Multiple equilibria in a monopoly market with heterogeneous agents and externalities

Jean-Pierre Nadal (1), Denis Phan (2,3)
Mirta B. Gordon(3) and Jean Vannimenus (1)

(1) Laboratoire de Physique Statistique, Ecole Normale Supérieure,
24 rue Lhomond, 75231 Paris cedex 05, France
(nadal@lps.ens.fr, http//www.lps.ens.fr/~nadal)
(2) ENST Bretagne, Economic Department, and ICI (Information, Coordination, Incitation), Université de Bretagne Occidentale, Faculté de Droit, Sciences Economiques et Gestion, Brest, France
(denis.phan@enst-bretagne.fr, http//www-eco.enst-bretagne.fr/~phan)
(3) Laboratoire Leibniz-IMAG,
46, Ave. Félix Viallet, 38031 Grenoble Cedex 1, France
(mirta.gordon@imag.fr, web: http//www-leibniz.imag.fr/Apprentissage)

Short title: Multiple equilibria in a monopoly market

IOP PACS/Classification: 89.65.Gh - A-B MOD - BOUND RAT - MARK D&P

Abstract

We explore the effects of social influence in a simple market model in which a large number of agents face a binary choice: to buy/not to buy a single unit of a product at a price posted by a single seller (monopoly market). We consider the case of *positive externalities*: an agent is more willing to buy if other agents make the same decision. We consider two special cases of heterogeneity in the individuals' decision rules, corresponding in the literature to the Random Utility Models of Thurstone, and of McFadden and Manski. In the first one the heterogeneity fluctuates with time, leading to a standard model in Physics: the Ising model at finite temperature (known as annealed disorder) in a uniform external field. In the second approach the heterogeneity among agents is fixed; in Physics this is a particular case of quenched disorder models known as random field Ising model, at zero temperature. We study analytically the equilibrium properties of the market in the limiting case where each agent is influenced by all the others (the mean field limit), and we illustrate some dynamic properties of these models making use of numerical simulations in an Agent based Computational Economics approach.

Considering the optimisation of the profit by the seller within the case of fixed heterogeneity with global externality, we exhibit a new regime where, if the mean willingness to pay increases and/or the production costs decrease, the seller's optimal strategy jumps from a solution with a high price and a small number of buyers, to another one with a low price and a large number of buyers. This regime, usually modelled with ad-hoc bimodal distributions of the idiosyncratic heterogeneity, arises here for general monomodal distributions if the social influence is strong enough.

1 Introduction

Following Kirman [9, 10], we view a market as a complex interactive system with a communication network. We consider a market with discrete choices [1], and explore the effects of localised externalities (social influence) on its properties. We focus on a simple case: a single homogeneous product sold by a single seller (monopoly), and a large number of heterogeneous customers. These customers are assumed to be myopic. The only cognitive agent in the model is the monopolist, who determines the price in order to optimise his profit.

Agents are assumed to have idiosyncratic willingness-to-pay (IWP) described by means of random variables, as suggested by Thurstone [20] and McFadden [12], leading to the so-called *Random Utility Models* [13] (hereafter referred to as RUM). We first compare and contrast two models: on one hand, the IWP are randomly chosen and remain fixed, on the other hand, the IWP present independent temporal fluctuations around a fixed (homogeneous) value. In the present paper we show that they correspond to two different statistical physics models, in the former case with *quenched disorder* (QRUM), and in the latter - the Thurstone case - with *annealed disorder* (TRUM). In both cases we assume that the heterogeneous preferences of the agents are drawn from the same (logistic) distribution. More generally, these models may be considered as the extreme limits of a single model where the IWPs are time dependent variables. If the time scale of the changes in the IWP is slow enough to allow myopic agents to reach the equilibrium upon repeated choices before the IWPs change, we are in the quenched disorder case (QRUM), whereas if the IWPs vary at the same pace as the individual decisions, we are in the annealed limit (TRUM). The equilibrium states of the two models generally differ, except in the special case of homogeneous interactions with complete connectivity. In this special situation, which corresponds to a *mean-field model* in physics, the expected aggregate steady-state is the same in both models.

Considering the optimisation of the profit by the seller in the *quenched disorder* case (QRUM), we exhibit a new regime where, if the mean willingness to pay increases and/or the production costs decrease, the seller's optimal strategy jumps from a solution with a high price and a small number of buyers, to another one with a low price and a large number of buyers. To our knowledge, this transition in the monopolist's startegy, which has the characteristics of what is called a *first order phase transition* in Physics, is usually modelled by assuming an ad-hoc bimodal heterogeneity in the agents'IWPs. We find that it exists even with a monomodal distribution, if the social influence is strong enough. We find also that it occurs not only in the domain of parameters where the demand itself, at a given price, presents two solutions, but also and more surprisingly, in a domain where the demand is uniquely defined.

In the following, we first present the demand side, and then consider the optimisation problem left to the monopolist, who is assumed to know the demand model and the distribution of the IWP over the population, but cannot observe the individual (private) values.

2 Simple models of discrete choice with social influence

We consider a set Ω_N of N agents with a classical linear IWP function. Each agent $i \in \Omega_N$ either buys ($\omega_i = 1$) or does not buy ($\omega_i = 0$) one unit of a homogeneous good sold by a unique seller. A rational agent chooses ω_i in order to maximise his *surplus function* V_i:

$$\max_{\omega_i \in \{0,1\}} V_i = \max_{\omega_i \in \{0,1\}} \omega_i (H_i + \sum_{k \in \vartheta_i} J_{ik} \omega_k - P), \tag{1}$$

where P is the price of one unit and H_i represents the idiosyncratic preference component: in the absence of social influence, H_i is the reservation price of agent i, i.e., the maximum price he is willing to pay for the good. In addition, in (1) we assume that the preferences of each agent i are influenced by the choices of a subset $\vartheta_i \subseteq \Omega_N$ of other agents, hereafter called neighbours of i. This social influence is represented by a weighted sum of these choices, ω_k with $k \in \vartheta_i$. The corresponding weight, denoted by J_{ik}, is the marginal social influence of the decision of agent $k \in \vartheta_i$ on agent i. When this social influence is assumed to be positive ($J_{ik} \geq 0$), its effect may be identified, following Durlauf [4], as a *strategic complementarity* in agents' choices [3].

For simplicity we consider here only the case of *homogeneous* influences, that is identical neighbourhood structures ϑ of cardinal n for all the agents, and identical positive weights, that we write $J_{ik} = J/n$. That is,

$$\forall i \in \Omega_N : \quad |\vartheta_i| = n, \text{ and: } \forall k \in \vartheta_i \quad J_{ik} = J/n > 0. \tag{2}$$

Hence for a given distribution of choices in the neighbourhood ϑ_i, and for a given price, the customer's behaviour is deterministic: an agent buys if

$$H_i > P - \frac{J}{n} \sum_{k \in \vartheta_i} \omega_k. \tag{3}$$

2.1 Psychological *versus* economic points of view

At the basis of *Ramdom Utility Models* (*RUM*) [11, 13], the discrete choice model (1) may represent two quite different situations, depending on the nature of the idiosyncratic term H_i. Following the typology proposed by Anderson *et al.* [1], we distinguish a "psychological" and an "economic" approach to individual choices. Within the psychological perspective of Thurstone [20], the utility has a *stochastic* aspect because "there are some qualitative fluctuations from one occasion to the next... for a given stimulus". Hereafter we refer to this model as the *Thurstone Random Utility Model* (*TRUM*). On the contrary, for McFadden [12] and Manski [11], each agent has a willingness to pay that is *invariable* in time, but may differ from one agent to the other. We call this perspective the *QRUM* case, where 'Q' stands for *Quenched* for reason explicited in the next section. Accordingly, the TRUM and the QRUM perspectives differ in the nature of the individual willingness to pay.

