

ference between the two is of more than semantic interest: in the list generated by Alg. 5.3 (`enumerate-ising`), we were able to pick out any information we wanted, for example the number of configurations of energy E and magnetization M , that is, the density of states $\mathcal{N}(E, M)$. In this subsection we discuss an alternative enumeration for the Ising model. It does not list the spin configurations, but rather all the loop configurations which appear in the high-temperature expansion of the Ising model. This program will then turn, in Subsection 5.1.4, into an enumeration of the second kind (Kac and Ward, 1954). It counts configurations and obtains $Z(\beta)$ for a two-dimensional Ising system of any size (Kaufman, 1949), and even for the infinite system (Onsager, 1944). However, it then counts without listing. For example, it finds the number $\mathcal{N}(E)$ of configurations with energy E but does not tell us how many of them have a magnetization M .

Van der Waerden, in 1941, noticed that the Ising-model partition function,

$$\begin{aligned} Z &= \sum_{\sigma} \exp\left(J\beta \sum_{\langle k,l \rangle} \sigma_k \sigma_l\right) \\ &= \sum_{\sigma} \prod_{\langle k,l \rangle} e^{J\beta \sigma_k \sigma_l}, \end{aligned} \quad (5.7)$$

allows each term $e^{J\beta \sigma_k \sigma_l}$ to be expanded and rearranged into just two terms, one independent of the spins and the other proportional to $\sigma_k \sigma_l$:

$$\begin{aligned} e^{\beta \sigma_k \sigma_l} &= 1 + \beta \sigma_k \sigma_l + \frac{\beta^2}{2!} \underbrace{(\sigma_k \sigma_l)^2}_{=1} + \frac{\beta^3}{3!} \underbrace{(\sigma_k \sigma_l)^3}_{=\sigma_k \sigma_l} + \cdots - \cdots \\ &= \underbrace{\left(1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots\right)}_{\cosh \beta} - \sigma_k \sigma_l \underbrace{\left(\beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \cdots\right)}_{\sinh \beta} \\ &= (\cosh \beta) (1 + \sigma_k \sigma_l \tanh \beta). \end{aligned}$$

Inserted into eqn (5.7), with $J = +1$, this yields

$$Z(\beta) = \sum_{\sigma} \prod_{\langle k,l \rangle} ((\cosh \beta) (1 + \sigma_k \sigma_l \tanh \beta)). \quad (5.8)$$

For concreteness, we continue with a 4×4 square lattice without periodic boundary conditions (with $J = 1$). This lattice has 24 edges and 16 sites, so that, by virtue of eqn (5.8), its partition function $Z_{4 \times 4}(\beta)$ is the product of 24 parentheses, one for each edge:

$$\begin{aligned} Z_{4 \times 4}(\beta) &= \sum_{\{\sigma_1, \dots, \sigma_{16}\}} \cosh^{24} \beta \overbrace{(1 + \sigma_1 \sigma_2 \tanh \beta)}^{\text{edge 1}} \overbrace{(1 + \sigma_1 \sigma_5 \tanh \beta)}^{\text{edge 2}} \\ &\quad \times \dots (1 + \sigma_{14} \sigma_{15} \tanh \beta) \underbrace{(1 + \sigma_{15} \sigma_{16} \tanh \beta)}_{\text{edge 24}}. \end{aligned} \quad (5.9)$$

We multiply out this product: for each edge (parenthesis) k , we have a choice between a “one” and a “tanh” term. This is much like the option of a spin-up or a spin-down in the original Ising-model enumeration, and can likewise be expressed through a binary variable n_k :

$$n_k = \begin{cases} 0 & (\equiv \text{edge } k \text{ in eqn (5.9) contributes } 1) \\ 1 & (\equiv \text{edge } k \text{ contributes } (\sigma_{s_k} \sigma_{s'_k} \tanh \beta)) \end{cases},$$

where s_k and s'_k indicate the sites at the two ends of edge k . Edge $k = 1$ has $\{s_1, s'_1\} = \{1, 2\}$, and edge $k = 24$ has, from eqn (5.9), $\{s_{24}, s'_{24}\} = \{15, 16\}$. Each factored term can be identified by variables

