"Antiferromagnetism" in social relations and Bonabeau model

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Abstract

We here present a fixed agents version of an original model of the emergence of hierarchies among social agents first introduced by Bonabeau *et al*. Having interactions occurring on a social network rather than among 'walkers' doesn't drastically alter the dynamics. But it makes social structures more stable and give a clearer picture of the social organisation in a 'mixed' regime.

1 Introduction

A number of models in socio- and econo-physics are inspired from magnetism, starting with the pioneering work of Föllmer (1974) [1]. They concern the modeling of opinion dynamics and assume for instance binary opinions of agents distributed on some social networks, with variants which include the mean field approximation. Agents adjust their opinion under the influence of their neighbours. In many models, agents follow the opinion of their neighbours via for instance

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some "voting" scheme: an agent would then check the opinions of its neighbours and adjust its own opinion according to the majority's opinion. The observed dynamics resemble ferromagnetism.

But a large number of other models start from the opposite assumption.

In the minority game [2], applied to financial markets, agents choose to behave in opposition with the majority: if the others are buying, you should sell (and the opposite). The minority game is only one of such models to which one can relate the model of the emergence of social classes based on the ultimatum game of Axtell *et al* [3], and Bonabeau *et al* model [4] of the emergence of social hierarchies. This Bonabeau *et al* approach, followed by the contributions of Ben-Naim and Redner [5] (taking a mean field approach) and of Stauffer [6], is the original inspiration for the work presented here.

We consider agents i whose internal state $h(i)$ is adjusted on the occasion of binary encounters. $h(i)$ can be interpreted as strength and encounters as fights. On the occasion of a fight, the winner, usually the stronger agent, wins and its strength is increased by one. The looser strength is decreased by one. By usually, we imply a probabilistic process with some "thermal" probability. Bonabeau *et al* original model is a walkers' model: agents are moving across a square lattice, and interact only when they encounter on the same lattice site. The main parameter of these models is the β discrimination parameter for the outcome of the fight process. In Bonabeau *et al* a phase transition is observed: at low β values, a "disordered" phase is observed with a Gaussian distribution of agent strengths, while at high β values, a large fraction of agents have extreme strength, positive or negative. Several variants yield a sharp phase transition, but under questionable assumptions. In Bonabeau *et al*, strength is allowed to go to infinity with time. In Stauffer's model [6], when a moving average process is introduced to limit strength, the transition is not sharp; Stauffer had to introduce some extra dynamics on the β discrimination parameter to get a sharp transition. The first motivation for the present research was to check the importance of the random "walkers" dynamics on the sharpness of the transition: would fixed agents, at the nodes of a social network, display a sharp transition towards some "anti-ferromagnetic" regime?

2 The model and its numerical implementation

We use a square lattice with periodic boundary conditions. Agents don't move and remain at lattice nodes. At each time step, an agent i, randomly selected, updates its internal variable $h_i(t)$ through a random process. The agent interacts with one of its four neighbours randomly chosen. He wins with a probability P related to its neighbour's strength by a logit (or thermal) function according to:

$$P = \frac{1}{1 + \exp(\beta(h_j(t) - h_i(t))} \tag{1}$$

As in thermal physics, a large β coefficient results in a nearly deterministic choice in favour of the stronger agent, and in a nearly random choice when β is small. As a result of the interaction, the winner's strength is increased by one, and the looser strength is decrease by one. Suppose, e.g., that i wins, the updating gives:

$$h_i(t) = (1-\gamma)h_i(t-1) + 1 \quad , \tag{2}$$

$$h_j(t) = (1-\gamma)h_j(t-1) - 1 \quad . \tag{3}$$

These equations correspond to a moving average of past gains. Gains increase strength, but past gains are discounted at a rate $1-\gamma$. $1/\gamma$ is the characteristic time of the dynamics. They imply one double update, since two sites i and j are involved; one time step corresponds to one double update per site.

We usually start simulations from a configuration where all strengths are 0, and run them for a long time to check the attractors of the dynamics. A typical time scale for the simulation is $10/\gamma$ updates per site, i.e. ten times the characteristic time. This choice is also motivated by the equilibrium amplitude of the strength of a constant winner:

$$h(eq) = (1-\gamma)h(eq) + 1 \tag{4}$$

$$h(eq) = \frac{1}{\gamma} \tag{5}$$

Times proportional to $1/\gamma$ are needed to allow strength to saturate.

