Simple models of firms emergence

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Abstract

We present a dynamical model of the emergence of firms as opposed to a flat labour market where entrepreneurs would recruit workers for each business opportunity. The model uses a preferential choice of partners based on previous collaborations experience. A sharp transition in the parameter space separates an ordered regime, where preferential links establish, from a disordered regime corresponding to a fast turnover of employees.

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1 Introduction

Standard micro-economics concentrate on the description of markets but is seldom interested in production. Several economists discussed the concept of a firm, as opposed to an open labour market where entrepreneurs would recruit workers on the occasion of each business opportunity. Coase [1] is one of them, who explains the existence of firms as institution because they reduce the transaction costs with respect to an open labour market. Several other aspects such as principal-agent theory, coalition theory, information processing, technical evolution of firms have been discussed by game theoretists and evolutionary economists (see the papers of Axtell [2] and Chang etal. [3] for a review).

Whatever the rationale proposed by economists to account for the existence of firms, their perspective is based on efficiency and cost analysis. Little attention is paid to the dynamics of emergence and evolution of firms, except in the references we now quote.

There is a literature in Computational Economics, i.e. based on computer simulations, which compares the performance of different firm structures, e.g. Marengo [4] and Miller [5] . Their inspiration, as ours, comes from the bounded rationality approach developped for instance in March and Simon [6]. But they consider firms and work relationship as given. Closer to our approach is the evolutionary perspective developped by Axtell [2] and the organisation of firms as reactions to business opportunities developped by Epstein [7].

The aim of the present manuscript is to check the global dynamical properties of a very simple model based on bounded rationality and reinforcement learning. Workers and managers are localised on a lattice and they choose collaborators on the basis of the success of previous work relations. The choice algorithm is largely inspired from the observation and modeling of long term customer/sellers relationships observed on perishable goods markets discussed in Weisbuch etal[8] and Nadal etal[9].

The model presented here is in no way an alternative to Coase. We describe the build-up of long term relationships which do reduce transaction costs, and we deduce the dynamical properties of networks built from our simple assumptions.

We first discuss in section 2 a multi-entrepreneur model which already exhibits long term relationships among historically selected sets of workers. A single entrepreneur model discussed in section 3 allows to quantify the conditions for the emergence of these long term relationships. But since these models only predict linear firm structures, we add in section 4 a further condition limiting the fraction of work load attributed by each node to its subordinate nodes to generate tree structures.

2 The multi-entrepreneurs model

2.1 The model

Let us imagine a production network of workers: we use the simplest structure of a lattice: at each node is localised a "worker" with a given production capacity of size 1. Business opportunities of size Q randomly strike "entrepreneur" sites at the surface of the lattice. The lattice can be of any dimension, but one dimension perpendicular to the surface, is oriented. Orders propagate inward from the entrepreneur, and production once achieved propagates outward (downstream). We then denote the lattice dimension as 1+1 for square lattices represented on figures 1, 2 and 6 but numerical simulations where also done in 1+2 and 1+3 dimensions.

The work load received by the entrepreneur is too large to be carried out by her: she then distributes it randomly to her nearest neighbours upstream, except for a work quantity of size one which she performs herself.

We here postulate that the choice reflects a re-enforcement learning mechanism. The entrepreneur uses a probabilistic choice rule based on preferences learned from past experience.

The probability of choosing neighbour j is given by the logit function:

$$p_j = \exp(\beta J_j) / \sum_{k=1}^{nb} \exp(\beta J_k) \tag{1}$$

where the sum extends to all subordinate neighbours of the node. The preferences J_j are updated at each time step according to:

$$J_j(t) = (1-\gamma) J_j(t-1) + q_j(t) \tag{2}$$

where $q_j(t)$ is the work load attributed to node j. Preferences are thus moving averages of the work load transfered across links. The rationale for equation (2) is that preferences reflect past gains obtained from the relation between the entrepreneur and the worker and that the entrepreneur gains are proportional to the work load. Equation 2 is the simplest form of a set of models where the gain term could be any function of $q_j(t)$, including eventually stochastic terms related the hazards of production.

