C

~~Notice that we~~

System of two vortices (see homework.

$$\epsilon_{8\sqrt{2}} = 26{,}383322$$
$$\epsilon_{4\sqrt{2}} = 22{,}971028$$
$$3{,}412294$$

$$U = -\pi J_K \left[\log 8\sqrt{2} - \log 4\sqrt{2}\right]$$

$$\Delta U = -\pi J_e \cdot \log 2. = 3.41229$$

Conclusion

$$g_2(r) \sim \text{~~exp~~}\; r^{(-k_BT/2\pi I)}$$

spin waves.

$\Pi(r)$
↑
distance of vortices

$$\sim \left(\frac{r_{ij}}{a}\right)^{-(\beta\pi\, J_R)}$$

$$\sim \underline{r^{-\frac{1}{T}\pi J_R}}$$

Exact results:

Fröhlich & Spencer 1981: Rigorous proof of the existence of the BKT Phase.

Nonuniversality

2J fixes Energy Scale:

For $k_BT \gg 2J$

$T_{\text{FIRST ORDER}} \sim J$

$T_{KT} \propto J_R$

$$F_R = \frac{1}{2} J_R (\nabla\theta)^2$$

~~Domany~~

Domany, Schick, Swendsen (1984)

Lecture 11

Kosterlitz-Thouless physics in two dimensions: KTHNY Melting theory (Transitions without order parameters 2/2)

Keim, Maret
von Gruenberg
PRE 2007.

LECTURE 8 MASTER 1 ICFP

Physics in two dimensions – Kosterlitz Thouless Physics 2/2

Literature:
Kosterlitz 2016
Kosterlitz & Thouless 1973
Kosterlitz 1974
Mermin 1968
Halperin Nelson 1978
Young 1979
Nelson & Kosterlitz 1977

Two-dimensional solid and superfluids

1) a) 1973 Kosterlitz–Thouless paper → Motivation
b) The crystalline state.

2) Review of lecture 7.:

Two harmonic models:

At low temperature, spin waves is all there is

I harmonic approximation to XY model
Wegner 1973

$\leftarrow$ 1D: $g(r) = \langle \cos(\varphi_0 - \varphi_{\vec r}) \rangle \sim \exp(-k_B T \frac{|r|}{2 J})$

NB Vortices $r^{-\beta\pi J_R}$ $\leftarrow$ 2D: $g(r) \sim r^{-k_B T}$

3D: $g(r) \underset{r\to\infty}{\sim} \exp(-k_B T f_3(\infty)/I)$ ....

II HARMONIC SOLID

Jancovici (1973)

The harmonic solid has no phase transition, but it has logarithmic correlations in 2d

Peierls, Landau!
JANCO: It is pointed out that the classical two-dimensional harmonic solid exhibits an infinite generalized susceptibility at low temperature, although there is no long-range order and no phase transition.

$\langle u_k^2 \rangle \omega_k^2 \sim k_B T$

$\langle u_k^2 \rangle \sim \frac{k_B T}{\omega_k^2} \sim \frac{k_B T}{k^2}$

$$\langle \Delta r^2 \rangle \sim k_B T \int_{k \geq 1/R}^{\leq 1/a} \frac{1 - \cos \vec k \vec R}{k^2} \, d^d k$$

$$\propto k_B T \int_{k \geq 1/R} k^{d-3} \, dk$$

$\propto k T R$ — $d = 1$
$\propto k T \log R$ — $d = 2$
$k T$ const — $d \geq 3$.

At low temperature, phonon is all there is.

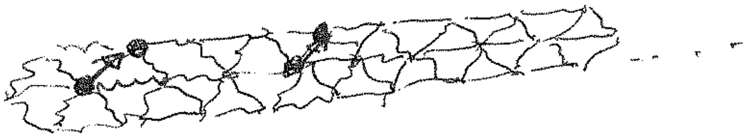
N.D. MERMIN (1968)

CRYSTALLINE ORDER IN TWO DIMENSIONS

$$\langle (u(\vec{R}) - u(\vec{R}'))^2 \rangle \sim \log |\vec{R} - \vec{R}'|$$

but

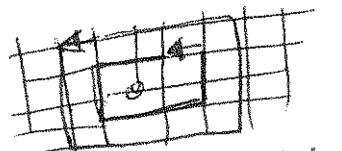
$$[\vec{r}(\vec{R}+\vec{a}_i) - \vec{r}(\vec{R})][\vec{r}(\vec{R}'+\vec{a}_i) - \vec{r}(\vec{R}')] \sim \text{const}$$



This shows that the harmonic solid, in two dimensions has quasi-long range positional order but TRUE long-range orientational order.