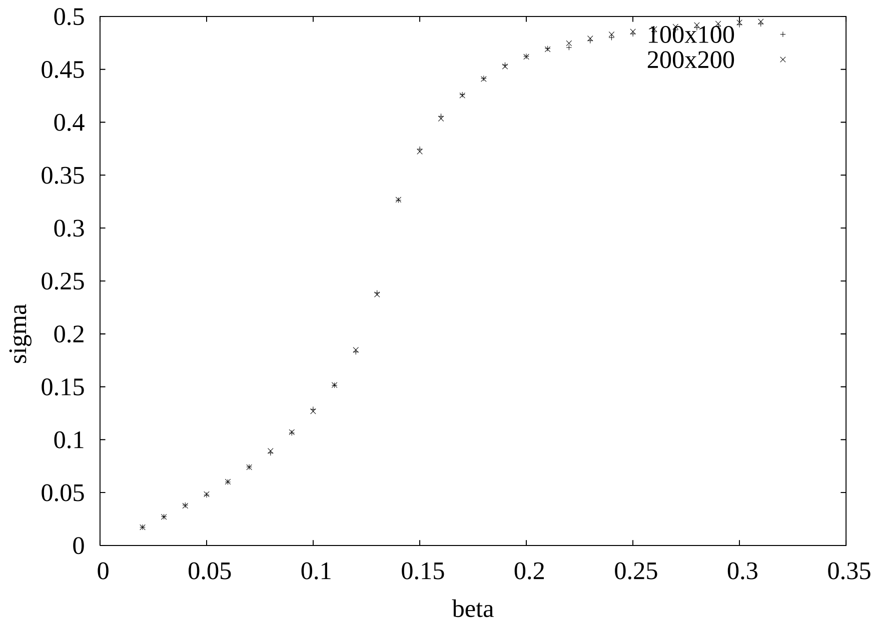

Figure 1: Standard deviation of the winning probability distribution as a function of β, observed at $\gamma = 0.1$ for networks of size 100×100 and 200×200. Simulated were 10 000 time steps.

3 Simulations results

Apart from the dimension of the lattice which might give rise to some size effects, two parameters control the dynamics, β the discrimination parameter and γ, the memory parameter. Our simulations display the system configuration at asymptotic time value or some kind of order parameter.

We have chosen the same order parameter as Bonabeau et al and followers, namely the standard deviation of the distribution of the probability to win, σ. We first checked at fixed γ the evolution of σ versus β, the discrimination coefficient (figure 1).

The observed crossover, starting at values of β close to γ is smooth and no size effect is detectable between networks of size 10,000 and 40,0000 which could account for the observed smoothness. We also checked, for the same reason, the influence of convergence times which again is not noticeable after a number of double updates per agent above $10/\gamma$ time steps (not represented here by a figure).

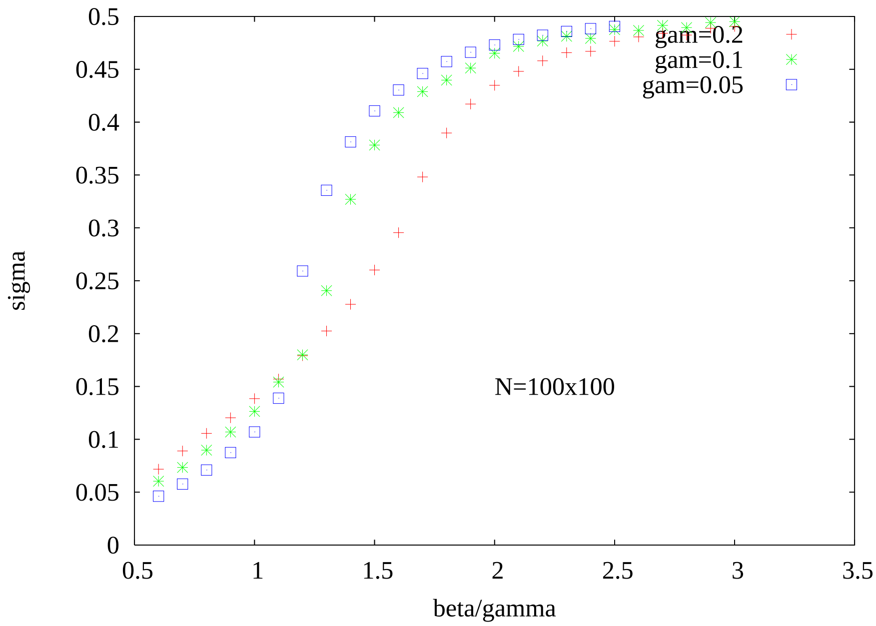

Figure 2: Standard deviation of the winning probability distribution as a function of β/γ, observed for networks of size 100×100. γ values are 0.05, 0.1 and 0.2. Simulation times per site are $10/\gamma$.