We assume that the seller is an external observer in a *risky* situation: he cannot observe the individual values of the IWPs. He considers that the heterogeneous set of N IWPs are a sample of N i.i.d. random variables, and we assume that he knows its statistical distribution over the population.

In the TRUM, the idiosyncratic preferences $H_i(t)$ are time-dependent i.i.d. random variables. The agents are identical in that the $H_i(t)$ are drawn at each time t

from a same probability distribution, which we characterise by its mean H and the cumulative distribution $F(z)$ of the deviations from the mean,

$$F(z) \equiv \mathcal{P}(H_i - H \leq z). \tag{4}$$

In the case of a logistic distribution with mean H, and variance $\sigma^2 = \pi^2/(3\beta^2)$ (where H and β are thus constant, independent of both i and t), the cumulative distribution $F(z)$ is:

$$F(z) = \frac{1}{1 + \exp(-\beta\, z)}. \tag{5}$$

In the QRUM, agents are heterogeneous: the private idiosyncratic terms H_i are randomly distributed over the agents, but remain fixed during the period under consideration. If we assume that the H_i are logistically distributed with mean H and variance $\sigma^2 = \pi^2/(3\beta^2)$, then at a given instant of time t, the empirical distribution of the IWP in the population is the same in the TRUM and the QRUM approaches: in the large N limit they are given by the same logistic distribution, with the same mean and same variance.

2.2 Static *versus* dynamic points of view

If the agents make repeated choices, in the standard TRUM the H_i are newly drawn from the logistic distribution, whereas in the QRUM the H_i remain the same.

The analysis of the dynamical evolution of the models in statistical physics, corresponds to the case of repeated choices and myopic agents. These apply at each time step the decision rule (3), based on the observations of the choices at the previous time. Under these hypothesis, an equilibrium or steady state of the overall system is reached. It may differ in both models, as shortly explained below.

2.3 "Annealed" *versus* "quenched" disorder

Since we have assumed isotropic (hence symmetric) interactions, there is a strong relation between these models and Ising type models in Statistical Mechanics, which is made explicit if we change the variables $\omega_i \in \{0, 1\}$ into *spin* variables $s_i \in \{\pm 1\}$ through $s_i = 2\omega_i - 1$. All the expressions in the present paper can be put in terms of either s_i or ω_i using this transformation. In the following we keep the encoding $\omega_i \in \{0, 1\}$. The economics assumption of *strategic complementarity* corresponds to having ferromagnetic couplings in physics (that is, the interaction J between Ising spins is positive). These couplings act as a bias favoring states where the spins s_i are aligned with each other, that is, they tend to take all the same value.

The TRUM corresponds to a case of *annealed disorder*. Having a time varying random idosyncratic component is equivalent to introducing a stochastic dynamics for the Ising spins. In the particular case where $F(z)$ is the logistic distribution, we obtain an Ising model in a uniform (non random) external field $H - P$, at temperature $T = 1/\beta$. Although the individual choices change at each time setp due to the randomness in the $H_i(t)$, the aggregate fraction of consumers in the long run fluctuates gently around a well defined stationary value. The QRUM has fixed heterogeneity; it is analogous to a *Random Field Ising Model* (RFIM) at zero temperature, that is, with deterministic dynamics. The RFIM belongs to the class of *quenched disorder* models: the values H_i are equivalent to random time-independent local fields. The "agreement" among agents favored by the ferromagnetic couplings may be broken by the influence of these heterogeneous external fields H_i. Due to the random distribution of H_i over the network of agents, the resulting organisation may be complex. Thus, from the physicist's point of view, the TRUM and the QRUM are quite different models: uniform field and finite temperature in the former, random field and zero temperature in the latter.

The properties of disordered systems have been and still are the subject of numerous studies in statistical physics. They show that annealed and quenched disorder can lead to very different behaviours. The standard Ising model (the TRUM case) is well understood. In the case where the agents are situated on the vertices of a 2-dimensional square lattice, and have four neighbours each, there is an exact analysis of the stationary states of the model for $P = P_n$ (the neutral case, see below equation (35)) due to Onsager [14]. Even if an analytical solution of the optimization problem (1) for an arbitrary neighbourhood does not exist, the *mean field* analysis gives approximate results that become exact in the limiting situation where every agent is a neighbour of every other agent (*i.e.* all the agents are interconnected through weights (2)). On the contrary, the properties of the RFIM (the QRUM case with externalities) are not yet fully understood. Since the first studies of the RFIM, which date back to Aharony and Galam [6, 5], a number of important results have been published in the physics literature (see e. g. [18]). Several variants of the RFIM have already been used in the context of socio-economic modelling [7, 15, 2, 21].

2.4 *Mean-field* version of the Random Utility Model with externalities

Hereafter, we restrict our investigation to the QRUM in the case of a global externality. That is, we consider *homogeneous interactions* and *full connectivity*, which is equivalent to considering the *mean field* version of the RFIM in physics. Within this general framework, we are interested in two different perspectives. First we consider a static point of view: by looking for the equality between demand and supply we determine the set of possible economic equilibria (the Nash equilibria). This will allow us to analyse in section 4 the optimal strategy of the monopolist, as a function of the model parameters. Then (section 5) we consider the market's dynamics assuming myopic agents: based on the observation of the behaviour of the other agents at time $t - 1$, each agent decides at time t to buy or not to buy (this corresponds to a myopic best-reply strategy). We show that, in general, the market converges towards the static equilibria, except for a precise range of the parameter values where interesting static as well as dynamic features are observed.

These two kinds of analysis correspond in Physics to the study of the thermal equilibrium properties within the *statistical ensemble* framework on the one hand, and the out of equilibrium dynamics (which, in most cases, approaches the static equilibrium through a relaxation process) on the other hand.

3 Aggregate demand

For the QRUM, it is convenient to decompose H_i as

$$H_i = H + \theta_i, \tag{6}$$

where H is the mean value of the H_i's over the population, and θ_i the idiosyncratic component which characterises how much the IWP of agent i deviates from the mean. We assume the θ_i to be i.i.d. random variables of zero mean and variance $\sigma^2 = \pi^2/(3\beta^2)$, with a logistic distribution so that their cumulative distribution $F(z) = \mathcal{P}(\theta_i \leq z)$ is given by (5).