$$\{n_1, \dots, n_{24}\} = \{\{0, 1\}, \dots, \{0, 1\}\}.$$

For $\{n_1, \dots, n_{24}\} = \{0, \dots, 0\}$, each parenthesis picks a “one”. Summed over all spin configurations, this gives 2^{16} . Most choices of $\{n_1, \dots, n_{24}\}$ average to zero when summed over spin configurations because the same term is generated with $\sigma_k = +1$ and $\sigma_k = -1$. Only choices leading to spin products $\sigma_s^0, \sigma_s^2, \sigma_s^4$ at each lattice site s remain finite after summing over all spin configurations. The edges of these terms form loop configurations, such as those shown for the 4×4 lattice in Fig. 5.8. The list of all loop configurations may be generated by Alg. 5.5 (`edge-ising`), a recycled version of the Gray code for 24 digits, coupled to an incremental calculation of the number of spins on each site. The $\{o_1, \dots, o_{16}\}$ count the number of times the sites $\{1, \dots, 16\}$ are present. The numbers in this vector must all be even for a loop configuration, and for a nonzero contribution to the sum in eqn (5.9).

Table 5.4 Numbers of loop configurations in Fig. 5.8 with given numbers of edges (the figure contains one configuration with 0 edges, 9 with 4 edges, etc). (From Alg. 5.5 (`edge-ising`)).

# Edges	# Configs
0	1
4	9
6	12
8	50
10	92
12	158
14	116
16	69
18	4
20	1

procedure edge-ising

input $\{(s_1, s'_1), \dots, (s_{24}, s'_{24})\}$

$\{n_1, \dots, n_{24}\} \leftarrow \{0, \dots, 0\}$

$\{\tau_0, \dots, \tau_{24}\} \leftarrow \{1, \dots, 25\}$

$\{o_1, \dots, o_{16}\} \leftarrow \{0, \dots, 0\}$

output $\{n_1, \dots, n_{24}\}$

for $i = 1, 2^{24} - 1$ **do**

$k \leftarrow \text{gray-flip}(\{\tau_0, \dots, \tau_{24}\})$
 $n_k \leftarrow \text{mod}(n_k + 1, 2)$
 $o_{s_k} \leftarrow o_{s_k} + 2 \cdot n_k - 1$
 $o_{s'_k} \leftarrow o_{s'_k} + 2 \cdot n_k - 1$
if ($\{o_1, \dots, o_{16}\}$ all even) **then**
 output $\{n_1, \dots, n_{24}\}$

Algorithm 5.5 `edge-ising`. Gray-code enumeration of the loop configurations in Fig. 5.8. The edge k connects neighboring sites s_k and s'_k .

For the thermodynamics of the 4×4 Ising model, we only need to keep track of the number of edges in each configuration, not the configurations themselves. Table 5.4, which shows the number of loop configurations