Kosterlitz - Thouless (1973). The "paper" but quite buggy.

Starting point: Dislocations:



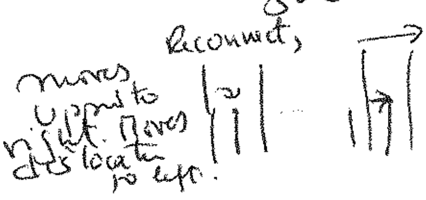
Just like in XY model

When there is a single dislocation, the strain

A dislocation in 2d is a point. (but with a Burgers vector, that is a lattice vector)

Free dislocation: no resistance to shear.

1934: Plastic deformations are to be explained in terms of dislocations. Dislocations are the "carriers" of plastic deformation.



The energy of an isolated dislocation with Burger's vector of magnitude b in a system of area A

$$E_{disloc} \sim \frac{\nu b^2 (1+\varepsilon)}{4\pi} \log \frac{A}{A_0}$$

Strain produced is inversely proportional to the distance from the dislocation, just like for the XY model.

This gives the critical temperature of the solid-liquid phase transition, in a similar way as it gives the XY transition temperature.

In XY model, the KT "theory" was essentially

Nobel-prize winning, but very wrong. ⚠

MAIN TRICK OF KT:

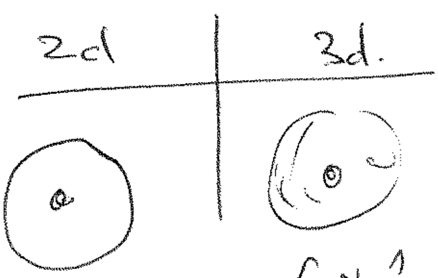
~~TRANSFORM~~ FORMULATE everything in terms of $\boxed{U(r_{ij}) = q_i q_j \log(r_{ij})}$.

As we did last week.

This is called coulomb potential in 2 dimensions.

2 dimensional coulomb gas.

Coulomb



$2\pi r E = 2\pi$

$\vec{E} \sim \frac{1}{r}$

$U(r) \sim \log r$

$E \sim \frac{1}{r^2}$

$U(r) \sim \frac{1}{r}$.

And it Really describes KT physics of the XY model, as it is this way.

HALPERIN, NELSON (1978, 1979)

Young (1979) → In the Kosterlitz–Thouless theory the harmonic approximation is exact at all temperatures below T_c, provided ~~that~~ one only investigates fluc

Melting and the vector Coulomb gas in two dimensions

2]

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_D$$

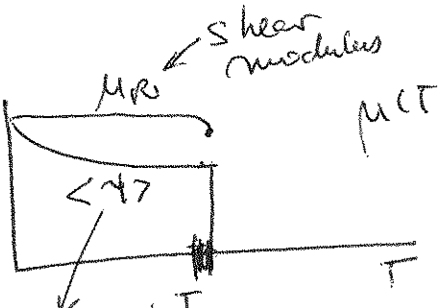
($\mathcal{H}_0$ ↑ phonon field; $\mathcal{H}_D$ ↑ dislocations)

1] A dislocation can be regarded as a tightly coupled pair of disclinations HN 78

$$\mathcal{H}_D = -\frac{1}{8\pi} \sum_{R,R'} K_1 \vec{b}(\vec{R}) \vec{b}(\vec{R}') \ln\left(\frac{|\vec{R}-\vec{R}'|}{a}\right) + K_2 . \frac{\vec{b}(\vec{R})\cdot(\vec{R}-\vec{R}') \cdot \vec{b}(\vec{R}')(\vec{R}-\vec{R}')}{(R-R')^2}.$$

Theory of dislocations.
NABARRO 1967

$K_2 = 0$: object of initial study by Nelson 1978



$$\mu(T) = \mu T(m) \left[1 + \text{const} (T_m - T)^{0,36963\ldots}\right].$$

* Picture of a free dislocation to show in ...