With (6) we can then rewrite the decision rule (3) as

$$\theta_i > P - H - \frac{J}{n} \sum_{k \in \vartheta_i} \omega_k \tag{7}$$

As discussed in the preceding section, we consider the full connectivity case with isotropic interactions given by (2) with cardinal $n = N - 1$, in the limit of a very large number of agents. Under these conditions the social influence term of the agents' surplus function (the coefficient of J in the above equation (7)), equal to $\sum_{k \in \vartheta} \omega_k/(N-1)$, can be approximated by the *penetration rate* η, defined as the fraction of customers that choose to buy at a given price:

$$\eta \equiv \lim_{N \to \infty} \sum_{k=1}^{N} \omega_k/N. \tag{8}$$

The condition of buying, given by equation (7) may thus be replaced by

$$\omega_i = 1 \quad \text{iff} \quad \theta_i \;\; > \;\; z, \tag{9}$$

where z is defined by

$$z \equiv P - H - J\,\eta. \tag{10}$$

Equations (9) and (10) allow us to obtain η as a fixed point:

$$\eta = 1 - F(z) \tag{11}$$

where z depends on P, H, and η, and $F(z)$ is the cumulative distribution of the IWP around the average value H. Note that this (macroscopic) equation is formally equivalent to the (microscopic) individual expectation that $\omega_i = 1$ in the TRUM case. Using the logistic distribution for θ_i, we have:

$$\eta = \frac{1}{1 + \exp(+\beta\, z)} \tag{12}$$

Equation (11) allows us to define η as an implicit function of the price through

$$\Phi(\eta, P) \equiv \eta(P) + F(P - H - J\,\eta(P)) - 1 = 0. \tag{13}$$

Since for a given P, equation (12) defines the penetration rate η as a fixed-point, inversion of this equation gives an *inverse demand function*:

$$P^d(\eta) = H + J\,\eta + \frac{1}{\beta}\ln\frac{1-\eta}{\eta} \tag{14}$$

At given values of β, J and H, for most values of P, (12) has a unique solution $\eta(P)$. However for $\beta J > \beta J_B \equiv 4$, there is a range of prices

$$P_1(\beta J, \beta H) < P < P_2(\beta J, \beta H) \tag{15}$$

such that, for any P in this interval, (12) has two stable solutions and an unstable one. The limiting values P_1 and P_2 are the particular price values obtained from the condition that eq. (12) has one degenerate solution:

$$\eta = 1 - F(z), \text{ and } \frac{d(1-F(z))}{d\eta} = 1.$$

The second equation gives $\beta J\eta(1-\eta) = 1$, which has two solutions, $\eta_2 \le 1/2 \le \eta_1$,

$$\eta_i = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{\beta J}}\right] \;;\; i \in \{1, 2\} \tag{16}$$

Note that η_i depends only on βJ. Then, the limiting prices P_i are equal to the inverse demands associated with these values η_i, which are, from (14):

$$P_i = H + J\eta_i + \frac{1}{\beta}\ln[\frac{1-\eta_i}{\eta_i}] \;;\; i \in \{1, 2\}. \tag{17}$$

Note that these limiting prices are not necessarily positive.

It is interesting to note that the set of equilibria is the same as what would be obtained if agents had rational expectations about the choices of the others: if every agent had knowledge of the distribution of the H_i, he could compute the equilibrium state compatible with the maximisation of his own surplus, taking into account that every agent does the same, and make his decision (to buy/not to buy) accordingly. For $\beta J < \beta J_B$ every agent could thus anticipate the value of η to be realized at the price P, and make his choice according to (9). For $\beta J > \beta J_B$, however, if the price is set within the interval $[P_1, P_2]$, the agents are unable to anticipate which equilibrium will be realized, even though the one with the largest value of η should be preferred by every one (it is the Pareto dominant equilibrium). Remark that such a situation is similar to the one arising in stag hunt-type coordination games.

4 Supply side

On the supply side, we consider a monopolist facing heterogeneous customers in a risky situation where he has perfect knowledge of the functional form of the agents' surplus functions and their maximisation behaviour (1). He also knows the statistical (logistic) distribution of the idiosyncratic reservation prices (H_i), but cannot observe any *individual* reservation price. He observes only the aggregated individual choices (which are either to buy or not to buy). The social influence on each individual decision is then close to $J\ \eta$, and the fraction of customers η is observed by the monopolist. That is, for a given price, the expected number of buyers is given by equation (11).

Remark that, although we have restricted our analysis to the QRUM case, the probability for an agent taken at random by the monopolist to be a customer would be formally the same in the TRUM case, as already pointed out by McFadden [13].

4.1 Profit maximisation

Let C be the monopolist cost for each unit sold, so that

$$p \equiv P - C \tag{18}$$

is his profit *per unit*. Since $P - H = (P - C) - (H - C)$, defining

$$h \equiv H - C, \tag{19}$$

we can rewrite z in (10) as:

$$z = p\ -\ h\ -\ J\ \eta. \tag{20}$$

Hereafter we write all the equations in terms of p and h (hence we will also make use of $p^d(\eta) \equiv P^d(\eta) - C$).

Since each customer buys a single unit of the good, the monopolist's total expected profit is $p\ N\ \eta$, which is proportional to the total number of customers. He is left with the following maximisation problem:

$$p_M = \arg\max_p \Pi(p), \tag{21}$$

where $N\ \Pi(p)$ is the expected profit, with:

$$\Pi(p) \equiv p\ \eta(p), \tag{22}$$

and where $\eta(p)$ is the solution to the implicit equation (13). If there is no discontinuity in the demand curve $\eta(p)$ (hence for $\beta J \leq 4$), p_M satisfies $d\Pi(p)/dp = 0$, which gives $d\eta/dp = -\eta/p$ at $p = p_M$. Using the implicit derivative theorem, from (13) one has:

$$\frac{d\eta(P)}{dP} = \frac{-\partial\Phi/\partial P}{\partial\Phi/\partial\eta} = \frac{-f(z)}{1 - J\,f(z)} \tag{23}$$

where $f(z) = dF(z)/dz$ is the probability density. Thus we obtain at $p = p_M$:

$$\frac{f(z)}{1 - Jf(z)} = \frac{\eta}{p}, \tag{24}$$

where z, defined in (20), has to be taken at $p = p_M$.

Because the monopolist observes the demand level η, we can use equation (11) to replace $1 - F(z)$ by η. Making use of the explicit form of $F(z)$ (the logistic (5)), one has $f(z) = \beta F(z)(1 - F(z)) = \beta(1-\eta)\eta$. Then equation (24) can be written as $p = p^s(\eta)$ where the function $p^s(\eta)$ is defined by:

$$p^s(\eta) \equiv \frac{1}{\beta\,(1-\eta)} - J\,\eta \tag{25}$$

A more general and formal analysis shows that, for an arbitrary function F, p^s is given by:

$$p^s(\eta) \equiv -\eta \frac{dp^d(\eta)}{d\eta}, \tag{26}$$

and the interesting structure of the optimisation program gives to this function $p^s(\eta)$ the role of an effective (inverse) supply function [22]. Since it is not a true supply function, but results from the monopolist's optimisation program based on the knowledge of the demand function, we will refer to $p^s(\eta)$ as the *implied* inverse supply function. We thus obtain p_M and η_M as the intersection between demand (14) and (implied) supply (25):

$$p_M = p^d(\eta_M) = p^s(\eta_M), \tag{27}$$

where $p^d = P^d - C$.