To check the influence of both β and γ it is interesting to use a reduced parameter. β/γ is a good candidate, by analogy with ferromagnetism. The mean field theory of ferromagnetism predicts a transition when $\beta = z \times J$, where J is the coupling constant and z the number of neighbours. We expect then 'something' to happen as a function of β/γ. Another model based on a logit choice and a moving average of a utility function based on past experience is Weisbuch *et* al model of Marseille fishmarket [7] which also gives β/γ as a reduced parameter. $1/\gamma$ being the characteristic strength involved in the present problem, β/γ is then a reasonable choice.

Figure 2 displays the standard deviation of the distribution of the probability to win, σ, as a function of β/γ,

As we guessed, the onset of a smooth crossover occurs slightly above $\beta/\gamma = 1$ where the three curves meet. And σ saturates at 0.5, which corresponds to all probabilities equally distributed between the two values 0 and 1. But the function σ vs. β/γ is not universal: the slope of the line near the crossover seems to increase with decreasing γ.

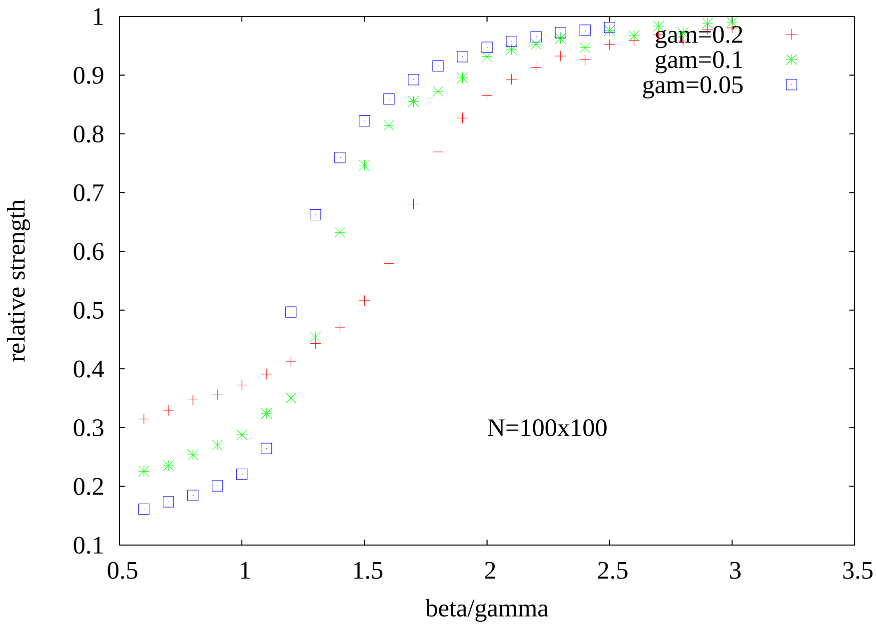

Figure 3: Normalised averaged strength amplitude $|h| \times \gamma$ as a function of β/γ, observed for networks of size 100×100. γ values are 0.05, 0.1 and 0.2. Simulation times per site are $10/\gamma$.

The Bonabeau *et* al result, a vertical transition for infinite strength, is then consistent with ours for small γ. The same is true for Stauffer's result: no sharp transition for finite strength values. In other words, our fixed fighters results are consistent with those obtained for walking fighters models.

A similar behaviour is observed in figure 3 for the relative average amplitude of sites strength, another possible order parameter. At large β/γ values, strength amplitudes saturate to $1/\gamma$. The curves cross around $\beta/\gamma = 1$, but they don't collapse.

The histograms displayed in figure 4 provide a more direct representation of the state of the system at different β values. Strengths vary between +10 and −10, respectively $\pm 1/\gamma$. In the low β region, say $\beta \leq \gamma$, the strength distribution is a Gaussian centered around 0. By contrast, when $\beta \gg \gamma$, two narrow peaks are observed at $\pm 1/\gamma$. In between the transition is smooth.

Figure 5 is a set of 50×50 strength patterns obtained after 10 000 time steps for different values of β below and above $1/\gamma$.