Every worker receiving a workload re-distribute it according to the previous principles, playing himself the role of an entrepreneur. Whenever the received load is one, the worker performs it and he has no children nodes. The work load distribution then proceeds up to a limited depth, obviously bounded by Q.

One time step corresponds to the distribution of the initial work load across the set of collaborators of the entrepreneur who received it.

A series of work loads randomly strike the entrepreneur sites at successive time steps. We want to characterise the asymptotic structure of work loads and preferences generated by a large number of work loads presented in succession to the entrepreneurs.

2.2 Simulations

Let us start from a lattice structure submitted to a random flow of business opportunities striking randomly entrepreneur sites. Each business opportunity results in flows of work load distributed from the entrepreneur site across the lattice according to rules 1 and 2 given above.

The initial conditions are all sites with zero work load and all preferences $J = 0$. Under the influence of incoming work loads, one can observe online the progressive build-up of preferences and the evolution of work load repartition from initial blobs flowing from surface hits towards linear paths after a large number of steps. Stability of the patterns is obtained after a transient period of the order of LQ/γ where L is the lattice width. This expression comes from the fact that a surface site is hit on the average every $1/L$ step and the build-up of preferences necessitates Q/γ strikes.

Figure 1 displays the moving average of the work loads at each lattice site after the transient period. The moving average tells us how much work load was taken (and accumulated for the purpose of the display) by each site over a period of the order of $1/\gamma$.

Although all surface sites have been randomly and uniformly stricken by business opportunities, the accumulated work loads are far from

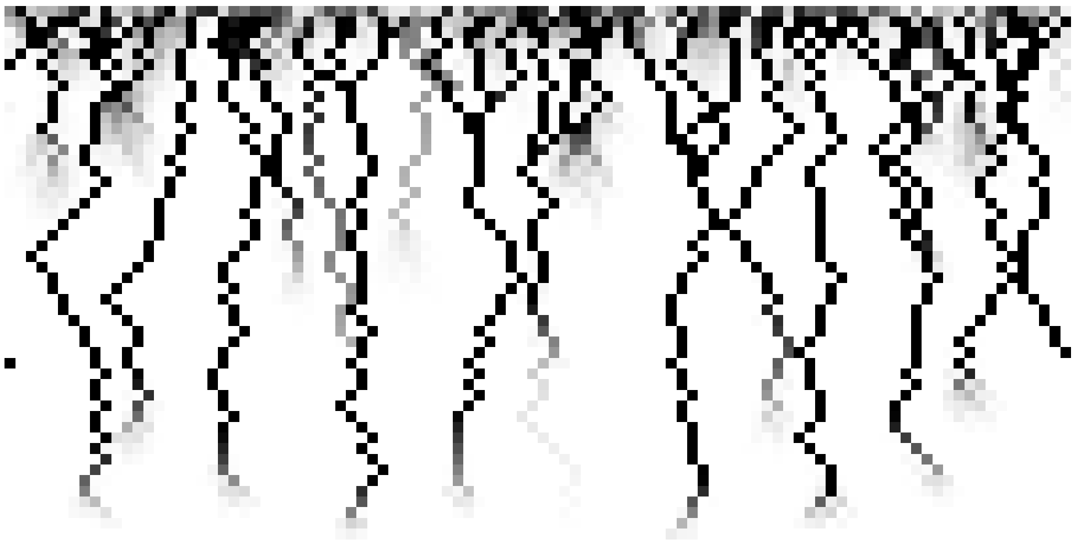

Figure 1: Moving average of work loads across the lattice after the transient period (100000 iterations per site). White sites correspond to zero load, gray levels to intermediate loads and black to maximum loads (50). Lattice depth=50, lattice width=100. $\gamma = 0.01, \beta = 0.2, Q = 50$.

uniform in depth. One notices linear paths of work distribution analogous to ants foraging trails; the analogy is not fortuitous, both structures result from similar re-enforcement learning mechanisms. More precisely the black (heavy loads) trails are very contrasted with respect to a background of unemployed nodes at intermediate depth. At extreme depth the light gray blobs correspond to nodes that are sometimes involved in work, but with less intensity and frequency. Just under the surface, inhomogeneities are less contrasted, due to the uniform probability of business opportunity arrivals; their structure resemble river basins where water is collected towards the main river of a valley. In our case, we are observing work flows and the underlying structure results from the shaping of the preference landscape by re-enforcement learning.