The (possibly local) maxima of the profit are the solutions of (27) for which

$$\frac{d^2\Pi}{dp^2} < 0. \tag{28}$$

It is straightforward to get the expression for the second derivative of the profit:

$$\frac{d^2\Pi}{dp^2} = -2\frac{\eta}{p}\left[1 + \frac{2\eta - 1}{2\beta p(1-\eta)^2}\right], \tag{29}$$

from which it is clear that the solutions with $\eta > 1/2$ are local maxima. For $\eta < 1/2$, condition (28) reads

$$\frac{1-2\eta}{2\beta p(1-\eta)^2} < 1. \tag{30}$$

Making use of the above equations, this can also be rewritten as

$$2\beta J\eta(1-\eta)^2 < 1. \tag{31}$$

For $\beta J > \beta J_B = 4$, the monopolist has to find $p = p_M$ which realises the programme:

$$p_M : \max\{\Pi_-(p_-^M), \Pi_+(p_+^M)\} \tag{32}$$

$$p_+^M = arg\max_p \Pi_+(p) \equiv p\,\eta_+(p), \tag{33}$$

$$p_-^M = arg\max_p \Pi_-(p) \equiv p\,\eta_-(p) \tag{34}$$

where the subscripts $+$ and $-$ refer to the solutions of (12) with a fraction of buyers larger, respectively smaller, than 1/2.

To illustrate the behaviour of these equations, figure 1 represents several examples of inverse supply and demand curves corresponding to different market configurations.

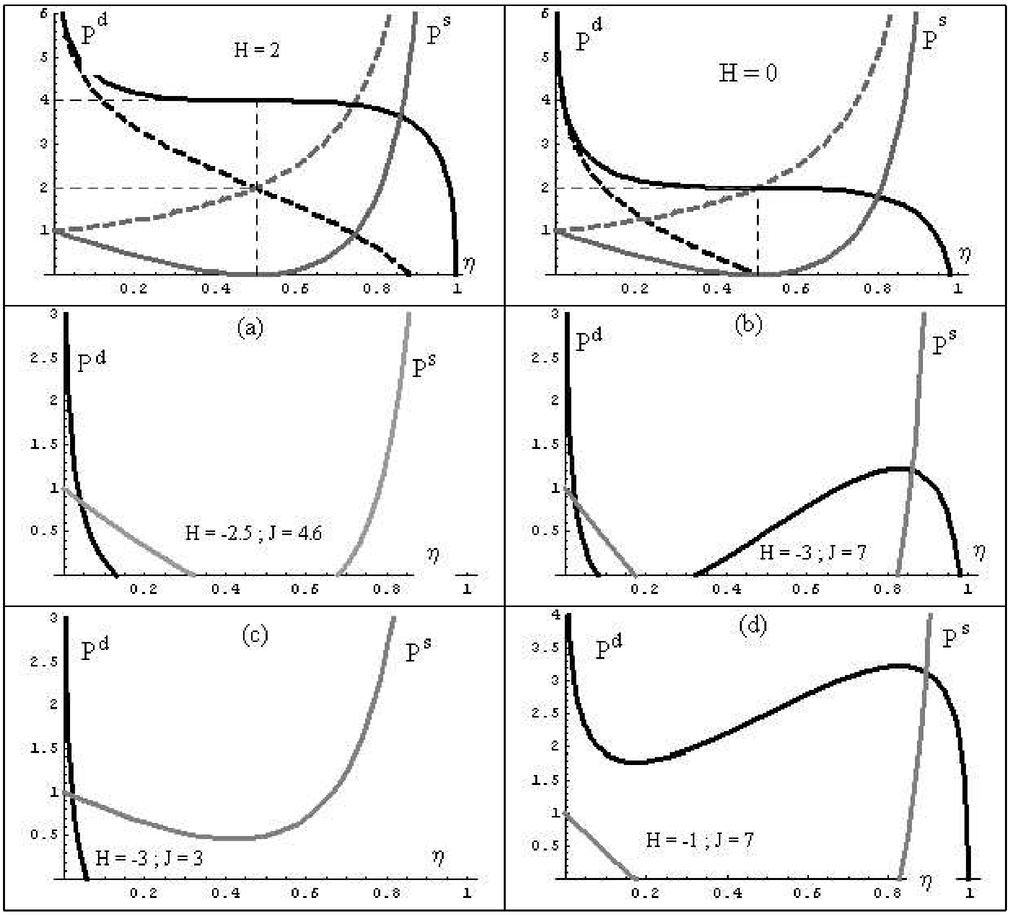


Figure 1: Inverse demand and *implied* supply curves $p^d(\eta)$ and $p^s(\eta)$, for different values of h and J ($\beta = 1$, $C = 0$ hence $h = H$). The equilibrium prices are obtained at the intersection between the demand (black) and the supply (grey) curves.
The two graphics on the top illustrate the difference between a complete absence of externality ($J = 0$, dashed lines) and a strong externality ($J = 4$, solid lines). The case $h = 2$ (left) corresponds to a strong positive average of the population's IWP ($h = 2$), whereas the population is neutral for $h = 0$ (right).
The values of h and J in the four graphics labelled (a) to (d) correspond to the points (a) to (d) in the phase diagram (figure 3). They all have negative values of the average of the population's IWP ($h < 0$), so that in the absence of externality only few consumers would be interested in the single commodity.
(a) corresponds to the *coexistence region* between two local market equilibria in figure 3; but one of them (not shown) is not relevant since it corresponds to a negative price solution. (b) lies also in the *coexistence region*; in this case, the optimal market equilibrium is the one with high η. (c) lies in the region with only one market equilibrium, with few buyers (small η). (d) corresponds to a large social effect; the single market equilibrium has large η and a high price.

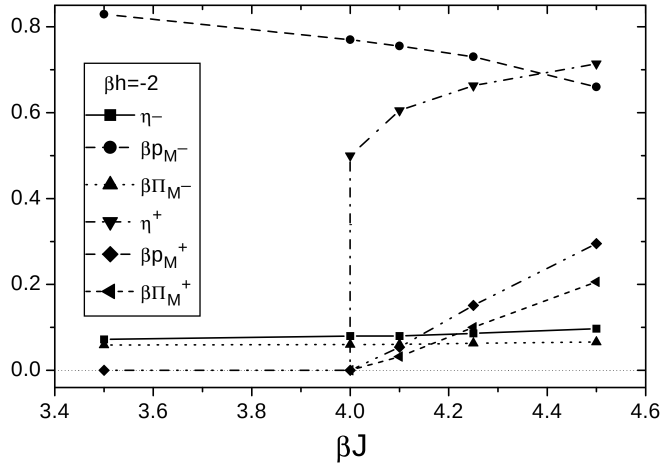


Figure 2: Fraction of buyers η, optimal price βp_M and monopolist profit Π_M, as a function of the social influence, for $\beta h = -2$. The superscripts $-$ and $+$ refer to the two solutions of equations (27) that are relative maxima.