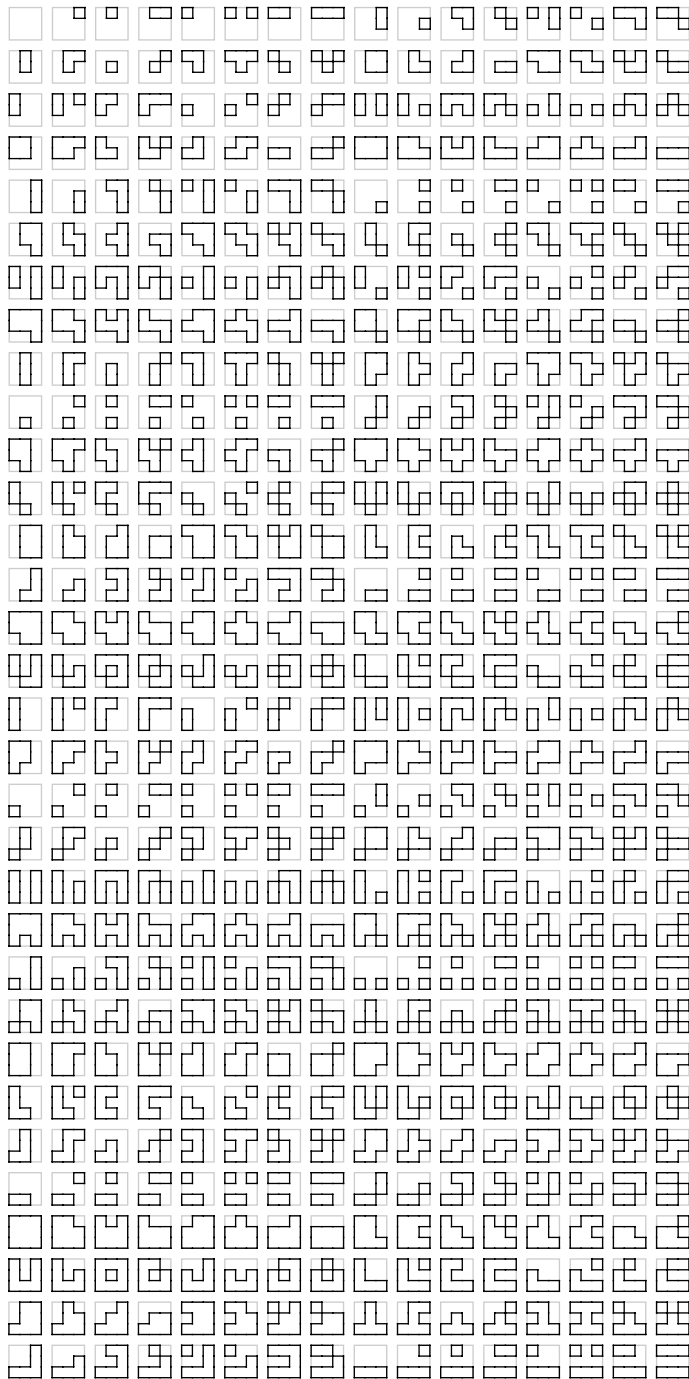


Fig. 5.8 The list of all 512 loop configurations for the 4×4 Ising model without periodic boundary conditions (from Alg. 5.5 (edge-ising)).

for any given number of edges, thus yields the exact partition function for the 4×4 lattice without periodic boundary conditions:

$$Z_{4 \times 4}(\beta) = (2^{16} \cosh^{24} \beta) (1 + 9 \tanh^4 \beta + 12 \tanh^6 \beta + \cdots + 4 \tanh^{18} \beta + 1 \tanh^{20} \beta). \quad (5.10)$$

Partition functions obtained from this expression are easily checked against the Gray-code enumeration.

5.1.4 Counting (not listing) loops in two dimensions

Following Kac and Ward (1952), we now construct a matrix whose determinant counts the number of loop configurations in Fig. 5.8. This is possible because the determinant of a matrix $U = (u_{kl})$ is defined by a sum of permutations P (with signs and weights). Each permutation can be written as a collection of cycles, a “cycle configuration”. Our task will consist in choosing the elements u_{kl} of the matrix U in such a way that the signs and weights of each cycle configurations correspond to the loop configurations in the two-dimensional Ising model. We shall finally arrive at a computer program which implements the correspondence, and effectively solves the enumeration problem for large two-dimensional lattices. For simplicity, we restrict ourselves to square lattices without periodic boundary conditions, and consider the definition of the determinant of a matrix U ,

$$\det U = \sum_{\text{permutations}} (\text{sign } P) u_{1P_1} u_{2P_2} \cdots u_{NP_N}.$$

We now represent P in terms of cycles. The sign of a permutation P of N elements with n cycles is $\text{sign } P = (-1)^{N+n}$ (see our detailed discussion in Section 1.2.2). In the following, we shall consider only matrices with even N , for which $\text{sign } P = (-1)^{\# \text{ of cycles}}$. The determinant is thus

$$\begin{aligned} \det U &= \sum_{\substack{\text{cycle} \\ \text{configs}}} (-1)^{\# \text{ of cycles}} \underbrace{u_{P_1 P_2} u_{P_2 P_3} \cdots u_{P_M P_1}}_{\text{weight of first cycle}} \underbrace{u_{P'_1 P'_2} \cdots}_{\text{other cycles}} \\ &= \sum_{\substack{\text{cycle} \\ \text{configs}}} \left(\left\{ \begin{array}{c} (-1) \cdot \text{weight of} \\ \text{first cycle} \end{array} \right\} \right) \times \cdots \times \left(\left\{ \begin{array}{c} (-1) \cdot \text{weight of} \\ \text{last cycle} \end{array} \right\} \right). \end{aligned}$$

It follows from this representation of a determinant in terms of cycle configurations that we should choose the matrix elements u_{kl} such that each cycle corresponding to a loop on the lattice (for example (P_1, \dots, P_M)) gets a negative sign (this means that the sign of $u_{P_1 P_2} u_{P_2 P_3} \cdots u_{P_M P_1}$ should be negative). All cycles not corresponding to loops should get zero weight.