In terms of business, we interpret the work load trails as stable business relationships, establishing the conditions for the emergence of firms. Whenever business opportunities strike in their neighborhood, the same set of workers gets regularly involved sharing the work load. Closer to the surface, the divergence of trails reflects the fact that workloads are collected from different surface sources. In depth the weakening of the averaged work load is due to the convergence of two factors: total workloads are attenuated in depth and their repartition is more balanced. We will discuss in the next section a even more simplified model with a single entrepreneur receiving loads at the surface.

3 A single entrepreneur

3.1 Simulation results: snakes and blobs

The single entrepreneur model gives a clearer picture and quantitative results. The model is the same as in the previous section, except that a single site receives business opportunities of size Q.

Work load propagate across the lattice, and after a transition period of the order of several Q/γ, stable structures such as those represented on Figure 2 appear. The figure displays workloads obtained in a (1+1)-dimensional lattice at the final stage of one simulation.

The work load path starts from the surface with a deterministic region (the snake) such that each node has only one child-node. The same child always gets all the charge distributed at this stage. The path ends with a blob part, where the charge is distributed to 2 or 3 nodes. In the blob, the repartition fluctuates from one instance to the next one.

On figure 2 the snakes extends from the initial load of 20 to the load of 7 followed by a small blob of height 3. Parameters for this simulation were $\gamma = 0.3, \beta = 0.3$.

More generally, according to β, γ and load values, two dynamical regimes are observed: a quasi-deterministic regime such that only one link out of $2d+1$ is systematically chosen resulting in a "snake" portion of the work pattern, and a random regime where all 3 links are evenly used, resulting in a "blob" portion of the work pattern. The interface between the two regimes corresponds to

$$\beta(q(z) - 1)/\gamma = \text{Constant} \tag{3}$$

where $q(z) - 1$ is the work load distributed by a node receiving charge $q(z)$ at depth z (the -1 term corresponds to the fact that the distributing node takes a work load of 1 for himself to accomplish). Because

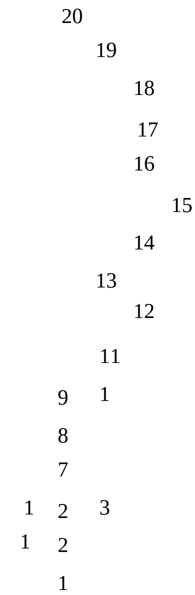

Figure 2: One instance of the work load repartition at equilibrium from an initial load of 20 at the top site. The load is first distributed with a strong preference for one neighbour out of three; it is further distributed more evenly starting from load 7. $\beta = 0.3, \gamma = 0.3, Q = 20$.

the work load to be distributed, $q(z) - 1$ decreases with increasing depth z, there is a given depth where the interface between the deterministic regime (the "snake") and the random regime (the "blob") is located as observed on figure 2.

3.2 The mean field solution

A mean field theory for multiple choices has been proposed in Weisbuch etal[8] and Nadal etal[9] to explain the two regimes and the transition.

At equilibrium equation 2 gives:

$$\gamma J_j = q_j \tag{4}$$

where J_j and q_j are equilibrium values. The mean field approximation consists in replacing q_j by its expectation $q.P_j$, where q is the charge to be distributed. We then obtain the following implicit equation for

J_j:

$$J_j = \frac{q}{\gamma} \frac{\exp(\beta J_j)}{\sum_{k=1}^{nb} \exp(\beta J_k)} \tag{5}$$

Solutions of equation 5, in fact of the system of equivalent equations for all J_j, is represented on figure 3, for a choice among 3 possible partners. We see that at lower β value, all three preferences are equal. But for larger β value, one preference is large and quickly reaches a maximum value of q/γ while the two others are small. The symmetric solution is unstable and separate three basins of attraction in which one preference is large and the two others are small.