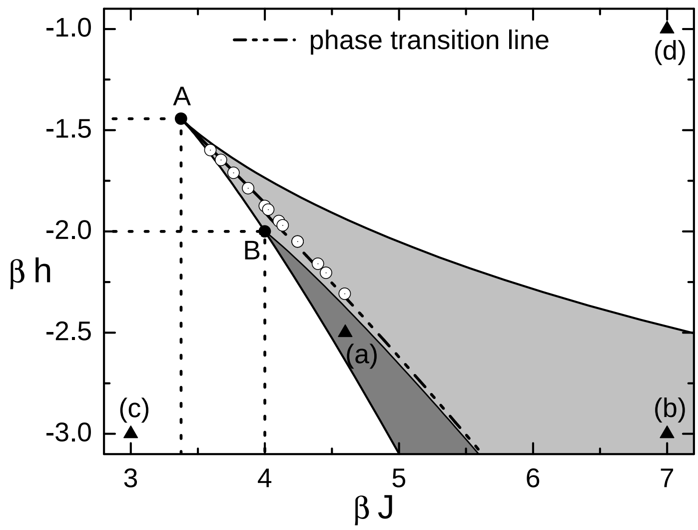


Figure 3: Phase diagram in the plane $\{\beta J, \beta h\}$: the grey region represents the domain in the parameter space where coexist two maxima of the monopolist's profit, a global one (the optimal solution) and a local one. Inside this domain, as βJ and/or βh increase, there is a (first order) transition where the monopolist's optimum jumps from a high price, low penetration rate solution $\eta = \eta_-$ to one with low price, large $\eta = \eta_+$. The circles on the transition line have been obtained numerically, the smooth curves are obtained analytically (see the Appendix and [22] for details). The points (a) to (d) correspond to the inverse supply and demand curves represented in figure 1. In the white region, for $\beta J < 27/8$, the fraction of buyers, η, increases continuously from 0 to 1 as βh increases from $-\infty$ to $+\infty$ (c-d). At the singular point A, ($\beta J = 27/8$, $\beta h = -3/4 - \log(2)$), $\eta_+ = \eta_- = 1/3$. At point B ($\beta J = 4$, $\beta h = -2$), the local maximum with η large appears with a null profit and $\eta_+ = 1/2$. In the dark-grey region below B, this local maximum exists with a negative profit, being thus non viable for the monopolist (a).

4.2 Phase transition in the monopolist's strategy

In this section we analyse and discuss the solution of the optimal supply-demand static equilibria, that is, the solutions of equations (27) and (31). As might be expected, the result for the product ηp_M depends only on the two parameters βh and βJ. That is, the variance of the idiosyncratic part of the reservation prices fixes the scale of the important parameters, and in particular that of the optimal price. This is why we present our results under the form of a phase diagram with axis $(\beta J, \beta h)$, as usual in physics (see figure 3). Each point in this diagram corresponds to a particular set of parameters of the customers-monopolist system. The lines represent boundaries between regions of qualitatively different equilibria, that we describe hereafter.

Let us first discuss the case where $h > 0$. It is straightforward to check that in this case there is a single solution η_M. It is interesting to compare the value of p_M with the value p_n corresponding to the neutral situation on the demand side. The latter corresponds to the *unbiased* situation where, on average, there are as many agents likely to buy as not to buy ($\eta = 1/2$). Since the expected willingness to pay of any agent i is $h + \theta_i + J/2 - p$, its average over the set of agents is $h + J/2 - p$. Thus, the neutral state is obtained for

$$p_n = h + J/2. \tag{35}$$

To compare p_M with p_n, it is convenient to rewrite equation (14) as

$$\beta(p^d - p_n) = \beta J(\eta - 1/2) + \ln[(1-\eta)/\eta]. \tag{36}$$

This equation gives $p^d = p_n$ for $\eta = 0.5$, as it should. For this value of η, equation (25) gives $p^s = p_n$ only if $\beta(h + J) = 2$: for these values of J and h, the monopolist maximises his profit when the buyers represent half of the population. When $\beta(h + J)$ increases above 2 (decreases below 2), the monopolist's optimal price decreases (increases) and the corresponding fraction of buyers increases (decreases).

Finally, if there are no social effects ($J = 0$) the monopolist optimal price is a solution of the implicit equation:

$$p_M = \frac{1}{\beta\, F(p_M - h)} = \frac{1 + \exp\left(-\beta(p_M - h)\right)}{\beta}. \tag{37}$$

The value of βp_M lies between 1 and $1 + \exp(\beta h)$. Increasing β lowers the optimal price: since the variance of the distribution of willingness to pay gets smaller, the only way to keep a sufficient number of buyers is to lower the prices.

Consider now the case with $h < 0$, that is, on average the population is not willing to buy. Due to the randomness of the individual's reservation price, $H_i = H + \theta_i$, the surplus may be positive but only for a small fraction of the population. Thus, we would expect that the monopolist will maximise his profit by adjusting the price to the preferences of this minority. However, if the social influence represented by J is strong enough, this intuitive conclusion is not supported by the solution to equations (27). The optimal monopolist's strategy shifts abruptly from a regime of high price and a small fraction of buyers to a regime of low price with a large fraction of buyers as βJ increases. Such a discontinuity might actually be expected for $\beta J > 4 = \beta J_B$, that is when the demand itself has a discontinuity. But, quite interestingly, the transition is also found in the range $\beta J_A \equiv 27/8 < \beta J < 4 = \beta J_B$, that is, in a domain of the parameters space $(\beta J, \beta h)$ where the demand $\eta(p)$ is a smooth function of the price.

Such a transition is analogous to what is called a *first order phase transition* in physics [19]: at a critical value $\beta J_c(\beta h)$ of the control parameter the fraction of buyers jumps from a low to a high value. Before the transition, above a value $\beta J_-(\beta h) < \beta J_c(\beta h)$ equations (27) already present several solutions. Two of them are local maxima of the monopolist's profit function, and one corresponds to a local minimum. The global maximum is the solution corresponding to a high price with few buyers for $\beta J < \beta J_c$, and that of low price with many buyers for $\beta J > \beta J_c$. Figure 2 presents these results for the particular value $\beta h = -2$, for which it can be shown analytically that $\beta J_- = 4$, and $\beta J_c \approx 4.17$ (determined numerically).

The detailed discussion of the full phase diagram in the plane $(\beta J, \beta h)$, shown on Figure 3, is presented in the Appendix, and a more general discussion will be presented elsewhere [22].

5 Dynamic features

In RUM models, the individual thresholds of adoption implicitly embody the number of agents each individual considers sufficient to modify his behaviour, as underlined in the field of social science [17, 8]. We briefly discuss here some dynamical aspects of the QRUM, considering a market with myopic customers: each agent makes its decision at time t based on the observation of the behaviour of the other agents at time $t-1$, that is, the agents have a myopic best-reply strategy. The adoption by very few agents in the population (the "direct adopters") may then lead to a significant change in the whole population through a chain reaction of "indirect adopters" [16]. This chain reaction depends on the type of dynamics considered, synchronous (all the agents take their decision at the same time, based on the previous decisions of their neighbours) or asynchronous (at each time step a single agent, picked at random, makes his decision). In synchronous dynamics, the fraction of adopters in the large N limit is then given by

$$\eta(t) = 1 - F(P - H - J\,\eta(t-1)) \tag{38}$$

and $\eta(t)$ converges to a solution of the fixed point equation (11). As we have seen, there are values of J, H and P (those of the shaded region in the phase diagram of Figure 3) for which (11) presents two stable and one unstable fixed points provided that β is large enough (small σ). At a given price, the stable solutions correspond to two possible levels of η (Figure 4a).

Consider the following monopolist strategy: start with a price sufficiently low (or sufficiently high) to be in the region where only one solution exists for the fraction of buyers. Then, by increasing (decreasing) the price smoothly, the equilibrium fraction of buyers at each price converges to the corresponding fixed point solution. At some price the system jumps abruptly to the other fixed point solution. However, if the price is decreased (increased) back, the jump occurs at a different price. This phenomenon, called *hysteresis*, is characteristic of the so called *first order phase transitions* in Physics. At such transitions, some extensive property of the system (here, the fraction of buyers) changes abruptly when the parameters (here, the price) are infinitesimally modified.