We must also address the problem that cycles in the representation of the determinant are directed. The cycle $(P_1, P_2, \dots, P_{M-1}, P_M)$ is

different from the cycle $(P_M, P_{M-1}, \dots, P_2, P_1)$, whereas the loop configurations in Fig. 5.8 have no sense of direction.

For concreteness, we start with a 2×2 lattice without periodic boundary conditions, for which the partition function is

$$Z_{2 \times 2} = (2^4 \cosh^4 \beta) (1 + \tanh^4 \beta). \quad (5.11)$$

The prefactor in this expression (2^N multiplied by one factor of $\cosh \beta$ per edge) was already encountered in eqn (5.10). We can find naively a 4×4 matrix $\hat{U}_{2 \times 2}$ whose determinant generates cycle configurations which agree with the loop configurations. Although this matrix cannot be generalized to larger lattices, it illustrates the problems which must be overcome. This matrix is given by

$$\hat{U}_{2 \times 2} = \begin{bmatrix} 1 & \gamma \tanh \beta & \cdot & \cdot \\ \cdot & 1 & \cdot & \gamma \tanh \beta \\ \gamma \tanh \beta & \cdot & 1 & \cdot \\ \cdot & \cdot & \gamma \tanh \beta & 1 \end{bmatrix}.$$

(In the following, zero entries in matrices are represented by dots.) The matrix must satisfy

$$Z_{2 \times 2} = (2^4 \cosh^4 \beta) \det \hat{U}_{2 \times 2},$$

and because of

$$\det \hat{U}_{2 \times 2} = 1 - \gamma^4 \tanh^4 \beta,$$

we have to choose $\gamma = e^{i\pi/4} = \sqrt[4]{-1}$. The value of the determinant is easily verified by expanding with respect to the first row, or by naively going through all the 24 permutations of 4 elements (see Fig. 4.16 for a list of them). Only two permutations have nonzero contributions: the unit permutation $\begin{pmatrix} 1234 \\ 1234 \end{pmatrix}$, which has weight 1 and sign 1 (it has four cycles), and the permutation, $\begin{pmatrix} 2431 \\ 1234 \end{pmatrix} = (1, 2, 4, 3)$, which has weight $\gamma^4 \tanh^4 \beta = -\tanh^4 \beta$. The sign of this permutation is -1 , because it consists of a single cycle.

The matrix $\hat{U}_{2 \times 2}$ cannot be generalized directly to larger lattices. This is because it sets u_{21} equal to zero because $u_{12} \neq 0$, and sets $u_{13} = 0$ because $u_{31} \neq 0$; in short it sets $u_{kl} = 0$ if u_{lk} is nonzero (for $k \neq l$). In this way, no cycles with hairpin turns are retained (which go from site k to site l and immediately back to site k). It is also guaranteed that between a permutation and its inverse (in our case, between the permutation $\begin{pmatrix} 1234 \\ 1234 \end{pmatrix}$ and $\begin{pmatrix} 2431 \\ 1234 \end{pmatrix}$), at most one has nonzero weight. For larger lattices, this strategy is too restrictive. We cannot generate all loop configurations from directed cycle configurations if the direction in which the edges are gone through is fixed. We would thus have to allow both weights u_{kl} and u_{lk} different from zero, but this would reintroduce the hairpin problem. For larger N , there is no $N \times N$ matrix whose determinant yields all the loop configurations.