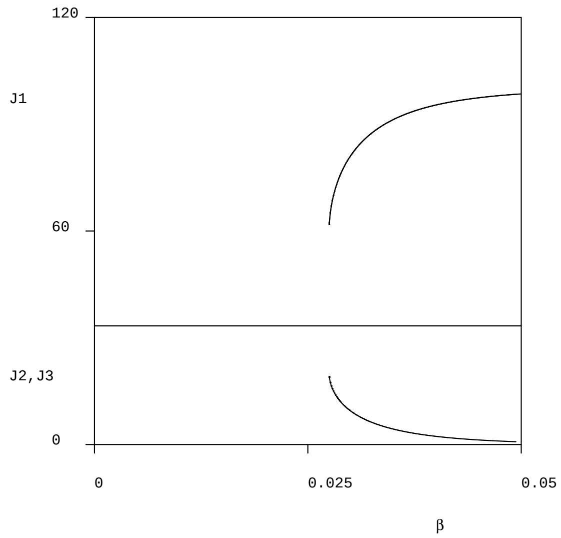

Figure 3: Solutions of the mean field equations (5), for 3 partners, $\gamma = 0.1, q = 10$. Below the critical value of $\beta = 0.03$, the three preferences are equal and constant. Above, one preference J_1 is large, and soon reaches the maximum value of q/γ, while the other two J_2, J_3 tend towards zero when β increases. In that region the symmetric root is unstable.

These results presented on figure 3 were obtained by numerical methods, but the transition point can be derived analytically. Summing equation 5 over all neighbours j yields:

$$\sum_{k=1}^{nb} J_j = q/\gamma \tag{6}$$

This equation has to be verified by any solution for the J_j; it is verified by the symmetric solution:

$$J_j = \frac{q}{\gamma \times nb} \tag{7}$$

which also satisfy equation 5. Is this solution stable? It is only the case when $(\beta \times q)/(\gamma \times nb)$, the derivative of the RHS of equation 5 with respect to J_j, is smaller than one (a graphical solution is based on checking staircases between the RHS expression and the first bisectrix).

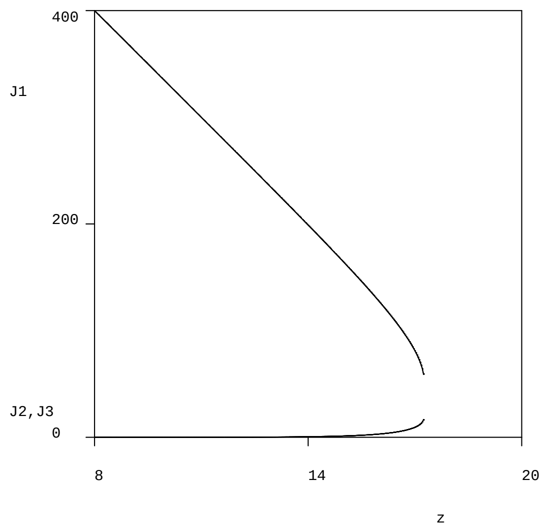

Figure 4: Solutions of the mean field equations (5) as a function of z for constant $\beta = 0.03$, and $\gamma = 0.03$, $Q = 20$. For lower z values, in the tail regime, the preferences are those obtained in the simulations (see figure 5).

In our case the number of choices nb is the number of neighbours $2d + 1$ in the lattice, and q at depth z is $q(z) - 1$. The transition between the head and tails regime at a depth z then obeys:

$$\frac{\beta * (q(z) - 1)}{\gamma} = 2d + 1 \tag{8}$$

For larger values of $q(z)$, all the work load is transfered to a single neighbour with a preference coefficient of $(q(z) - 1)/\gamma$, and all other coefficients are 0. For lower values of $q(z)$, all preference coefficients are equal, with possible fluctuations around the interface.