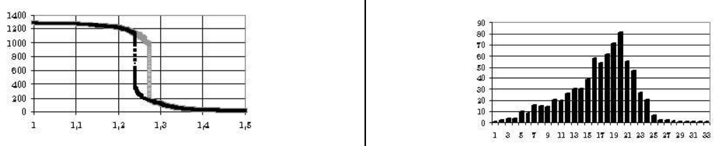
In the region of the phase diagram where two solutions exist, the monopolist is not guaranteed that the fraction of customers will be the fraction expected by his profit optimization program. Generally, when the price is slightly changed, the number of customers between two fixed point solutions evolves through a series of clustered flips (between $\omega_i = 1$ and $\omega_i = 0$). The resulting global change is referred to as an *avalanche*. Notice that several successive updates of the customers' decisions are needed to reach the corresponding fixed point solution. In the case of bounded agents' neighbourhoods (not discussed in this paper), when the system is at one equilibrium point on the hysteresis cycle, secondary inner hysteresis loops, that start and end at the same point, may be produced by changing back and forth the price. This complex behaviour of RFIMs, first described by Sethna [18], has been discussed within the economics context of QRUM with externalities by Pajot et al [16]. Note that these secondary hysteresis phenomena are specific to the QRUM; they are not present in the TRUM.

In the present case of a global neighborhood, there is a single huge avalanche involving all the agents that change their states, at the corresponding transition. Figure 4 illustrates the hysteresis phenomenon. The curves in Figure 4a, represent the number of customers as a function of the price, obtained through a simulation of the whole demand system with synchronous dynamics. The black (grey) curve corresponds to the "upstream" (downstream) trajectory, when prices decrease (increase) in steps of 10^{-4}, within the interval $[0.9, 1.6]$. We observe the *hysteresis* phenomenon with discontinuous transitions around the theoretical neutral price $P_n = 1.25$, defined by (35). Typically, along the downstream trajectory (with increasing prices, grey curve) the externality effect induces a strong resistance of the demand system against a decrease in the number of customers. In both cases, large avalanches occur at the first order phase transition. Figure 4b represents the sizes of the dramatic induced effects at the successive updates in the avalanche from one fixed point to the other, as a function of time (see [16] for more details).

For small enough values of β (large σ), there is always a single fixed point for all the values of P, and no hysteresis at all. This simpler behaviour is obtained for parameter values situated in the white region of the phase diagram (Figure 3).

6 Conclusion

In this paper, we have first compared two extreme special cases of discrete choice models, the Random Utility Model (QRUM) of Manski and McFadden, and the Thurstone model (TRUM), in which the individuals bear a local positive social influence on their willingness to pay, and have random heterogeneous idiosyncratic preferences. In the QRUM the latter remain fixed, and give rise to a complex market organisation. For physicists, this model with fixed heterogeneity belongs to the class



a - Hysteresis in the trade-off between prices and customers: upstream (black) and downstream (grey) trajectories	b - Chronology and sizes of induced effects in an avalanche at the phase transition for $P = 1.2408$ ($P_n = 1.25$)

Figure 4: Number of customers as a function of the price, at the discontinuous phase transition (full connectivity, synchronous activation regime; source: Phan *et al.* [16]; parameters: $N = 1296$, $H = 1$, $J = 0.5$, $\beta = 10$).

of "quenched" disorder models; the QRUM is equivalent to a "Random Field Ising Model" (RFIM). In the TRUM, all the agents share a homogeneous component of the willingness to pay, but have an additive, time varying, random (logistic) idiosyncratic characteristic. In physics, this problem corresponds to a case of "annealed" disorder. The random idiosyncratic component results in a stochastic dynamics, because each agent decides to buy according to the logit choice function at each time step, making this model formally equivalent to an Ising model at temperature $T \neq 0$ in a uniform (non random) external field. From the physicist's point of view, the QRUM and the TRUM are quite different: random field and zero temperature in the QRUM case, uniform field and non zero temperature in the TRUM case. An important result in statistical physics is that quenched and annealed disorders can lead to very different behaviours. In this paper we have briefly discussed some consequences on the market's behaviour.

Next, we have considered the QRUM case with a global externality corresponding to a positive social influence, and with a single seller (the case of a monopoly market). Studying the optimisation of the profit by the seller, we have exhibited a new "first order phase transition": when the social influence is strong enough, there is a regime where, upon increasing the mean willingness to pay, or decreasing the production costs, the optimal monopolist's solution jumps from one with a high price and a small number of buyers, to one with a low price and a large number of buyers. It is worth to stress that the multi equilibria domain exists as a consequence of a positive externality, *without assuming any bimodal distribution* of the IWPs. Moreover, as we will show with more details in a forthcoming paper [22], the phase diagram derived here under the hypothesis of a logit distribution is generic of any smooth monomodal distribution.

We have only considered fully connected systems: the theoretical analysis of systems with finite connectivity is more involved, and requires numerical simulations. The simplest configuration is one where each customer has only two neighbours, one on each side. The corresponding network, which has the topology of a ring, has been analysed numerically by Phan *et al.* [16]) who show that the optimal monopolist's price increases both with the degree of the connectivity graph and the range of the interactions (in particular, in the case of "small world" networks). Buyers'clusters of different sizes may form, so that it is no longer possible to describe the externality with a single parameter, like in the mean field case. Further studies in computational economics are required in order to explore such situations.

Acknowledgements

We thank Annick Vignes for a critical reading of the manuscript. This work is part of the project 'ELICCIR' supported by the joint program "Complex Systems in Human and Social Sciences" of the French Ministry of Research and of the CNRS.

References

[1] Anderson S.P., DePalma A, Thisse J.-F. (1992) *Discrete Choice Theory of Product Differentiation*, MIT Press, Cambridge MA.

[2] Bouchaud J-Ph. (2000) "Power-laws in Economics and Finance: some ideas from physics", Quantitative Finance 1, 105

[3] Bulow J.I., Geanalopolos J.D., Klemperer P.D. (1985) "Multimarket oligopoly: strategic substitutes and complements", Journal of Political Economy, 93/3 pp.488-511.

[4] Durlauf S.N. (1997) "Statistical Mechanics Approaches to Socioeconomic Behavior", in Arthur, Durlauf, Lane (eds), *The Economy as an Evolving Complex System II*, Santa Fe Institute Studies in the Sciences of Complexity, Volume XXVII, Addison-Wesley.

[5] Galam S., (1982) "A New Multicritical Point in Anisotropic Magnets. III. Ferromagnets in both a Random and a Uniform Longitudinal Field", Journal of Physics C, 15, pp. 529-545.

[6] Galam S., Aharony A. (1980) "New multicritical points in anisotropic magnets, I. Ferromagnets in a random longitudinal field", J. Phys. C: Solid State Phys. 13, 1065-1081; Galam S. and Aharony A. (1981) "A new multicritical point in anisotropic magnets, II. Ferromagnet in a random skew field", J. Phys. C: Solid State Phys. 14, 3603-3619.

[7] Galam S., Gefen Y., Shapir Y. (1982) "Sociophysics: A Mean Behavior Model for the Process of Strike", Mathematical Journal of Sociology, 9, pp. 1-13.

[8] Granovetter M. (1978) "Threshold Models of Collective Behavior", American Journal of Sociology, 83(6), p. 1360-1380.

[9] Kirman A..P. (1997a), "The Economy as an Interactive System", in Arthur B., Durlauf S., Lane D. (eds.), *The Economy as an Evolving Complex System II*, Santa Fe Institute Studies in the Sciences of Complexity, Volume XXVII, Addison-Wesley, pp. 491-531.