Kac and Ward's (1951) solution to this problem associates a matrix index, not with each lattice site, but with each of the four directions

Table 5.5 Correspondence between lattice sites and directions, and the indices of the Kac–Ward matrix U

Site	Direction	Index
1	→	1
	↑	2
	←	3
	↓	4
2	→	5
	↑	6
	←	7
⋮	↓	8
	⋮	⋮
	→	$4k - 3$
	↑	$4k - 2$
k	←	$4k - 1$
	↓	$4k$

only trivial cycles. It is characterized by an empty nontrivial cycle configuration c_1 . Other permutations with nonzero weights are

$$c_2 \equiv \begin{pmatrix} \text{site} & 1 & 2 & 4 & 3 \\ \text{dir.} & \rightarrow & \uparrow & \leftarrow & \downarrow \\ \text{index} & 1 & 6 & 15 & 12 \end{pmatrix}$$

and

$$c_3 \equiv \begin{pmatrix} \text{site} & 1 & 3 & 4 & 2 \\ \text{dir.} & \uparrow & \rightarrow & \downarrow & \leftarrow \\ \text{index} & 2 & 9 & 16 & 7 \end{pmatrix}.$$

Finally, the permutation c_4 is put together from the permutations c_2 and c_3 , so that we obtain

$$\begin{aligned} c_1 &\equiv 1, \\ c_2 &\equiv u_{1,6}u_{6,15}u_{15,12}u_{12,1} = \alpha^4 = -\tanh^4 \beta, \\ c_3 &\equiv u_{2,9}u_{9,16}u_{16,7}u_{7,2} = \bar{\alpha}^4 = -\tanh^4 \beta, \\ c_4 &\equiv c_2c_3 = \alpha^4\bar{\alpha}^4 = \tanh^8 \beta. \end{aligned}$$

We thus arrive at

$$\det U_{2 \times 2} = 1 + 2 \tanh^4 \beta + \tanh^8 \beta = \underbrace{(1 + \tanh^4 \beta)^2}_{\text{see eqn (5.11)}}, \quad (5.13)$$

and this is proportional to the square of the partition function in the 2×2 lattice (rather than the partition function itself).

The cycles in the expansion of the determinant are oriented: c_2 runs anticlockwise around the pad, and c_3 clockwise. However, both types of cycles may appear simultaneously, in the cycle c_4 . This is handled by drawing two lattices, one for the clockwise, and one for the anticlockwise cycles (see Fig. 5.10). The cycles $\{c_1, \dots, c_4\}$ correspond to all the loop configurations that can be drawn simultaneously in both lattices. It is thus natural that the determinant in eqn (5.13) is related to the partition function in two independent lattices, the square of the partition function of the individual systems.

Before moving to larger lattices, we note that the matrix $U_{2 \times 2}$ can be written in more compact form, as a matrix of matrices:

$$U_{2 \times 2} = \begin{bmatrix} \mathbb{1} & u_{\rightarrow} & u_{\uparrow} & \cdot \\ u_{\leftarrow} & \mathbb{1} & \cdot & u_{\uparrow} \\ u_{\downarrow} & \cdot & \mathbb{1} & u_{\rightarrow} \\ \cdot & u_{\downarrow} & u_{\leftarrow} & \mathbb{1} \end{bmatrix} \quad \left(\begin{array}{l} \text{a } 16 \times 16 \text{ matrix,} \\ \text{see eqn (5.15)} \end{array} \right), \quad (5.14)$$

where $\mathbb{1}$ is the 4×4 unit matrix, and furthermore, the 4×4 matrices

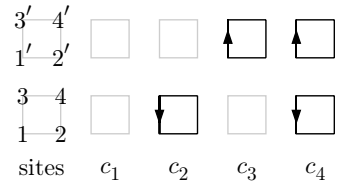


Fig. 5.10 Neighbor scheme and cycle configurations in two independent 2×2 Ising models.

$u_{\rightarrow}, u_{\uparrow}, u_{\leftarrow},$ and u_{\downarrow} are given by

$$\begin{aligned}
 u_{\rightarrow} &= \begin{bmatrix} \nu & \alpha & \cdot & \bar{\alpha} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, & u_{\uparrow} &= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \bar{\alpha} & \nu & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \\
 u_{\leftarrow} &= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{\alpha} & \nu & \alpha \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, & u_{\downarrow} &= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha & \cdot & \bar{\alpha} & \nu \end{bmatrix}.
 \end{aligned} \tag{5.15}$$

The difference between eqns (5.12) and (5.14) is purely notational.

