econ. Assoc.* **12**(2–3), 440–449.

Bourguignon, F., Nunez, J. & Sanchez, F. (2003*b*) What part of the income distribution does matter for explaining crime? The case of Colombia. *J. Eur. Econ. Assoc.* **1**(2–3), 440–449.

Brunel, N. & van Rossum, M. C. W. (2007) Lapicques 1907 paper: From frogs to integrate-and-fire. *Bio. Cybern.* **97**(5–6), 337–339.

Calvó-Armengol, A. & Zenou, Y. (2004) Social networks and crime decisions: The role of social structure in facilitating delinquent behavior. *Int. Econ. Rev.* **45**, 939–958.

Campbell, M. & Ormerod, P. (1998) Social interactions and the dynamics of crime. Volterra Consulting Preprint. URL: http://www.paulormerod.com/pdf/CRIME42.pdf

Chow, C., Gutkin, B., Hansel, D., Meunier, C. & Dalibard, J. (editors) (2004) *Methods and Models in Neurophysics. Session LXXX*, Lecture Notes of the Les Houches Summer School 2003, Elsevier.

Clarke, R. V. & Felson, M. (editors) (1993) *Routine Activity and Rational Choice. Advances in Criminological Theory*, Vol. 5, Transaction Books, New Brunswick, NJ.

Cohen, L. E. & Felson, M. (1979) Social change and crime rate trends: A routine activity approach. *Am. Sociol. Rev.* **44**, 588–608.

Crane, J. (1991) The epidemic theory of ghettos and neighborhood effects on dropping out and teenage childbearing. *Am. J. Sociol.* **96**(5), 1226–1259.

Eck, J. & Liu, L. (2008) Contrasting simulated and empirical experiments in crime. *J. Exp. Criminol.* **4**, 195–213.

Glaeser, E. L., Sacerdote, B. & Scheinkman, J. A. (1996) Crime and social interactions. *Q. J. Econ.* **111**, 507–548.

Gordon, M. B. (2010) A random walk in the literature on criminality: A partial and critical view on some statistical analyses and modeling approaches. *Eur. J. Appl. Math.*, doi:10.1017/S0956792510000069 (in press).

Gordon, M. B., Iglesias, J. R., Semeshenko, V. & Nadal, J.-P. (2009a) Crime and punishment: The economic burden of impunity. *Eur. Phys. J.* B – *Condens. Matter Complex Syst.* **68**(1), 133–144.

Gordon, M. B., Nadal, J.-P., Phan, D. & Semeshenko, V. (2009b) Discrete choices under social influence: Generic properties. *Math. Models Methods Appl. Sci. (M3AS)* **19**(Supplementary Issue 1), 1441–1481.

Groff, L. (2007) Simulation for theory testing and experimentation: An example using routine activity theory and street robbery. *J. Quant. Criminol.* **23**(2), 75–103.

Johnson, S. D., Summers, L. & Pease, K. (2009) Offender as forager? A direct test of the boost account of victimization. *J. Quant. Criminol.* **25**(2), 181–200.

Nadal, J.-P., Gordon, M. B., Semeshenko, V. & Iglesias, J. R. (2010) Modelling the individual and collective dynamics of the propensity to offend. *Eur. J. Appl. Math.*, doi:10.1017/S0956792510000173 (in press).

Nadal, J.-P., Phan, D., Gordon, M. B. & Vannimenus, J. (2006) Multiple equilibria in a monopoly market with heterogeneous agents and externalities. *Quant. Finance* **5**(6), 557–568.

Nagin, D. & Paternoster, R. (1993) Enduring individual differences and rational choice theories of crime. *Law Soc. Rev.* **467**.

Nuño, J. C., Herrero, M. A. & Primicerio, M. (2008) A triangle model of criminality. *Physic. A: Stat. Mech. Appl.* **387**(12), 2926–2936.

Ormerod, P. (2005) *Crime: Economic Incentives and Social Networks*, Insitute of Economic Affairs, London.

Pease, K. (1998) Repeat victimization: Taking stock. Crime Detection and Prevention Series Paper 90. The Home Office: Police Research Group.

Pitcher, A. B. (2010) Adding police to a mathematical model of burglary. *Eur. J. Appl. Math.*, doi:10.1017/S0956792510000112 (in press).

Sethna, J. P. (2009) *Statistical Mechanics. Entropy, Order Parameters and Complexity*, Oxford University Press.

Short, M. B., Brantingham, P. J., Bertozzi, A. L. & Tita, G. E. (2010) Dissipation and displacement of hotspots in reaction-diffusion models of crime. *Proc. Natl. Acad. Sci. (PNAS)* **107**(9), 3961–3965.

Short, M., D'Orsogna, M., Pasour, V., Tita, G., Brantingham, P., Bertozzi, A. & Chayes, L. (2008) A statistical model of criminal behavior. *Math. Models Methods Appl. Sci. (M3AS)* **18** (Suppl.), 1249–1267.