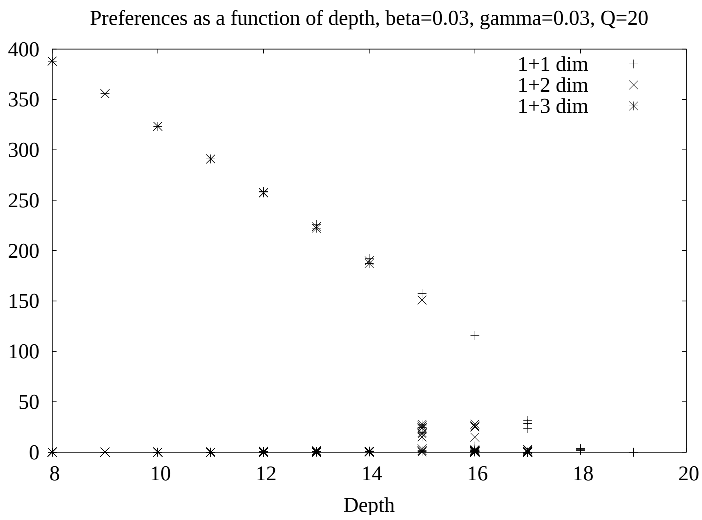

Figure 5: Evolution of the preference coefficient with distance from the "surface" observed in simulations. For smaller depths, in the tail region, the preference coefficients are either strong ($(q(z) - 1)/\gamma$) and independent of the dimension, or zero. Transition are observed around res. charges of 5, 4 and 3 rather than for $2d + 1 = 7$, 5 and 3 resp. as predicted by the mean field theory. $\beta = 0.03$, $\gamma = 0.03$, $d = 1, 2, 3$.

We plotted in figure 4 the theoretical solution of equation 6 when z varies; q in equation 6 is replaced by $q - 1 - z$. The theoretical curves are similar to the preferences observed in the simulations (figure 5) for the smaller values of z. These simulations where done in 1+1, 1+2 and 1+3 dimensions.

3.3 Partial conclusions: what is missing?

The simple reinforcement learning presented here does end-up in a stable path in the worker space represented here by the snake structure, which we interpret as a firm. On the other hand we would rather imagine firms as hierarchical structures such as trees [10, 11] . Because of the blob-snake sharp transition as a function of z, we never observe a well balanced tree with a selection at each node of several preferred collaborators, but rather either a nearly complete preference for one neighbour or roughly equal preference for all.

In conclusion, the present model explains the stability of employment relations in the firm, but something has to be added to it to explain the more balanced workload repartition observed in real firms.

4 Forcing link multiplicity

In real life link multiplicity arises for many reasons:

- A boss is able handle several subordinates and wishes it: several subordinates bring more profit to him. Furthermore they might have different skills and handle different tasks.
- The boss is risk averse to having a single subordinated line: in a production context, it is better for him not to face a single provider who would be in a strong position to get most of the added value of production; furthermore a single line has some chances to get disrupted: one would rather have several lines to lower chances of total shortage.

A short circuit to link multiplicity is to impose it from the beginning: we can simply extend our model by saying that each node recruits a minimum number of subordinate nodes: the standard algorithm (moving average of flux + logit choice) is then used, except that we limit the transfered charge to each sub-node to $(q-1)/m$, where m is the desired minimum number of subnodes. We thus impose a number of subnodes between m and the maximum number of virtual subnodes, say n.

In fact because, since each node ends up in having several children, some care has to be taken to avoid that nodes have several parents. We then started from a tree, rather than a lattice, as the underlying structure.

In figure 6, each node has $nb = 3$ possible children nodes, but a higher limit of $q/2$ for the transfered charged was imposed for each link. The figure represents the preference acquired after 1000 iteration steps, for $\gamma = 0.1, \beta = 0.1$, and $Q = 60$. The color code is black=60 for nodes and 30 for links, to white=0 for both. We observe that the nodes in the first layers display a strong preference for two links and that the load distribution starts being more balanced from the third layer.

Figure 6: One instance of the work load repartition on a 3-tree when fluxes across each link are limited to one half of the incoming charge. Darkness codes the magnitude of fluxes and workloads. $\beta = 0.1, \gamma = 0.1, Q = 60$.

The variation of the fluxes $J(i,j)$ as a function of depth is represented in figure 7 obtained for a 5-tree, namely a tree where each node is connected to 5 possible children. Again the same algorithm is used to reinforce connections, with a distribution of work load limited to $(q-1)/2$ at each subnode. (1000 iteration steps, $\lambda = 0.1, \beta = 0.1$, and $Q = 120$).