Note that the orientation is not disrupted.

* Picture of a dislocation pair

~~below~~ Above T_m, there are free dislocations.
This means that to perform shear, one can create pairs of dislocations.

→ More detailed study shows that the correlations are ~~not~~ algebraically decaying....

Hexatic phase → (Halperin, Nelson 1978).

Free dislocations destroy positional order

$$H_{hex} \sim \frac{1}{2} K_A \int [\nabla \theta(r)]^2 d^2r.$$

↑ Frank constant

It is this phase that is analogous to the low temperature phase of the XY model...

Behind this is a hexatic-liquid transition which is no longer...

+ Picture of a negative disclination (7 neighbors)

+ Picture of a positive disclination (5 neighbors).

Is the KTHNY scenario truly realized in two-dimensional particle systems?

⇒ <u>hard disks</u>: No

⇒ <u>soft disks</u>: It depends

+ improvisation

Lecture 12

The renormalization group - an introduction

The present chapter is inspired by the paper by Maris and Kadanoff [23]. Let's resume the situation of equilibrium statistical mechanics, and second-order phase transitions:

- Phase transitions are non-analyticities of the free energy with respect to parameters as the temperature, the magnetic field, etc. This is what we studied in earlier chapters using the transfer matrix. We studied in detail how these non-analyticities can arise from the transfer-matrix formulation.
- Landau theory states that the free energy *is* analytic in parameters such as M, ∇M, T, H, etc. The mechanism of how nature constructs a non-analyticity from an analytic function is symmetry breaking: The correct value of the free energy is the minimum of the analytic function. Above T_c, for example, this minimum is at $M = 0$, but below T_c, this is no longer the case.
- In the chapter on the Ginzburg criterium, we have seen that the predictions of mean-field theory (in other words, of Landau theory) are self-consistent above a critical dimension, but below this dimension they are not consistent, and therefore they are wrong. This has to be understood.
- Finally, there is the subject of universality: Many microscopic models have the same critical exponents, which is a subject to be understood.

Revolutionary clarification was brought about by the renormalization group.

Let us take the example of the one-dimensional Ising model, with the partition function

$$Z = \sum_{\boldsymbol{\sigma}} \exp\left[K\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \cdots +\right)\right] \tag{12.1}$$

This can be written as

$$= \sum_{\sigma_1,\sigma_3,\sigma_5\ldots} \left\{\exp\left[k\left(\sigma_1+\sigma_3\right)\right] + \exp\left[-k\left(\sigma_1+\sigma_3\right)\right]\right\}\left\{\exp\left[k\left(\sigma_3+\sigma_5\right)\right] + \exp\left[-k\left(\sigma_3+\sigma_5\right)\right]\right\}, \tag{12.2}$$

where we have written out explicitly the $\pm$ terms of σ_2, σ_4, etc.

Each of the terms in the above product equals terms as in $2\cosh\left[k\left(\sigma_1+\sigma_3\right)\right]$, which can be written as $f(k)\exp\left(k'\sigma_1\sigma_3\right)$, where $f(k)$ does not depend on σ_1 nor on σ_3. The solution of this little equation is

$$\sigma_1=\sigma_3: \quad \mathrm{e}^{2k}+\exp-2k=f\mathrm{e}^{k'} \tag{12.3}$$

$$\sigma_1=-\sigma_3: \quad 2=f\mathrm{e}^{-k'} \tag{12.4}$$

$$\tag{12.5}$$

with the solution $f^2=2\left[\exp(2k)+\exp(-2k)\right]=4\cosh(2k)$ so that we have $f=2\cosh^{1/2}(2k)$. Entering this into the previous equation, we find $1=\cosh^{1/2}(2k)\exp(-k')$ or, in other words:

$$k'=\frac{1}{2}\log\left[\cosh(2k)\right] \tag{12.6}$$

... and we have the exact relationship:

$$Z=f(k)^{N/2}\mathrm{e}^{k'(\sigma_1\sigma_3+\sigma_3\sigma_5+\sigma_5\sigma_7+\cdots+)} \tag{12.7}$$