The 2×2 lattice is less complex than larger lattices. For example, one cannot draw loops in this lattice which sometimes turn left, and sometimes right. (On the level of the 2×2 lattice it is unclear why left turns come with a factor α and right turns with a factor $\bar{\alpha}$.) This is what we shall study now, in a larger matrix. Cycle configurations will come up that do not correspond to loop configurations. We shall see that they sum up to zero.

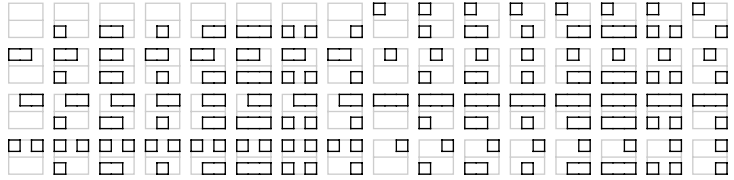


Fig. 5.11 All 64 loop configurations for two uncoupled 4×2 Ising models without periodic boundary conditions (a subset of Fig. 5.8).

For concreteness, we consider the 4×2 lattice (without periodic boundary conditions), for which the Kac–Ward matrix can still be written down conveniently. We understand by now that the matrix and the determinant describe pairs of lattices, one for each sense of orientation, so that the pair of 4×2 lattices corresponds to a single 4×4 lattice with a central row of links eliminated. The 64 loop configurations for this case are shown in Fig. 5.11. We obtain

$$U_{4 \times 2} = \begin{bmatrix} \mathbb{1} & u_{\rightarrow} & \cdot & \cdot & u_{\uparrow} & \cdot & \cdot & \cdot \\ u_{\leftarrow} & \mathbb{1} & u_{\rightarrow} & \cdot & \cdot & u_{\uparrow} & \cdot & \cdot \\ \cdot & u_{\leftarrow} & \mathbb{1} & u_{\rightarrow} & \cdot & \cdot & u_{\uparrow} & \cdot \\ \cdot & \cdot & u_{\leftarrow} & \mathbb{1} & \cdot & \cdot & \cdot & u_{\uparrow} \\ u_{\downarrow} & \cdot & \cdot & \cdot & \mathbb{1} & u_{\rightarrow} & \cdot & \cdot \\ \cdot & u_{\downarrow} & \cdot & \cdot & u_{\leftarrow} & \mathbb{1} & u_{\rightarrow} & \cdot \\ \cdot & \cdot & u_{\downarrow} & \cdot & \cdot & u_{\leftarrow} & \mathbb{1} & u_{\rightarrow} \\ \cdot & \cdot & \cdot & u_{\downarrow} & \cdot & \cdot & u_{\leftarrow} & \mathbb{1} \end{bmatrix}. \tag{5.16}$$

Written out explicitly, this gives a 32×32 complex matrix $U_{4 \times 2} = (u_{k,l})$


```

procedure combinatorial-ising
input  $\{u_{\rightarrow}, u_{\uparrow}, u_{\leftarrow}, u_{\downarrow}\}$  (see eqn (5.15))
 $\{U(j, j')\} \leftarrow \{0, \dots, 0\}$ 
for  $k = 1, \dots, N$  do
  for  $n = 1, \dots, 4$  do
     $j \leftarrow 4 \cdot (k - 1) + n$ 
     $U(j, j) \leftarrow 1$ 
    for  $n' = 1, \dots, 4$  do
       $k' \leftarrow \text{Nbr}(1, k)$ 
      if  $(k' \neq 0)$  then
         $j' \leftarrow 4 \cdot (k' - 1) + n'$ 
         $U(j, j') \leftarrow u_{\rightarrow}(n, n')$ 
       $k' \leftarrow \text{Nbr}(2, k)$ 
      if  $(k' \neq 0)$  then
         $j' \leftarrow 4 \cdot (k' - 1) + n'$ 
         $U(j, j') \leftarrow u_{\uparrow}(n, n')$ 
       $k' \leftarrow \text{Nbr}(3, k)$ 
      if  $(k' \neq 0)$  then
         $j' \leftarrow 4 \cdot (k' - 1) + n'$ 
         $U(j, j') \leftarrow u_{\leftarrow}(n, n')$ 
       $k' \leftarrow \text{Nbr}(4, k)$ 
      if  $(k' \neq 0)$  then
         $j' \leftarrow 4 \cdot (k' - 1) + n'$ 
         $U(j, j') \leftarrow u_{\downarrow}(n, n')$ 
output  $\{U(j, j')\}$ 

```

Algorithm 5.6 *combinatorial-ising*. The $4N \times 4N$ matrix U , for which $\sqrt{\det U} \propto Z(\beta)$ (Ising model without periodic boundary conditions).