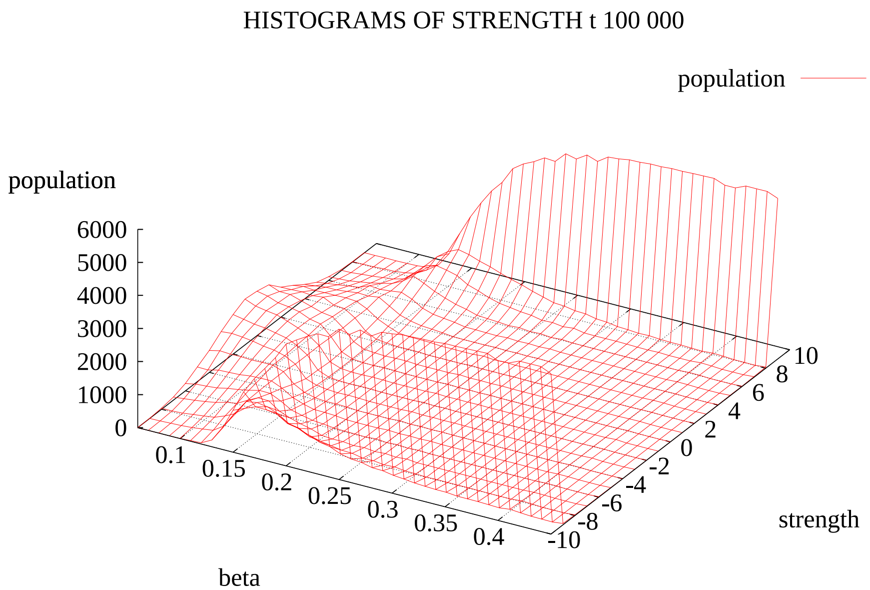

Figure 4: Strength distribution as a function of β, observed for networks of size 100×100. $\gamma = 0.1$. Simulation times were 100 000 time steps.

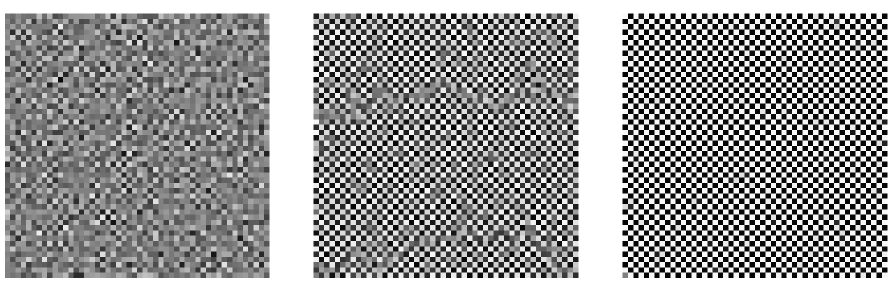

Figure 5: Strength patterns obtained on 50×50 lattices after 10 000 time steps for $\beta = 0.11$, 0.15 and 0.3; $\gamma = 0.1$. Black corresponds to –10 strength, grey to intermediate values, and white to 10 strength. When β increases from 0 to 0.3, 'anti-ferromagnetic' islands with checkerboard structure increase in size and invade the whole lattice

The checkerboard pattern obtained when $\beta = 0.3$ is a clear picture of the analogous of an "anti-ferromagnetic" configuration: all sites have a maximum strength $1/\gamma$, alternately positive and negative. For the intermediate values of β, anti-ferromagnetic islands of size increasing with β invade a sea of low-strength sites coloured in grey. At intermediate β values, the checkerboard islands do not grow in time since the low strength agents at their boundary have one neighbour within the checkerboard structure against 3 outside. They tend to relax towards the average strength after their interactions with checkerboard sites.

This intermediate regime can be compared to type II superconductors which also exhibit a mixed regime between two values of the external magnetic field, where magnetisation increases continuously with the field. In the mixed regime, 'normal' vortices regions are surrounded by superconductive regions.

As observed in figure 6, the time evolution of the patterns in the anti-ferromagnetic region implies two characteristic times: first, ferromagnetic domains separated by low strength lines appear quite fast. Two domains separated by a line are shifted by one site; in one dimension the structure would be:

$$+ - + - + - + \; 0 - + - + - + - + -$$

(where $+$ is positive, $-$ negative and 0 weak strength). The two antiferromagnetic domains are phase shifted by π with respect to the lattice structure.

The subsequent annealing process with coarsening of anti-ferromagnetic domains into a unique domain is much longer.