The structure of this equation is $Z(N,k)=f(k)^{N/2}Z(N/2,k')$. Let us now write $\log Z=N\xi$, which takes us to

$$\xi(k)=\frac{1}{2}\log f(k)=\frac{1}{2}\xi(k') \tag{12.8}$$

$$\xi(k')=2\xi(k)-\log\left[2\cosh^{1/2}(2k)\right] \tag{12.9}$$

If the partition function is known for one value of k (or for the temperature T), then it is konwn for another value of k, namely k', where it corresponds to the identical Ising model on a twice larger lattice. From eq. (**??**), we see that $k'=\frac{1}{2}\log\left[\cosh(2k)\right]<k$ for $k>0$. This means that the "flow goes towards smaller k" or, in other words, towards $T\to\infty$.

It is interesting that we may write the recursion relation for the free energy per particle going from $k'\to k$. This leads us to

$$\xi(k)=\frac{1}{2}\log 2+\frac{1}{2}k'+\frac{1}{2}\xi k' \tag{12.10}$$

Now, at $k'=0.01$, we can expect that the free energy per particle equals $\log 2$ to extremely good approximation. We then deduce the free energy per particle at temperature $k=0.100334$, etc. where we find that the free energy per particle is 0.698147 instead of the exact value 0.698172, and eventually, at $k=2.702146$, we find a $\xi=2.706633$ instead of the exact 2.706634.

The flow diagram which gives k' as a function of k shows how k moves under recursions, and we notice that there are only two points where $k'=k$, namely 0 and ∞. $k=0$ is a stable fix point and $k=\infty$ an unstable one.

We now use the same procedure for the two-dimensional Ising model, leaving out every other spin in the operation that resembles our approach in one dimensions. Applying a diagonal elimination, at each spin (that we call 0), we have four neighboring spins $\sigma_1,\sigma_2,\sigma_3,\sigma_4$.

$$\mathrm{e}^{k(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}+\mathrm{e}^{-k(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} \tag{12.11}$$

Now it would be great if one could write this in terms of $\sigma_1\sigma 2$ and $\sigma_2\sigma_3$ and so on, but this is impossible. What is possible is to write

$$Z = f(k)^{N/2} \sum \exp\left(k_1 \sum_{nn} \sigma_p\sigma_q + k_2 \sum_{nnn} \sigma_p\sigma_q + k_3 \sum_{plaquette} \sigma_p\sigma_q\sigma_r\sigma_s\right) \tag{12.12}$$

where

$$f(k) = 2\cosh^{1/2}(2k)\cosh^{1/8}(4k) \tag{12.13}$$

$$k_1 = \frac{1}{4}\log\left[\cosh(4k)\right] \tag{12.14}$$

$$k_2 = \frac{1}{8}\log\left[\cosh(4k)\right] \tag{12.15}$$

$$k_3 = \frac{1}{8}\log\left[\cosh(4k)\right] - \frac{1}{2}\log\left[\cosh(2k)\right] \tag{12.16}$$

(Here we see that k_1 is twice larger than k_2, because the nn terms arise from two plaquettes rather than from a single one.)

We see that on a larger scale, the Ising model on a two-dimesional square lattice is not described through an Ising model.

What to do?

1/ we could simply neglect the values of k_2 and of k_3, in order to force that on the larger scale, the Ising model is once again described by an Ising model. This leads to the recursion:

$$k' = \frac{1}{4}\log\left[\cosh(4k)\right] \tag{12.17}$$

which, when compared with eq. (**??**), gives the same flow as the one-dimensional Ising model, and leads to the absence of a phase transition.

