Transfer Matrix for the 2 x M Ising model (stripe of height 2 without periodic boundary conditions in the y-direction).
Material for the 6th ENS-ICFP lecture on Statistical Physics, 14 October 2019 (Werner Krauth).

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T = {{Exp[3 K], 1, 1, Exp[-K]}, {1, Exp[K], Exp[-3 K], 1},
  { 1, Exp[-3 K], Exp[K], 1}, {Exp[-K], 1, 1, Exp[3 K]}}
```

$$\left\{\left\{e^{3K}, 1, 1, e^{-K}\right\}, \left\{1, e^{K}, e^{-3K}, 1\right\}, \left\{1, e^{-3K}, e^{K}, 1\right\}, \left\{e^{-K}, 1, 1, e^{3K}\right\}\right\}$$

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T.T
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$$\begin{aligned}&\left\{\left\{2+e^{-2K}+e^{6K}, e^{-3K}+e^{-K}+e^{K}+e^{3K}, e^{-3K}+e^{-K}+e^{K}+e^{3K}, 2+2e^{2K}\right\},\right.\\ &\left\{e^{-3K}+e^{-K}+e^{K}+e^{3K}, 2+e^{-6K}+e^{2K}, 2+2e^{-2K}, e^{-3K}+e^{-K}+e^{K}+e^{3K}\right\},\\ &\left\{e^{-3K}+e^{-K}+e^{K}+e^{3K}, 2+2e^{-2K}, 2+e^{-6K}+e^{2K}, e^{-3K}+e^{-K}+e^{K}+e^{3K}\right\},\\ &\left.\left\{2+2e^{2K}, e^{-3K}+e^{-K}+e^{K}+e^{3K}, e^{-3K}+e^{-K}+e^{K}+e^{3K}, 2+e^{-2K}+e^{6K}\right\}\right\}\end{aligned}$$

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Eigenvalues[T]
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$$\begin{aligned}&\left\{e^{-3K}\left(-1+e^{4K}\right), e^{-K}\left(-1+e^{4K}\right),\right.\\ &\frac{1}{2}e^{-3K}\left(1+e^{2K}+e^{4K}+e^{6K}-\left(1+e^{2K}\right)\sqrt{1-4e^{2K}+10e^{4K}-4e^{6K}+e^{8K}}\right),\\ &\left.\frac{1}{2}e^{-3K}\left(1+e^{2K}+e^{4K}+e^{6K}+\left(1+e^{2K}\right)\sqrt{1-4e^{2K}+10e^{4K}-4e^{6K}+e^{8K}}\right)\right\}\end{aligned}$$

```
V2 = {{Exp[2 K], 1, 1, Exp[-2 K]}, {1, Exp[2 K], Exp[-2 K], 1},
  {1, Exp[-2 K], Exp[2 K], 1}, {Exp[-2 K], 1, 1, Exp[2 K]}}
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$$\left\{\left\{e^{2K}, 1, 1, e^{-2K}\right\}, \left\{1, e^{2K}, e^{-2K}, 1\right\}, \left\{1, e^{-2K}, e^{2K}, 1\right\}, \left\{e^{-2K}, 1, 1, e^{2K}\right\}\right\}$$

```
V1sq = {{Exp[K / 2], 0, 0, 0},
  {0, Exp[-K / 2], 0, 0}, {0, 0, Exp[-K / 2], 0}, {0, 0, 0, Exp[K / 2]}}
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$$\left\{\left\{e^{K/2}, 0, 0, 0\right\}, \left\{0, e^{-K/2}, 0, 0\right\}, \left\{0, 0, e^{-K/2}, 0\right\}, \left\{0, 0, 0, e^{K/2}\right\}\right\}$$

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V1sq . V2 . V1sq
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$$\left\{\left\{e^{3K}, 1, 1, e^{-K}\right\}, \left\{1, e^{K}, e^{-3K}, 1\right\}, \left\{1, e^{-3K}, e^{K}, 1\right\}, \left\{e^{-K}, 1, 1, e^{3K}\right\}\right\}$$

Now use periodic boundary conditions

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V1sq = {{Exp[K], 0, 0, 0}, {0, Exp[-K], 0, 0}, {0, 0, Exp[-K], 0}, {0, 0, 0, Exp[K]}}
```

$$\left\{\left\{e^{K}, 0, 0, 0\right\}, \left\{0, e^{-K}, 0, 0\right\}, \left\{0, 0, e^{-K}, 0\right\}, \left\{0, 0, 0, e^{K}\right\}\right\}$$

$$\left\{\left\{e^{K}, 0, 0, 0\right\}, \left\{0, e^{-K}, 0, 0\right\}, \left\{0, 0, e^{-K}, 0\right\}, \left\{0, 0, 0, e^{K}\right\}\right\}$$

```
TT = V1sq . V2 . V1sq
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$$\left\{\left\{e^{4K}, 1, 1, 1\right\}, \left\{1, 1, e^{-4K}, 1\right\}, \left\{1, e^{-4K}, 1, 1\right\}, \left\{1, 1, 1, e^{4K}\right\}\right\}$$

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B = TT . TT
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$$\begin{aligned}&\left\{\left\{3+e^{8K}, 2+e^{-4K}+e^{4K}, 2+e^{-4K}+e^{4K}, 2+2e^{4K}\right\},\right.\\ &\left\{2+e^{-4K}+e^{4K}, 3+e^{-8K}, 2+2e^{-4K}, 2+e^{-4K}+e^{4K}\right\},\\ &\left\{2+e^{-4K}+e^{4K}, 2+2e^{-4K}, 3+e^{-8K}, 2+e^{-4K}+e^{4K}\right\},\\ &\left.\left\{2+2e^{4K}, 2+e^{-4K}+e^{4K}, 2+e^{-4K}+e^{4K}, 3+e^{8K}\right\}\right\}\end{aligned}$$

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Eigenvalues[TT]
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$$\left\{e^{-4K}\left(-1+e^{4K}\right),\ -1+e^{4K},\ \frac{1}{2}e^{-4K}\left(1+2e^{4K}+e^{8K}-\sqrt{1+14e^{8K}+e^{16K}}\right),\ \frac{1}{2}e^{-4K}\left(1+2e^{4K}+e^{8K}+\sqrt{1+14e^{8K}+e^{16K}}\right)\right\}$$