Figure 7 shows that at level resp. 1, 2 ,3 the number of non-zero fluxes are resp. 2, 4, 8, as compared to resp. 5, 25, 125, the total number of subnodes at these levels: the selection of two subnodes among five possible has then been achieved, and this selection is stable.

In conclusion, limiting the size of the work-load passed to subnodes allows to induce a tree-structure, but does not change the regime transition: when the work load is limited to $(q-1)/m$, it is divided into only m subnodes as long the workload is large enough to satisfy the ordered regime condition (equation 6). When the work load becomes too small, it is randomly distributed to all neighbours.

5 Conclusion

We have shown that simple re-enforcement learning coupled to a logit choice is sufficient to generate long term work relationships among historically selected sets of agents. We interpret these connected sets as prototypes for firms. A supplementary mechanism limiting the fraction of transfered work loads to subordinates allows to generalise to the familiar tree structure observed in real firms.

Equation 6 of the model predicts the conditions for the emergence of firms: large work loads (Q), selectivity of partner choice (large β) and frequent enough business opportunities (large $1/\gamma$). (For the last condition, we take into account the fact that the time between business opportunities was set to 1. More precisely, $1/\gamma$ is the ratio between the frequency of business opportunities and the rate of preference updating).

The model is admittedly very simple and does not takes into account many important features of modern economies such as the role of capital investment (machines), technological investment (in the widest sense), workers and services specialisation etc. Furthermore, business opportunities are here presented as independent, since workers recruitment and production are occurring sequentially. They are not in real life, and one function of the firm is the scheduling of many tasks in order to realise parallel processing of many orders.

But even as simple as it is, the present model is a good candidate to incorporate more features and to be applied to present day issues such as merger/acquisition or outsourcing.

Ackowledgments

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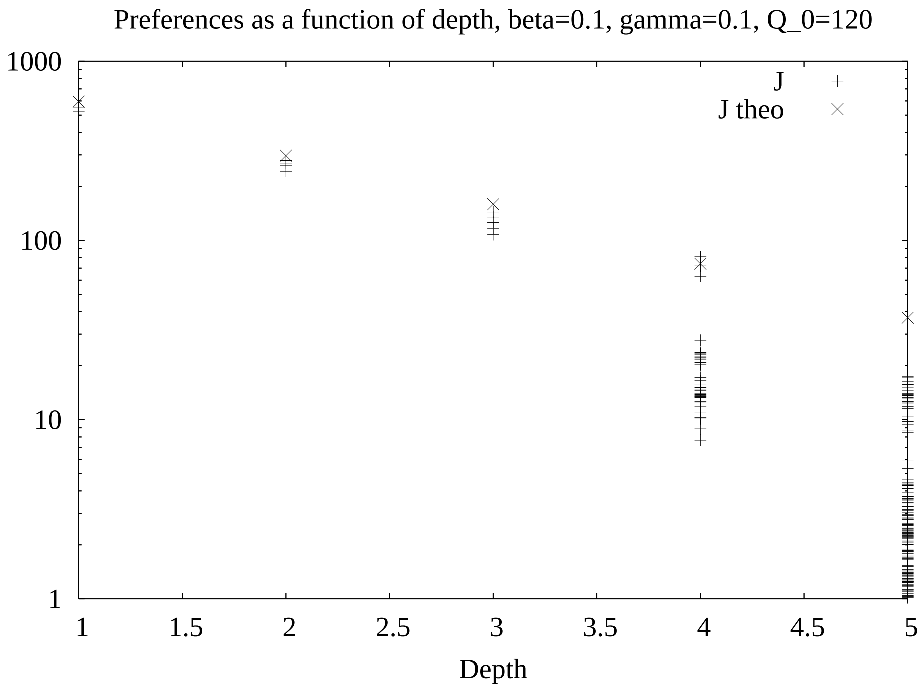

Figure 7: Preferences on a 5-tree as a function of depth z, in logarithmic scale. Fluxes are limited to one third of the incoming charge. + are the fluxes obtained from numerical simulation, x correspond to the theoretical prediction in the ordered regime. Fluxes less than one are not shown. $\beta = 0.1, \gamma = 0.1, Q = 120$.