A much more interesting (but ad-hoc) case is realized as follows: See that k_1 and k_2 are ferromagnetic. Therefore, let us simply add the generated nnn interaction to the new nn interaction (there are as many nearest neighors as there are next-nearest neighbor interactions). This leaves us with

$$k' = k_1 + k_2 \tag{12.18}$$

$$= \frac{3}{8}\log\left[\cosh(4k)\right] \tag{12.19}$$

Note that $3/8 > 1/4$. Solving for the fixed point $k = k'$ leaves us with

$$k = \frac{1}{4}\cosh^{-1}\left[\exp(frac83k)\right] = 0.506981 \tag{12.20}$$

which compares favorably with the exact value 0.44069. The solution with adds k_1 and k_2 is deceptively simple, and not very good physics. In fact, this is the starting point of what is called "real-space renormalization", and of the renormalization in general.

It is also interesting, as was done by Wilson 1975, to compute the values of $k_1 = 1.00376$, $k_2 = 0.137327$, and $k_3 = -0.035960$ at the exactly known critical point.

Here we have that:

- Singularities in the free energy correspond to nontrivial unstable fixed points of the RG flow.
- Critical exponents correspond to the linearized RG flow close to the fixed point.
- Universality means that the behavior of the system described by the fixpoint of the recursion relation (or a differential equation) close to the fixed point.

Further reading is Kadanoff and Houghton 1975, and especially Wilson in his 1975 RMP[24] where, instead of 4 spins, he used 15 spins.

ICFP 2016 LECTURE 11/15

Renormalization Group, an Introduction

MARIS & Kadanoff AJP 1978

Kadanoff 1966

Plischke + Bergersen

Wilson '71

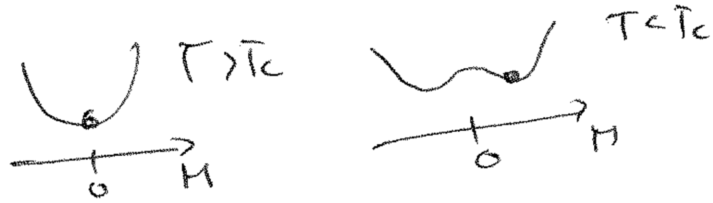
Wilson RMP ... 1975

Wilson RMP 1983

Reminder:

* Phase transition as non-analyticity of free energy with temperature, magnetic field, etc. → TRANSFER MATRIX → Exact solutions

* Landau theory: Free energy is analytic in M, ∇M, T, H; non-analyticity stems from



* Above d crit predictions of MFT are correct, below they are not

* Universality: Many models have the same exponents, how to explain

Example: One-dimensional Ising model:

$$Z = \sum_{\{\sigma\}} \exp\left[K(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_5 + \cdots\right]$$

$$= \sum \left[\exp K[\sigma_1+\sigma_3] + \exp[-K[\sigma_1+\sigma_3]]\right] \times \left[\exp K[\sigma_3+\sigma_5] + \exp[-K[\sigma_3+\sigma_5]]\right] \times \cdots$$

Write ~~exp[K~~ $\boxed{2\cosh[K[\sigma_1+\sigma_3]]} = f(K)\cdot\exp[K'\sigma_1\sigma_3]$.

$f(K)$: not depend on $\sigma_1\sigma_3$

Solution:

$\sigma_1 = 1 = \sigma_3$

$$e^{2K} + e^{-2K} = f \exp K'$$
$$2 = f \exp(-K')$$

$$\Rightarrow f^2 = 2\left[\exp(2K) + \exp(-2K)\right] = 4\cosh(2K)$$
$$f = 2 \cdot \cosh^{1/2}(2K).$$

$$1 = \cosh^{1/2}(2K)\exp(-K')$$
$$\exp(K') = \cosh^{1/2}(2K)$$
$$\boxed{K' = \tfrac{1}{2}\log\left(\cosh(2K)\right)} \quad *$$

$$Z = f(K)^{N/2} e^{K'(\sigma_1\sigma_3 + \sigma_3\sigma_5 + \sigma_5\sigma_7 + \cdots)}$$