On larger than 4×2 lattices, there are more elaborate loops. They can, for example, have crossings (see, for example, the loop in Fig. 5.13). There, the cycle configurations c_1 and c_2 correspond to loops in the generalization of Fig. 5.11 to larger lattices, whereas the cycles c_3 and c_4 are superfluous. However, c_3 makes six left turns and two right turns, so that the overall weight is $\alpha^4 = -1$, whereas the cycle c_4 makes three left turns and three right turns, so that the weight is $+1$, the opposite of that of c_3 . The weights of c_3 and c_4 thus cancel.

For larger lattices, it becomes difficult to establish that the sum of cycle configurations in the determinant indeed agrees with the sum of loop configurations of the high-temperature expansion, although rigorous proofs exist to that effect. However, at our introductory level, it is more rewarding to proceed heuristically. We can, for example, write down the 144×144 matrix $U_{6 \times 6}$ of the 6×6 lattice for various temperatures (using Alg. 5.6 (*combinatorial-ising*)), and evaluate the determinant $\det U_{6 \times 6}$ with a standard linear-algebra routine. Partition functions thus obtained are equivalent to those resulting from Gray-code enumeration, even though the determinant is evaluated in on the order

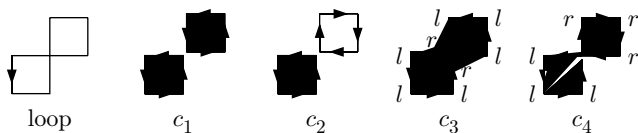


Fig. 5.13 Loop and cycle configurations. The weights of c_3 and c_4 cancel.

of $144^3 \simeq 3 \times 10^6$ operations, while the Gray code goes over $2^{35} \simeq 3 \times 10^{10}$ configurations. The point is that the determinant can be evaluated for lattices that are much too large to go through the list of all configurations.

The matrix $U_{L \times L}$ for the $L \times L$ lattice contains the key to the analytic solution of the two-dimensional Ising model first obtained, in the thermodynamic limit, by Onsager (1944). To recover Onsager's solution, we would have to compute the determinant of U , not numerically as we did, but analytically, as a product over all the eigenvalues. Analytic expressions for the partition functions for Ising models can also be obtained for finite lattices with periodic boundary conditions. To adapt for the changed boundary conditions, one needs four matrices, generalizing the matrix U (compare with the analogous situation for dimers in Chapter 6). Remarkably, evaluating $Z(\beta)$ on a finite lattice reduces to evaluating an explicit function (see Kaufman (1949) and Fisher and Ferdinand (1969); see also Exerc. 5.9).

The analytic solutions of the Ising model have not been generalized to higher dimensions, where only Monte Carlo simulations, high-temperature expansions, and renormalization-group calculations allow to compute to high precision the properties of the phase transition. These properties, as mentioned, are universal, that is, they are the same for a wide class of systems, called the Ising universality class.

5.1.5 Density of states from thermodynamics

The direct and indirect enumeration algorithms in this chapter differ in the role played by the density of states. In Alg. 5.3 (`enumerate-ising`), it was appropriate to first compute $\mathcal{N}(E)$, and later determine partition functions, internal energies, and specific heat capacities at any temperature, in $\propto N$ operations. In contrast, the indirect enumerations in Section 5.1.4 determine the partition function $Z(\beta)$, not the density of states. Computing $Z(\beta)$ from $\mathcal{N}(E)$ is straightforward, but how to recover $\mathcal{N}(E)$ from $Z(\beta)$ requires some thought:

$$\mathcal{N}(E) \begin{array}{c} \xrightarrow{\text{Subsection 5.1.2}} \\ \xleftarrow{\text{this subsection}} \end{array} Z(\beta).$$

The mathematical problem of the present section is common to many basic problems in statistical and solid state physics, and appears also in the interpretation of experimental or Monte Carlo data. In the presence