[10] Kirman A.P. (1997b), "The Economy as an Evolving Network", Journal of Evolutionary Economics, 7, pp. 339-353.

[11] Manski, C. (1977), "The Structure of Random Utility Models", Theory and Decision, 8, 229-254.

[12] McFadden D. (1974) "Econometric Analysis of Qualitative Response Models", In Griliches Intrilligator (eds.), *Handbook of Econometrics* Vol II, Elsevier Science B.V, Amsterdam.

[13] McFadden D. (1976) "Quantal Choice Analysis: A Survey," Annals of Economic and Social Measurement, Vol. 5, No. 4, 363-390.

[14] Onsager, L.(1944) "A 2D model with an order-disorder transition", Physical Review, 65:117-149.

[15] Orléan A. (1995) "Bayesian Interactions and Collective Dynamics of Opinion: Herd Behaviour and Mimetic Contagion", Journal of Economic Behavior and Organization, 28, pp. 257-74.

[16] Phan D., Pajot S., Nadal J.P. (2003) "The Monopolist's Market with Discrete Choices and Network Externality Revisited: Small-Worlds, Phase Transition and Avalanches in an ACE Framework". Ninth annual meeting of the Society of Computational Economics, University of Washington, Seattle, USA, July 11 - 13, 2003.
http://econpapers.hhs.se/paper/scescecf3/150.htm

[17] Schelling T.S. (1978) *Micromotives and Macrobehavior*, Norton and Co, N.Y.

[18] Sethna J.P., Dahmen K., Kartha S., Krumhansl J.A., Roberts B.W., Shore J.D. (1993), "Hysteresis and Hierarchies: Dynamics of Disorder-Driven First-Order Phase Transformations", Physical Review Letters, 70, pp. 3347-3350.

[19] Stanley H.E. (1971) *Introduction to phase transitions and critical phenomena*, Oxford University Press.

[20] Thurstone L.L. (1927) "Psychological Analysis", American Journal of Psychology, 38, pp.368-398.

[21] Weisbuch G., Stauffer D. (2003) "Adjustment and social choice", Physica A: Statistical Mechanics and its Applications, Vol. 323 , pp. 651-662

[22] A more detailed and general analysis, that is for a large family of IWP distributions, will be published elsewhere (Gordon M. B., Nadal J.-P., Phan D. and Vannimenus J., in preparation).

Appendix: Phase Diagram

In this Appendix we detail the derivation of the phase diagram in the plane (βJ, βh), presented in figure 3. The phase diagram shows the domain in the parameter space where coexist two maxima of the monopolist's profit, one global maximum (the optimal solution) and one local maximum. Inside this domain there is a (first order) transition line where, as βJ and/or βh increases, the optimal solution jumps from a solution '-' with a low value $\eta = \eta_-$ to a solution '+' with a large value $\eta = \eta_+$, η being the fraction of buyers. In figure 3, the circles 'o' are points on the transition line obtained numerically; all the other curves being obtained analytically as explained below.

In the following without loss of generality we set $\beta = 1$, which is equivalent to say that we measure J and h in units of $1/\beta$.

To explain the phase diagram in more detail, it is convenient to parametrise every quantity/curve as a function of η. First the (*per unit*) profit p is given by

$$p = \frac{1}{1-\eta} - J\eta \tag{39}$$

(hence the profit $N\Pi = Np(\eta)\eta$), and η is a fixed point of the equation

$$\eta = G(h, J, \eta) \tag{40}$$

with

$$G(h, J, \eta) = \frac{1}{1 + \exp(-h - 2J\eta + \frac{1}{1-\eta})} \tag{41}$$

We will also make use of an alternative form of (40), (41), that is

$$h = -2J\eta + \frac{1}{1-\eta} + \log(\frac{\eta}{1-\eta}) \tag{42}$$

One can also show that the fixed point equation (40) is equivalent to

$$\frac{d(\eta P^d(\eta))}{d\eta} = 0, \tag{43}$$

and that the condition for having a maximum, (29), is equivalent to

$$\frac{d(\eta P^s(\eta))}{d\eta} > 0. \tag{44}$$

As we will see, there are two singular points of interest:
A: $J_A = 27/8$, $h_A = -3/4 - \log(2)$;
B: $J_B = 4$, $h_B = -2$.

Let us describe the phase diagram considering that, at fixed J, one increases h starting from some low (strongly negative) value.

If $J < 27/8 = J_A$, the optimal solution changes continuously, the fraction of buyers increasing with no discontinuity from a low to a high value as h increases. More generally, outside the grey domain in figure 3 there is a unique solution of the optimisation of the profit.

For $J > 27/8 = J_A$, as h increases one will first hit the lower boundary of the grey region on the phase diagram, $h = h_-(J)$. On this line, a local maximum of the profit appears, corresponding to a value $\eta = \eta_+ > 1/3$. As shown in figure 5a, the curve $y = G(h, J, \eta)$ intersects $y = \eta$ at some small value $\eta = \eta_-$ and is tangent to it at $\eta = \eta_+$. For $h_-(J) < h < h_+(J)$ $y = G(h, J, \eta)$ has three intersects with the diagonal $y = \eta$, $h = h_+(J)$ being the upper boundary of the grey region on the phase diagram. The stability analysis shows that the two extreme intersects correspond to maxima of the profit, giving the solutions $\eta = \eta_-$ and $\eta = \eta_+$. On the upper boundary $h = h_+(J)$, it is the solution with a small value of η which disappears, with $y = G(h, J, \eta)$ becoming tangent to $y = \eta$ for $\eta = \eta_-$, see figures 5c1 and 5c2. These lower and upper boundaries are obtained by writing that the second derivative of the profit with respect to p is zero, giving

$$2J\eta(1-\eta)^2 = 1 \tag{45}$$

Together with (42) this gives the curves parametrised by η,

$$\begin{aligned} J &= \frac{1}{2\eta(1-\eta)^2} \\ h &= -\frac{1}{(1-\eta)^2} + \frac{1}{1-\eta} + \log(\frac{\eta}{1-\eta}) \end{aligned} \tag{46}$$

the lower curve $h_-(J)$ corresponding to the branch $\eta = \eta_+ \in [1/3,\ 1]$, and the upper curve $h_+(J)$ corresponding to the branch $\eta = \eta_- \in [0,\ 1/3]$.

The two curves merge at the singular point A, at which $\eta_+ = \eta_- = 1/3$, $J_A = 27/8, h_A = -3/4 - \log(2)$. Expanding the above equations (46) near $\eta = 1/3$ we find that the two curves are cotangent at A, with a slope $-2/3$. This common tangent is thus also tangent to the transition line at A. A straight segment of slope $-2/3$ starting from A is plotted on the phase diagram, figure 3, and one can see that this is a very good approximation of the transition line for $J < 4 = J_B$.

On the lower boundary, for $J > 4 = J_B$, the local maximum with $\eta = \eta_+$ appears with a negative profit (zero profit at point B where $\eta_+ = 1/2$). The profit becomes positive on the curve starting at point B, on which the profit is zero with $\eta_+ > 1/2$. This curve is obtained by writing $p = 0$, $\eta_+ > 1/2$, that is $\eta_+ = \eta_+^0(J)$,

$$\eta_+^0(J) \equiv \frac{1}{2}\left[1 + \sqrt{1 - \frac{4}{J}}\right] \tag{47}$$

and h is obtained as a function of J, by replacing η in (42) by the above expression (47). In this domain of negative profit for the local maximum, the distance to the transition line (at a given value of J) is equal to the amount by which the production cost per unit of good must be lowered in order to make the solution viable.