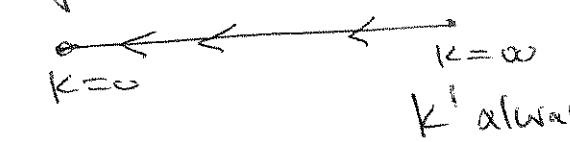
$$Z(N,K) = f(K)^{N/2} Z(N/2, K')$$

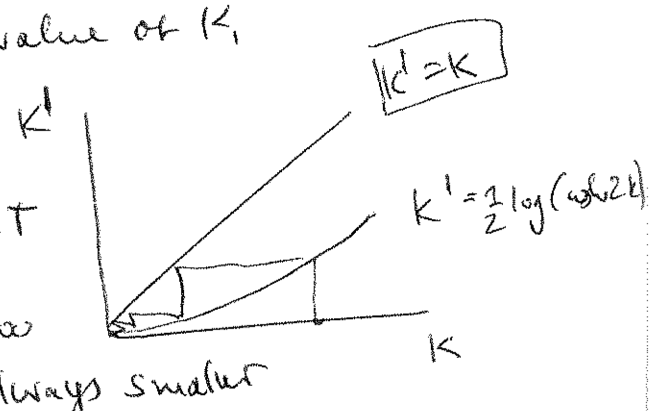
↑
original partition function

$$\ln Z = N\zeta$$

$$\zeta(K) = \tfrac{1}{2}\log f(K) + \tfrac{1}{2}\zeta(K')$$
$$\zeta(K') = 2\zeta(K) = \log\left(2\cosh^{1/2}(2K)\right).$$

If the partition function is known for one value of K (or T), then it is known for any other value of K, namely K'





K' always smaller than K. This means "Flow goes towards $T \to \infty$"

Practical Matters: One can write the recursion relations in the other way

$$\frac{1}{2}\cosh^{-1} \exp(2K') = K.$$

Using

$$\zeta(K) = \frac{1}{2}\log 2 + \frac{1}{2}K' + \frac{1}{2}\zeta(K').$$

$$K \leftarrow K'$$

$K' \approx 0.01$

$\zeta(0.01) \approx \log 2.$

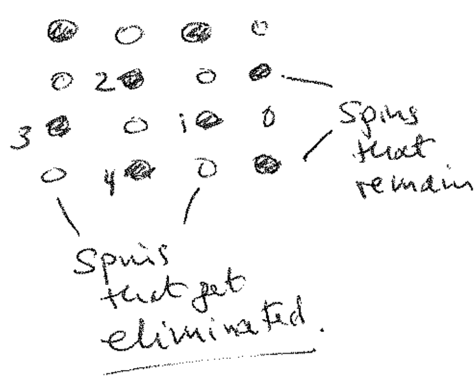
$\downarrow$

$K = 0.100334$

$K = 0.01.$

Let's use the same procedure for the two-dimensional Ising model

$$Z = \sum_{\dots} e^{K \sum_{nn} \sigma_i \sigma_j}$$



$$e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}$$

$$\prod_{\text{plaquettes}} 2\cosh\big(K(\underbrace{\sigma_1+\sigma_2+\sigma_3+\sigma_4}_{\text{sums of spins around plaquette}})\big)$$

Just as before, where we wrote

$$2\cosh(K(\sigma_1+\sigma_3)) = f(K)\exp(K'\sigma_1\sigma_3)$$

We now write

$\sigma_1+\sigma_2+\sigma_3+\sigma_4$ in terms of

1, $\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4$

and $\sigma_1\sigma_2\sigma_3\sigma_4$

$$\Rightarrow e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} =$$

$$f \exp\Big[\tfrac{1}{2}K_1(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)$$
$$+K_2(\sigma_1\sigma_3+\cdot \quad \dots$$

$f(K) = 2\cosh$ e^{4K}

$$K_1 = \frac{1}{4}\log\cosh(4K)$$
$$K_2 = \frac{1}{8}\log\cosh(4K)$$
$$K_3 = \frac{1}{8}\log\cosh(4K) - \frac{1}{2}\log\cosh(2K)$$

$$f(K) = 2\cosh^{1/2}(2K)\cosh^{1/8}(4K)$$

$$K_1 = \frac{1}{4}\log\cosh 4K$$

$$K_2 = \frac{1}{8}\log\cosh(4K)$$

$$K_3 = \frac{1}{8}\log\cosh(4K) - \frac{1}{2}\log\cosh(2K)$$

$$Z = f(K)^{N/2}\sum \exp\Big(K_1\sum_{nn}\sigma_p\sigma_q + K_2\sum_{nnn}\sigma_p\sigma_q + K_3\sum_{sq}\sigma_p\sigma_q\sigma_r\sigma_s\Big).$$

~~There is a difference in~~

K_1 is twice larger than K_2, because the nn terms arrive for two plaquettes.