The online observation of the sites dynamics shows that two factors account for the slow convergence dynamics and for the finite width of the crossover:

- The structuring process into antiferromagnetic patches begins early in time and when $\beta/\gamma \simeq 1$. But patches growing from different regions of the lattice don't have any reason to be in phase. Whenever intermediate strength ($h \simeq 0$) agents interact against agents from the antiferromagnetic patches, their strength is increased (in absolute value) by interaction with one side but it is decreased when they interact with the other side.
- Since the strength difference between intermediate agents and agents inside patches is half of the strength difference inside the patches, a higher β value is necessary to maintain long range

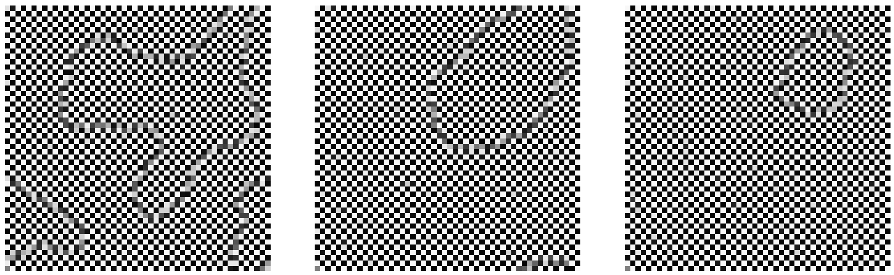

Figure 6: Time evolution of strength patterns obtained on 50×50 lattices for $\beta = 0.3$ and $\gamma = 0.1$, after 1000, 5000 and 8000 time steps. The colour code is the same as above. One first observes a fast evolution towards a partial anti-ferromagnetic structure with low strength line boundaries. Annealing of the boundaries takes a longer time and is achieved after 10000 time steps as observed in the previous figure.

order.

Finally, figure 7 returns to the question posed already by Fig.2: Would we get for $\gamma \to 0$ a sharp phase transition in the sense that the slope of σ versus the ratio β/γ would become infinite at some critical value of this ratio? For this purpose we simulated much larger lattices, for at least $100/\gamma$ time steps and $0.001 \leq \gamma < 1$. We also used regular updating, instead of the above random updating, and made minor changes. There are long-time effects preventing complete equilibration, presumably due to reptation of domain boundaries. Nevertheless this figure confirms our guess from Fig.2 about a sharp transition in the limit $\gamma \to 0$. The critical ratio β/γ seems to be 1.

4 Conclusions

Our first aim was to compare a fixed agents model of 'fighters' with results obtained for 'walkers' models of Bonabeau *e*t al. We wanted to know whether the smoothness of the crossover observed when strength remains finite was due to the extra randomness introduced by walkers diffusion. We demonstrated that a fixed agent model also displays a smooth crossover.

The observation of equilibrium patterns shows that small anti-ferromagnetic domains start to develop in the intermediate mixed re-

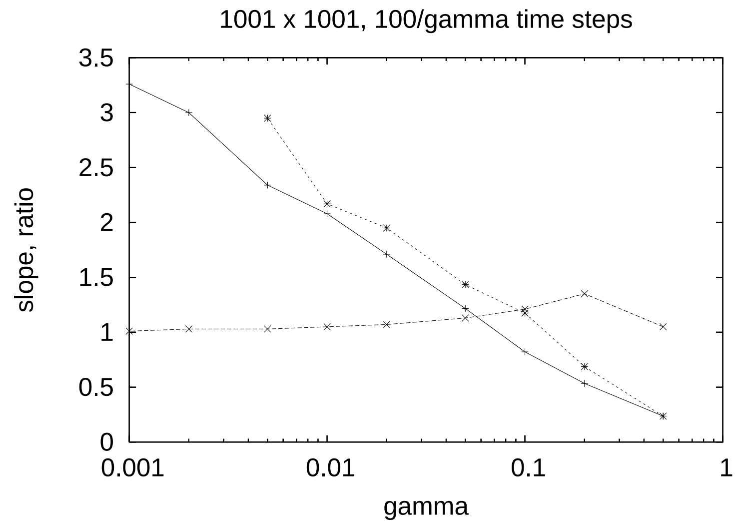

Figure 7: Maximum slope (+,*), of the σ versus β/γ curves like figure 2, on 1000×1000 lattices for $100/\gamma$ (+) and $1000/\gamma$ (*) time steps. The x symbols indicate the ratio β/γ for which this slope is reached.

gion, when $\beta \simeq \gamma$, but infinite long range order is only achieved for values of the discrimination parameter β above the onset of the mixed region. The width of the mixed region appears to increase with γ.

Interpretations of the lattice in terms of social networks show that the hierarchy is re-enforced by the fact each agent interacts with the same neighborhood. On the other hand, social networks are more random than lattices. They are not totally random as Erdös-Rényi nets and display more betweenness (small loops), than regular lattices. But the stability of the ordered regime obtained with square lattices containing only non-frustrated loops cannot be expected from more random structures: frustrated loops introduce glassy dynamics, with wider crossover and more irreducible fluctuations in the ordered regime.

Acknowledgments

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