In the domain $J > 4 = J_B$, the transition line, computed numerically, appears to be just above this null profit line. This suggests that an expansion for p small for the solution η_+, and for η small for the solution η_- should provide good approximations. The transition is obtained when $\Pi_+ = \Pi_-$ as explained below, and this allows to display the curve of figure 3, which turns out to be a very good approximation of the transition line for large values of J (or small values of h, typically $h < -4.5$).

Let us first consider the vicinity of the point B at which $p_+ = 0$, $\eta_+ = 1/2$, and the second derivative of the profit is zero for this '+' local solution. Expanding near $J = 4$, h just above $h_-(4) = -2$ (p small), one gets the behaviour of the '+' solution:

$$\begin{aligned}
\epsilon &\equiv h + 2, \quad 0 < \epsilon << 1 \\
\eta_+ &= \frac{1}{2}(1 + \sqrt{\frac{\epsilon}{2}}) \\
p_+ &= \epsilon + o(\epsilon^{3/2}) \\
\Pi_+ &= \frac{\epsilon}{2} + o(\epsilon^{3/2})
\end{aligned} \tag{48}$$

The singular, square-root, behaviour of η is specific to point B. For any $J > 4$, just above the null curve η_+ increases linearly with $\epsilon \equiv h - h_+^0(J)$, where $h_+^0(J)$ is the value of h on the null curve (obtained by replacing η in (42) by $\eta_+^0(J)$ defined in (47)). The price and the profit have, however, the same behaviour as for $J = 4$. More precisely, at lowest order in ϵ, one gets:

$$\begin{aligned}
0 &< \epsilon = h - h_+^0(J) << 1 \\
\eta_+ &= \eta_+^0(J) + \epsilon \frac{\eta_+^0(1 - \eta_+^0)^2}{1 - 2J\eta_+^0(1 - \eta_+^0)^2} \\
p_+ &= \epsilon, \\
\Pi_+ &= \eta_+^0(J)\, \epsilon.
\end{aligned} \tag{49}$$

One can see from the expression of η_+ in (49) how the singularity at point B appears: the coefficient of ϵ diverges when condition (45) is fulfilled, that is when the solution is marginally stable, which is the case at B.

Similarly one can get the behaviour of the '+' solution near point B at $h = -2$, increasing J from $J = 4$, as shown in figure 2

$$\begin{aligned}
\eta_+ &= \frac{1}{2}(1 + \sqrt{\frac{J-4}{2}}) \\
p_+ &= \frac{1}{2}(J - 4) + \frac{\sqrt{2}}{12}(J - 4)^{3/2} \\
\Pi_+ &= \frac{1}{4}(J - 4) + \frac{1}{3\sqrt{2}}(J - 4)^{3/2}
\end{aligned} \tag{50}$$

Coming back to the behaviour at a given value of J, one can get an approximation of the '-' solution. The fixed point equation for η for given values of J and h, is

$$\eta = H(\eta) \equiv 1/\left[1 + \exp(1 - h - 2J\eta + \frac{\eta}{1-\eta})\right] \tag{51}$$

The '-' solution corresponding to a small value of η can be found by iterating $\eta(k+1) = H(\eta(k))$ starting with $\eta(0) = 0$, and $\eta(k)$ is an increasing sequence of approximations of η_-. The lowest non trivial order is then given by

$$\eta_-^0 = H(0) = 1/[1 + \exp(1 - h)] \tag{52}$$

which is indeed small for h strongly negative. At the next order

$$\eta_-^1 = H(\eta_-^0) \tag{53}$$

Taking η_-^0 as the small parameter, the expansion of η_-^1 gives

$$\eta_-^1 = \eta_-^0(1 + (2J - 1)\eta_-^0) \tag{54}$$

and this gives the corresponding approximations for the price and the profit,

$$\begin{aligned} p_-^1 &= 1 - (J-1)\eta_-^0 \\ \Pi_-^1 &= \eta_-^0 + J(\eta_-^0)^2. \end{aligned} \tag{55}$$

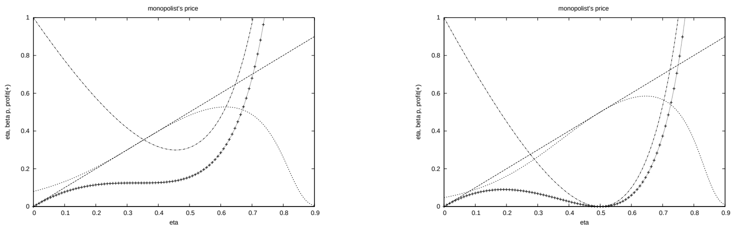
It is clear from the above equation that the dependency on J is weak since η_-^0 is small, in agreement with the exact behaviour computed numerically, shown on figure 2.

Now we consider the neighbourhood of $(J, h_+^0(J))$, that is $h = h_+^0(J) + \epsilon$. Denoting by $\eta_0(J)$ the value of η_-^0 at $h = h_+^0(J)$, $\eta_-^0(h) = \eta_0(J) + \epsilon\eta_0(J)(1 - \eta_0(J))$. Taking this expression for computing the profit of the '-' solution, and writing that at the transition the two solutions '+' and '-' give the same profit, one gets the following approximation for the value $\epsilon_c(J)$ of ϵ at the transition (hence the value of h at the transition, $h_c(J) = h_+^0(J) + \epsilon_c(J)$):

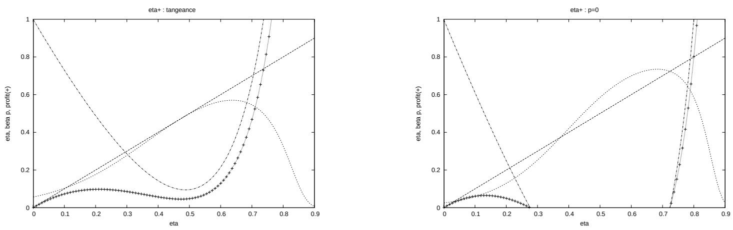
$$\epsilon_c(J) = \frac{\eta_0(J)}{\eta_+^0(J)} \tag{56}$$

where $\eta_+^0(J)$ is given by equation (47). It is this curve $h_c(J) = h_+^0(J) + \epsilon_c(J)$ which is plotted in figure 3 for $J > J_B = 4$.

Figure 5: Functions $y = G(h, J, \eta)$, $y = \eta$, price $p(\eta)$ and profit $\Pi(\eta)$ $(+-)$. The intersects of $y = G(h, J, \eta)$ with $y = \eta$ give the extrema of the profit; the (possibly local) maxima are those for which $d\Pi/d\eta > 0$. Shown here are marginal cases where for one solution $d\Pi/d\eta = 0$, that is $y = G(h, J, \eta)$ is tangent to $y = \eta$ (points on the lower or upper curves of the phase diagram, figure 3).



(a) $J = 27/8$ $h = -3/4 - \log(2)$ (b) $J = 4$ $h = -2$



(c1) Point on the lower curve, $\eta < 1/2$ (c2) Point on the lower curve, $\eta > 1/2$

(d1) Point on the upper curve, $J > 4$ (d2) Point on the upper curve, $J < 4$