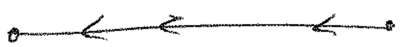
we obtain from a hamiltonian with nn interactions → hamiltonian with nn, nnn + sq interactions.

What to do?

(1) Simply ignore K_2, K_3.

→ $K' = \frac{1}{4}\log\cosh(4K)$

(compare, earlier $\frac{1}{2}\log\cosh(2K)$)

→  Absence of phase transition.

Better result:

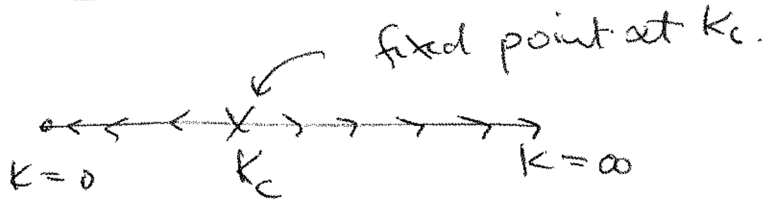
K_1 and K_2 are ferromagnetic.

Therefore, let us simply add the generated nnn interaction to the new nn interaction (there are as many nearest-neighbor as ~~next-nearest-~~ next-nearest-neighbor interactions!

$$K' = K_1 + K_2$$

$$= \frac{3}{8} \log \cosh(4K).$$

Ad hoc



Solving for fixed point: $K \to \tilde{K}$

$$\tilde{K} = \frac{1}{4} \operatorname{acosh}\left[\exp \frac{8}{3} K'\right] = 0.50698$$

It is at this point that the correlation function is infinite, and that one has an infinite correlation length.

A fixed point can only appear when the

In Stat Mechanics, ~~a fixed point~~

A renormalization tra....

Solution which adds K_1 and K_2 is deceptively simple.
In fact, this is the Starting point of what is called "Real-Space renormalization".

• Singularities of the free energy	⟺	nontrivial unstable fixed points of the ~~real~~ RG Flow
• CRITICAL EXPONENTS	⟺	Behavior of the linearized ~~renorm~~ RG Flow close to the Fixed Point
• Universality	⟺	Behavior of the system described by the fix of a recursion relation (or a differential equation) (close to the fixed Point).

Think globally; act locally

Further reading:

Kadanoff & Houghton 1975

And especially Wilson RMP 1975

⇓

Instead of using 4 spins

⇓

15 spins in the 2d Ising model

⇓

15 spins + 65000 configurations

Exact exponents to 0.2%.

```
import math
K0 = 1.0
Knn = 2.0
for k in [-1, 1]:
    for l in [-1, 1]:
        for m in [-1, 1]:
            for n in [-1, 1]:
                S0 = 1.0
                S2 = k * l + k * m + k * n + l * m + l * n + m * n
                S4 = k * l * m * n
                L0 = (2.0 * K0 + math.log(2.0) + math.log(math.cosh(4.0 * Knn)) / 8.0
+
                       math.log(math.cosh(2.0 * Knn)) / 2.0)
                L2 =  math.log(math.cosh(4.0 * Knn)) / 8.0
                L4 = math.log(math.cosh(4.0 * Knn)) / 8.0 - math.log(math.cosh(2.0 * K
nn)) / 2.0
                first_term = 2.0 * math.exp(2.0 * K0) * \
                                   math.cosh(Knn * (k + l + m + n) )
                second_term = math.exp(L0 * S0 + L2 * S2 + L4 * S4)
                print first_term, second_term
```

```
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Lecture 13

Quantum statistics 1/2: Ideal Bosons

Lecture 14

Quantum statistics 2/2: 4He and the 3D Heisenberg model, Non-classical rotational inertia

Lecture 15

The Fluctuation–Dissipation theorem (an introduction)

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